JORDAN'S EXPANSION OF THE RECIPROCAL OF THETA FUNCTIONS AND 2-DENSELY DIVISIBLE NUMBERS

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Abstract
We prove that the expansion of some classical infinite product is closely related to the sequence of 2-densely divisible numbers.

1. Introduction

Motivated by a mechanical problem, C. G. J. Jacobi [6] wrote the expression[1]

$$\frac{\sum(-1)^{\frac{\mu-1}{2}}\mu q^{\frac{1}{4}\mu^2}\cdot\sum(-1)^{\frac{\nu-1}{2}}q^{\frac{1}{4}\nu^2}\left(x^\nu y^\nu - x^{-\nu}y^{-\nu}\right)}{\sum(-q)^{m^2}x^{2m}\cdot\sum(-q)^{n^2}y^{2n}},$$

where μ and ν run through the positive odd integers, whereas m and n run by the integers.

Let $z = xy$, $q = e^{\tau\pi i}$ and $z = e^{\zeta\pi i}$. Using complex integration techniques, L. Kronecker [11] found several identities related to Jacobi's expression, e.g.,

$$2\pi i x y\sqrt{q}F(q,x,y) = \frac{\vartheta'(0)\,\vartheta(\xi+\eta)}{\vartheta\left(\xi+\frac{\tau}{2}\right)\,\vartheta\left(\eta+\frac{\tau}{2}\right)},$$

where

$$\begin{aligned}\vartheta(\zeta) &:= -i\sum_{\nu}(-1)^{\frac{\nu-1}{2}}q^{\frac{1}{4}\nu^2}\left(z^\nu - z^{-\nu}\right),\\ F(q,x,y) &:= \sum_{\mu}\sum_{\nu}q^{\frac{1}{2}\mu\nu}\left(x^\mu y^\nu - x^{-\mu}y^{-\nu}\right).\end{aligned}$$

[1] We omit the range of summation of μ, ν, m and n in order to preserve the notation from our main historical reference [11]. Throughout this introduction, we will keep this convention.

C. Jordan [7] made some substitutions in Kronecker's identities in order to obtain the expansion of the multiplicative inverse of some θ-functions[2], e.g.,

$$\frac{1}{\theta(w)} = \frac{1}{1-w} + \sum_{\substack{n, m > 0 \\ n \not\equiv m \pmod 2}} (-1)^n q^{\frac{nm}{2}} w^{\frac{m-n-1}{2}}, \tag{1}$$

where $\theta(w)$ is associated to the formal powers series given by the initial condition $\theta(0) = 1$ and the functional equations

$$\begin{aligned} \theta(q\,w) &= -w^{-1}\,\theta(w), \\ \theta(w^{-1}) &= -w^{-1}\,\theta(w). \end{aligned}$$

Using a powerful tool from algebraic geometry known as Gttsche's formula, T. Hausel, E. Letellier and F. Rodriguez-Villegas [5] studied a polynomial $C_n(q)$ which contains important topological information[3] about a rather mysterious space named the Hilbert scheme[4] of n points on the algebraic torus $(\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})$. This polynomial, named the E-polynomial of the space, is given by the infinite product

$$\prod_{m=1}^{\infty} \frac{(1-t^m)^2}{(1-qt^m)(1-q^{-1}t^m)} = 1 + \sum_{n=1}^{\infty} \frac{C_n(q)}{q^n} t^n,$$

which is equivalent to Jordan's identity (1) after a change of variables.

C. Kassel and C. Reutenauer [8, 9, 10] studied some number theoretical properties of $C_n(q)$ and of the polynomial $P_n(q) \in \mathbb{Z}[q]$ satisfying $C_n(q) = (q-1)^2 P_n(q)$. J. M. R. Caballero [1, 2, 3] referred to $P_n(q)$ as the *Kassel–Reutenauer q-analog of the sum of divisors* because $P_n(1)$ is the sum of divisors of n, as it was shown by C. Kassel and C. Reutenauer [8].

In virtue of (1), we can explicitly express $P_n(q)$ as

$$P_n(q) = \frac{1}{q-1} \sum_{\substack{d \mid n \\ d \equiv 1 \pmod 2}} \left(q^{n + \frac{1}{2}\left(\frac{2n}{d} - d - 1\right)} - q^{n - \frac{1}{2}\left(\frac{2n}{d} - d + 1\right)} \right). \tag{2}$$

It easily follows from (2) that the coefficients of $P_n(q)$ are non-negative. Motivated by this fact, the author wrote the code

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R.<q> = PolynomialRing(ZZ)
```

[2] Jordan never wrote this formula in [7], he only explained how to derive it from Kronecker's identities. Nevertheless, this identity can be found in [5].

[3] The information is about the so-called Hodge structure of the space.

[4] An informal way to visualize this space is to think that it is the set of all possible configurations of n sugar grains on the surface of a doughnut: each configuration is a point. The technical definition, due to A. Grothendieck, is based on the so-called functor of points.

```
def P(n):
    return R(1/(q-1)*sum([q^(n + ZZ( ( 2*n/d - d - 1 )/2  ))
    - q^(n-ZZ( ( 2*n/d - d + 1 )/2  ))
     for d in n.divisors() if d%2 == 1]))

print [n for n in [1..256] if len(P(n).coefficients()) == 2*n-1]
```

in SageMath in order to find the list of $n \geq 1$ for which all the coefficients of $P_n(q)$ are non-zero. Surprisingly, this integer sequence already exists: it is A174973 in [12], i.e., the sequence of 2-densely divisible numbers. T. Tao [14] defined these numbers in a more general way as follows.

Definition 1. If $y > 1$ and n is a natural number, we say that n is *y-densely divisible* if, for every $1 \leq R \leq n$, one can find a factor of n in the interval $[y^{-1}R, R]$.

The y-densely divisible numbers were used by the Polymath8 project [13], led by T. Tao, in order to improve the Zhang's bounded gaps between primes [15].

The aim of this paper is to prove the above-mentioned empirical connection between $P_n(q)$ and the sequence of 2-densely divisible numbers.

Theorem 2. *For any integer $n \geq 1$, all the coefficients of $P_n(q)$ are non-zero if and only if n is 2-densely divisible.*

Our method of proof will be rather atypical in number theory, because of the use of well-matched parentheses following our previous approach [4]. Despite the use of some terminology borrowed from the theory of formal languages[5], our proof will be completely elementary and no advanced knowledge from language theory will be required to follow the argument step by step.

2. A Language-theoretic Approach

Let L be a finite set of real numbers. Consider the set[6]

$$\mathcal{T}(L;t) := \bigcup_{\ell \in L} [\ell, \ell + t], \tag{3}$$

endowed with the topology inherited from $\mathbb{R}$, where $t > 0$ is an arbitrary real number. It is natural to associate any integer $n \geq 1$ with the topological space

$$\mathcal{T}_\lambda(n) := \mathcal{T}(L;t),$$

[5] We use commutative diagrams in a trivial, set-theoretic, way. No knowledge of category theory is required.

[6] As usual, we use the notation $[a,b]$, $]a,b]$, $[a,b[$ and $]a,b[$ for the sets of real numbers x satisfying $a \leq x \leq b$, $a < x \leq b$, $a \leq x < b$ and $a < x < b$ respectively.

where $L := \{\ln d : d|n\}$ and $t := \ln \lambda$. It follows that an integer $n \geq 1$ is λ-densely divisible if and only if $\mathcal{T}_\lambda(n)$ is connected (see Proposition 3).

We will show a relationship between the number of connected components of $\mathcal{T}(L;t)$ and the factorization of the Dyck word $\langle\langle S \rangle\rangle_\lambda$ introduced in [4], provided that $L = \{\ln s : s \in S\}$ and $t = \ln \lambda$. From this general result, we will derive a characterization of λ-densely divisible numbers in terms of the Dyck word $\langle\langle n \rangle\rangle_\lambda$, also introduced in [4]. We recall the definitions of $\langle\langle S \rangle\rangle_\lambda$ and $\langle\langle n \rangle\rangle_\lambda$ given in [4].

Definition 3. Consider a real number $\lambda > 1$ and a 2-letter alphabet $\Sigma = \{a, b\}$.

(i) Given a finite set of positive real numbers S, the λ-*class* of S is the word

$$\langle\langle S \rangle\rangle_\lambda := w_0\, w_1\, w_2 \dots w_{k-1} \in \Sigma^*, \tag{4}$$

such that each letter is given by

$$w_i := \begin{cases} a & \text{if } \mu_i \in S, \\ b & \text{if } \mu_i \in \lambda\, S, \end{cases} \tag{5}$$

for all $0 \leq i \leq k-1$, where μ_0, μ_1, ..., μ_{k-1} are the elements of the symmetric difference $S \triangle \lambda\, S$ written in increasing order, i.e.,

$$\begin{aligned} \lambda\, S &:= \{\lambda\, s : s \in S\}, \\ S \triangle \lambda\, S &= \{\mu_0 < \mu_1 < \dots < \mu_{k-1}\}. \end{aligned} \tag{6}$$

(ii) If S is the set of divisors of n, then we will write $\langle\langle n \rangle\rangle_\lambda := \langle\langle S \rangle\rangle_\lambda$. The word $\langle\langle n \rangle\rangle_\lambda$ will be called the λ-*class* of n.

The proof that $\langle\langle n \rangle\rangle_\lambda$ and $\langle\langle S \rangle\rangle_\lambda$ are Dyck words was given in [4]. Also, the height of the Dyck path associated to $\langle\langle n \rangle\rangle_\lambda$ coincides with the generalized Hooley's Δ_λ-function

$$\Delta_\lambda(n) := \max_{R>0} \# \left\{ d|n : d \in \left] \lambda^{-1} R, R \right] \right\},$$

where R runs over the positive real numbers (see [4]).

The main language-theoretic will be the following theorem.

Theorem 4. *Let $\lambda > 1$ be a real number.*

(i) For any integer $n \geq 1$, the number of connected components of $\mathcal{T}_\lambda(n)$ is precisely $\Omega\left(\langle\langle n \rangle\rangle_\lambda\right)$.

(ii) An integer $n \geq 1$ is λ-densely divisible if and only if $\langle\langle n \rangle\rangle_\lambda$ is an irreducible Dyck word.

The function $\Omega(w)$, formally defined using diagram (7), is just the number of irreducible Dyck words needed to obtain the Dyck word w as a concatenation of them[7]. We will derive Theorem 4 taking S to be the set of divisors of n in the following more general result.

Proposition 1. *Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers S. Define $L := \{\ln s \,:\, s \in S\}$ and $t := \ln \lambda$. The number of connected components of $\mathcal{T}(L;t)$ is $\Omega(\langle\langle S \rangle\rangle_\lambda)$.*

3. Preliminaries

Consider a 2-letter alphabet $\Sigma = \{a, b\}$. The *bicyclic semigroup*[8] $\mathcal{B}$ is the monoid given by the presentation

$$\mathcal{B} := \langle a, b | a\, b = \varepsilon \rangle\,,$$

where ε is the empty word.

Let $\pi : \Sigma^* \longrightarrow \mathcal{B}$ be the canonical projection. The *Dyck language* $\mathcal{D}$ is the kernel of π, i.e.,

$$\mathcal{D} := \pi^{-1}\left(\pi\left(\varepsilon\right)\right).$$

Interpreting the letters a and b as the displacements $1+\sqrt{-1}$ and $1-\sqrt{-1}$ in the complex plane $\mathbb{C}$, we can represent each word $w \in \mathcal{H}$ by means of a Dyck path, i.e., a lattice path from 0 to $|w|$, using only the above-mentioned steps and always keeping the imaginary part on the upper half-plane $\{z \in \mathbb{C} \,:\, \text{Im } z \geq 0\}$. For an example of a Dyck path, see Fig 1. It is easy to check that $\mathcal{D}$ can be described as the language corresponding to all possible Dyck paths.

The language of *reducible Dyck words* is the submonoid

$$\widetilde{\mathcal{D}} := \{\varepsilon\} \cup \{u\, v \,:\, u, v \in \mathcal{D} \backslash \{\varepsilon\}\}$$

of $\mathcal{D}$. The elements of the complement of $\widetilde{\mathcal{D}}$ in $\mathcal{D}$, denoted

$$\mathcal{P} := \mathcal{D} \backslash \widetilde{\mathcal{D}},$$

are called *irreducible Dyck words*.

It is well-known that $\mathcal{D}$ is freely generated by $\mathcal{P}$, i.e., every word in $\mathcal{D}$ may be formed in a unique way by concatenating a sequence of words from $\mathcal{P}$. So, there is a

[7]We use the notation $\Omega(w)$ in analogy to the arithmetical function $\Omega(n)$ which is equal to the number of prime factors of n counting their multiplicities.

[8]In this paper, the bicyclic semigroup is not just a semigroup, but also a monoid. We preserved the word "semigroup" in the name for historical reasons.

unique morphism of monoids[9] $\Omega : \mathcal{D} \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the monoid of non-negative integers endowed with the ordinary addition, such that the diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{P}^* \\ & \overset{\Omega}{\searrow} & \downarrow \\ & & \mathbb{N} \end{array} \tag{7}$$

commutes, where $\mathcal{D} \longrightarrow \mathcal{P}^*$ is the identification of $\mathcal{D}$ with the free monoid $\mathcal{P}^*$ and $\mathcal{P}^* \longrightarrow \mathbb{N}$ is just the length of a word in $\mathcal{P}^*$ considering each element of the set $\mathcal{P}$ as a single letter (of length 1). In other words, $\Omega(w)$, with $w \in \mathcal{D}$, is the number of irreducible Dyck words that we need to obtain w as a concatenation of them.

We will use the following result proved in [4].

Proposition 2. *Let S be a finite set of positive real numbers. For any real number $\lambda > 1$ we have that $\langle\langle S \rangle\rangle_\lambda \in \mathcal{D}$, i.e., $\langle\langle S \rangle\rangle_\lambda$ is a Dyck word.*

4. Generic Case

Given a finite set of positive real numbers S, we says that a real number $\lambda > 1$ is *regular* (with respect to S) if S and $\lambda\, S$ are disjoint. Otherwise, we say that $\lambda > 1$ is *singular* (with respect to S). This notion was already introduced in [4].

It is easy to check that the number of singular values (corresponding to a finite set S) is finite. In this section we will prove Proposition 1 under the additional hypothesis that λ is regular. The proof that this proposition also holds true for singular values of λ will be deduced from the case for regular values in next section.

Lemma 1. *Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers S. Suppose that λ is regular. Define $L := \{\ln s : s \in S\}$ and $t := \ln \lambda$. The space $\mathcal{T}(L;t)$ is disconnected if and only if $\langle\langle S \rangle\rangle_\lambda$ is a reducible Dyck word.*

Proof. Define $L + t := \{\ell + t : \ell \in L\}$. We have $L \cup (L+t) = \{\ln \mu_i : 0 \leq i \leq k-1\}$ because λ is regular. Here μ_0, μ_1, ..., μ_{k-1} are the numbers appearing in (6). Consider the word $\langle\langle S \rangle\rangle_\lambda = w_0\, w_1 \ldots w_{k-1}$ as defined in (4).

Suppose that $\mathcal{T}(L;t)$ is disconnected. In virtue of (3), for some $0 \leq j < k-1$, we have $\ln \mu_j + t < \ln \mu_{j+1}$, i.e., $\lambda\, \mu_j < \mu_{j+1}$. Indeed, if for any $0 \leq j < k-1$, we have $\ln \mu_{j+1} \leq \ln \mu_j + t$, then the space $\mathcal{T}(L;t) = [\ln \mu_0, \ln(\mu_{k-1}) + t]$ will be a connected.

So, the list μ_0, μ_1, ..., μ_j contains as many elements from S as elements from $\lambda\, S$. It follows from (5) that $u := w_0\, w_1 \ldots w_j$ satisfies $|u|_a = |u|_b$. So, u is Dyck word. Therefore, $\langle\langle S \rangle\rangle_\lambda$ is a reducible Dyck word, because its nonempty proper prefix u is a Dyck word.

[9] A morphism of free monoids is just a fancy name for substitution.

By Proposition 2, $\langle\langle S\rangle\rangle_\lambda$ is a Dyck word. Suppose that $\langle\langle S\rangle\rangle_\lambda$ is reducible. For some $0 \leq j < k-1$ we have that the nonempty proper prefix $u := w_0\, w_1 \ldots w_j$ of $\langle\langle S\rangle\rangle_\lambda$ is a Dyck word. The relation $|u|_a = |u|_b$ and (5) imply that the list μ_0, μ_1, ..., μ_j contains as many elements from S as elements from $\lambda\, S$. So, $\lambda\, \mu_j < \mu_{j+1}$, i.e., $\ln \mu_j + t < \ln \mu_{j+1}$. Using (3) we conclude that $\mathcal{T}\,(L; t)$ is disconnected. □

Lemma 2. *Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers S. Suppose that λ is regular. Define $L := \{\ln s \, : \, s \in S\}$ and $t := \ln \lambda$. The number of connected components of $\mathcal{T}\,(L; t)$ is $\Omega\,(\langle\langle S\rangle\rangle_\lambda)$.*

Proof. Let μ_0, μ_1, ..., μ_{k-1} be the numbers appearing in (6). Consider the word $\langle\langle S\rangle\rangle_\lambda = w_0\, w_1 \ldots w_{k-1}$ as defined in (4). By Proposition 2, $\langle\langle S\rangle\rangle_\lambda$ is a Dyck word. We proceed by induction on the number $c \geq 1$ of connected components of $\mathcal{T}\,(L; t)$.

Consider the case $c = 1$. Suppose that $\mathcal{T}\,(L; t)$ is connected. By Lemma 1, $\langle\langle S\rangle\rangle_\lambda$ is irreducible. Then $c = \Omega\,(\langle\langle S\rangle\rangle_\lambda) = 1$.

Suppose that the number of connected components of $\mathcal{T}\,(L; t)$ is $\Omega\,(\langle\langle S\rangle\rangle_\lambda)$, provided that $\mathcal{T}\,(L; t)$ has at most $c-1$ connected components for some $c > 1$. Assume that $\mathcal{T}\,(L; t)$ has precisely c connected components. By Lemma 1, $\langle\langle S\rangle\rangle_\lambda$ is reducible. Let p_1, p_2, ..., p_h be irreducible Dyck words satisfying $\langle\langle S\rangle\rangle_\lambda = p_1\, p_2 \ldots p_h$.

For some $0 \leq j < k-1$ we have $p_1 = w_0\, w_1 \ldots w_j$. Notice that $\lambda\, \mu_i \leq \mu_j < \mu_{j+1}$ for all $0 \leq i \leq j$ such that $\mu_i \in S$. Indeed, this follows from the fact that both p_1 and $p_2 \ldots p_h$ are Dyck words.

Setting $R = \{\mu_0, \mu_1, ..., \mu_j\}$, it follows that $\langle\langle S \backslash R\rangle\rangle_\lambda = p_2\, p_3 \ldots p_h$.

The space $\mathcal{T}\,(L \backslash \ln(R); t)$, where $\ln(R) := \{\ln s \, : \, s \in R\}$, has precisely $c-1$ connected components, because $\ln \mu_j + \ln \lambda < \ln \mu_{j+1}$. Applying the induction hypothesis, $c - 1 = \Omega\,(\langle\langle S \backslash R\rangle\rangle_\lambda) = h - 1$. Hence, $c = \Omega\,(\langle\langle S\rangle\rangle_\lambda) = h$.

By induction, we conclude that the number of connected components of $\mathcal{T}\,(L; t)$ is $\Omega\,(\langle\langle S\rangle\rangle_\lambda)$. □

5. General Case

Consider a 3-letter alphabet $\Gamma = \{a, b, c\}$. We define the *Hooley monoid* $\mathcal{C}$ to be the monoid given by the presentation

$$\mathcal{C} := \langle a, b, c | a\, b = \varepsilon, a\, c\, b = a\, b, c\, c = c \rangle\,.$$

Let $\varphi : \Gamma^* \longrightarrow \mathcal{C}$ be the canonical projection. The *Hooley-Dyck language* $\mathcal{H}$ is the kernel of φ, i.e.,

$$\mathcal{H} := \varphi^{-1}\,(\varphi\,(\varepsilon))\,.$$

Associating the letters a, b and c to the displacements $1+\sqrt{-1}$, $1-\sqrt{-1}$ and 1, respectively, in the complex plane $\mathbb{C}$, it follows that each word $w \in \mathcal{H}$ can be represented by Schröder path, i.e., a lattice path from 0 to $|w|$, using only the above-mentioned steps and always keeping the imaginary part on the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$. For an example of Schröder path, see Fig 2.

Notice that the language $\mathcal{H}$ corresponds to all possible Schröder paths having all the horizontal displacements (corresponding to c) strictly above the real axis.

The language of *reducible Hooley-Dyck words* is the submonoid

$$\widetilde{\mathcal{H}} := \{\varepsilon\} \cup \{u\, v : \ u, v \in \mathcal{H}\backslash\{\varepsilon\}\}$$

of $\mathcal{H}$. The elements of the complement of $\widetilde{\mathcal{H}}$ in $\mathcal{H}$, denoted

$$\mathcal{Q} := \mathcal{H}\backslash\widetilde{\mathcal{H}}$$

are called *irreducible Hooley-Dyck words.*

It is easy to check that $\mathcal{Q}$ freely generates $\mathcal{H}$. So, there is a unique morphism of monoids $\Theta : \mathcal{H} \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the monoid of non-negative integers endowed with the ordinary addition, such that the diagram

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{Q}^* \\ & \overset{\Theta}{\searrow} & \downarrow \\ & & \mathbb{N} \end{array} \tag{8}$$

commutes, where $\mathcal{H} \longrightarrow \mathcal{Q}^*$ is the identification of $\mathcal{Q}$ with the free monoid $\mathcal{Q}^*$ and $\mathcal{Q}^* \longrightarrow \mathbb{N}$ is just the length of a word in $\mathcal{Q}^*$ considering each element of the set $\mathcal{Q}$ as a single letter (of length 1).

Lemma 3. *Let $\gamma : \Gamma^* \longrightarrow \Sigma^*$ be the morphism of monoids given by $a \mapsto a$, $b \mapsto b$ and $c \mapsto \varepsilon$. We have that $\gamma(\mathcal{H}) \subseteq \mathcal{D}$.*

Proof. Notice that the diagram

$$\begin{array}{ccc} \Gamma^* & \xrightarrow{\varphi} & \mathcal{C} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Sigma^* & \xrightarrow{\pi} & \mathcal{B} \end{array} \tag{9}$$

commutes, where ψ is the morphism of monoids given by $\psi(C) := \gamma(C)$, for each equivalence class $C \in \mathcal{C}$.

Take $w \in \gamma\,(\ker\varphi)$. By definition, $w = \gamma(v)$ for some $v \in \ker\varphi$. Using the equalities

$$\begin{aligned}\pi(w) &= \pi\left(\gamma(v)\right)\\ &= \psi\left(\varphi(v)\right)\\ &= \psi\left(\varphi(\varepsilon)\right)\\ &= \pi(\varepsilon),\end{aligned}$$

we obtain that $w \in \ker \pi$. Hence, $\gamma\left(\ker \varphi\right) \subseteq \ker \pi$, i.e., $\gamma\left(\mathcal{H}\right) \subseteq \mathcal{D}$. □

Lemma 4. *The morphism γ defined in Lemma 3 satisfies $\gamma\left(\mathcal{Q}\right) \subseteq \mathcal{P}$.*

Proof. Take $q \in \mathcal{Q}$. By Lemma 3, we have $\gamma(q) \in \mathcal{D}$. Also, we have $\gamma(q) \neq \varepsilon$, because c^* and $\mathcal{Q}$ are disjoint, where $c^* := \{\varepsilon, c, cc, ccc, ...\}$.

Suppose that $\gamma(q) = u\, v$, for some $u, v \in \mathcal{D}\backslash\{\varepsilon\}$. It follows that $q = \hat{u}\, \hat{v}$ for some $\hat{u}, \hat{v} \in \Gamma^*$ satisfying $\gamma\left(\hat{u}\right) = u$ and $\gamma\left(\hat{v}\right) = v$. Using the commutative diagram 9, the fact that ψ is an isomorphism and the equalities,

$$\begin{aligned}\varphi(\hat{u}) &= \psi^{-1}\left(\pi\left(\gamma(\hat{u})\right)\right)\\ &= \psi^{-1}\left(\pi\left(u\right)\right)\\ &= \psi^{-1}\left(\pi\left(\varepsilon\right)\right)\\ &= \varphi\left(\varepsilon\right),\end{aligned}$$

we obtain that $\hat{u} \in \ker \varphi = \mathcal{H}$. Similarly, $\hat{v} \in \ker \varphi = \mathcal{H}$. Hence, $q \notin \mathcal{Q}$, contrary to our hypothesis. By *reductio ad absurdum*, $\gamma\left(\mathcal{Q}\right) \subseteq \mathcal{P}$. □

Lemma 5. *Given $w \in \mathcal{H}$, we have $\Theta\left(w\right) = \Omega\left(\gamma(w)\right)$, where γ is the morphism defined in Lemma 3, Θ is given by diagram (8) and Ω is given by diagram (7).*

Proof. Notice that the diagram

$$\begin{array}{ccc}\mathcal{H} & \longrightarrow & \mathcal{Q}^*\\ \downarrow & & \downarrow\\ \mathcal{D} & \longrightarrow & \mathcal{P}^*\end{array}$$

commutes, where $\mathcal{D} \longrightarrow \mathcal{P}^*$ is the identification of $\mathcal{D}$ with the free monoid $\mathcal{P}^*$, $\mathcal{H} \longrightarrow \mathcal{Q}^*$ is the identification of $\mathcal{H}$ with the free monoid $\mathcal{Q}^*$, $\mathcal{Q}^* \longrightarrow \mathcal{P}^*$ is the morphism of monoids given by $w \mapsto \gamma(w)$ for all $w \in \mathcal{Q}$ (this function is well-defined in virtue of Lemma 4) and $\mathcal{H} \longrightarrow \mathcal{D}$ is given by $w \mapsto \gamma(w)$ (this function is well-defined in virtue of Lemma 3). It follows that $\Theta\left(w\right) = \Omega\left(\gamma(w)\right)$ holds for each $w \in \mathcal{H}$. □

Lemma 6. *Let $\alpha : \Gamma^* \longrightarrow \Sigma^*$ be the morphism of monoids given by $a \mapsto a$, $b \mapsto b$ and $c \mapsto a\, b$. We have that $\alpha(\mathcal{H}) \subseteq \mathcal{D}$.*

Proof. Notice that the diagram

$$\begin{array}{ccc} \Gamma^* & \xrightarrow{\varphi} & \mathcal{C} \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\chi}\downarrow \\ \Sigma^* & \xrightarrow{\pi} & \mathcal{B} \end{array} \tag{10}$$

commutes, where χ is the morphism of monoids given by $\chi(C) := \alpha(C)$, for each equivalence class $C \in \mathcal{C}$.

Take $w \in \alpha(\ker\varphi)$. By definition, $w = \alpha(v)$ for some $v \in \ker\varphi$. Using the equalities

$$\begin{aligned} \pi(w) &= \pi(\alpha(v)) \\ &= \chi(\varphi(v)) \\ &= \chi(\varphi(\varepsilon)) \\ &= \pi(\varepsilon), \end{aligned}$$

we obtain that $w \in \ker\pi$. Hence, $\alpha(\ker\varphi) \subseteq \ker\pi$, i.e., $\alpha(\mathcal{H}) \subseteq \mathcal{D}$. □

Lemma 7. *The morphism α defined in Lemma 6 satisfies $\alpha(\mathcal{Q}) \subseteq \mathcal{P}$.*

Proof. Take $q \in \mathcal{Q}$. By Lemma 6, we have $\alpha(q) \in \mathcal{D}$. Using the fact that α does not decrease length, we have that $\alpha(q) \neq \varepsilon$, because $\varepsilon \notin \mathcal{Q}$.

Suppose that $\alpha(q) = u\,v$ for some $u, v \in \mathcal{D}\backslash\{\varepsilon\}$. It follows that $q = \hat{u}\,\hat{v}$ for some $\hat{u}, \hat{v} \in \Gamma^*$ satisfying $\alpha(\hat{u}) = u$ and $\alpha(\hat{v}) = v$. Using the commutative diagram 10, the fact that χ is an isomorphism and the equalities,

$$\begin{aligned} \varphi(\hat{u}) &= \chi^{-1}(\pi(\alpha(\hat{u}))) \\ &= \chi^{-1}(\pi(u)) \\ &= \chi^{-1}(\pi(\varepsilon)) \\ &= \varphi(\varepsilon), \end{aligned}$$

we obtain that $\hat{u} \in \ker\varphi = \mathcal{H}$. Similarly, $\hat{v} \in \ker\varphi = \mathcal{H}$. Hence, $q \notin \mathcal{Q}$, contrary to our hypothesis. By *reductio ad absurdum*, $\alpha(\mathcal{Q}) \subseteq \mathcal{P}$. □

Lemma 8. *Given $w \in \mathcal{H}$, we have $\Theta(w) = \Omega(\alpha(w))$, where α is the morphism defined in Lemma 6, Θ is given by diagram (8) and Ω is given by diagram (7).*

Proof. Notice that the diagram

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{Q}^* \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{P}^* \end{array}$$

commutes, where $\mathcal{D} \longrightarrow \mathcal{P}^*$ is the identification of $\mathcal{D}$ with the free monoid $\mathcal{P}^*$, $\mathcal{H} \longrightarrow \mathcal{Q}^*$ is the identification of $\mathcal{H}$ with the free monoid $\mathcal{Q}^*$, $\mathcal{Q}^* \longrightarrow \mathcal{P}^*$ is the morphism of monoids given by $w \mapsto \alpha(w)$ for all $w \in \mathcal{Q}$ (this function is well-defined in virtue of Lemma 7) and $\mathcal{H} \longrightarrow \mathcal{D}$ is given by $w \mapsto \alpha(w)$ (this function is well-defined in virtue of Lemma 6). It follows that $\Theta(w) = \Omega(\alpha(w))$ holds for each $w \in \mathcal{H}$. □

The following construction was previously used in [4].

Definition 5. Given a finite set of positive real numbers S, let ν_0, ν_1, ..., ν_{r-1} be the elements of the union $S \cup \lambda S$ written in increasing order, i.e.,

$$S \cup \lambda S = \{\nu_0 < \nu_1 < ... < \nu_{r-1}\}.$$

Consider the word

$$[\![S]\!]_\lambda := u_0\, u_1\, u_2 \ldots u_{r-1} \in \Gamma^*,$$

where each letter is given by

$$u_i := \begin{cases} a & \text{if } \nu_i \in S \setminus (\lambda S), \\ b & \text{if } \nu_i \in (\lambda S) \setminus S, \\ c & \text{if } \nu_i \in S \cap \lambda S, \end{cases}$$

for all $0 \le i \le r-1$.

Example 6. The Dyck path corresponding to $\langle\!\langle 126 \rangle\!\rangle_2 = aabaababbabb$ is shown in Fig 1. The Schröder path corresponding to $[\![126]\!]_2 = acabcaabccabbcabcb$ is shown in Fig 2.

Lemma 9. *Consider a finite set of positive real numbers S. For any real number $\lambda > 1$ we have $[\![S]\!]_\lambda \in \mathcal{H}$.*

Proof. We proceed by induction on the number of elements of S, denoted $m := \#S$.

For $m = 0$, we have $[\![S]\!]_\lambda = \varepsilon \in \mathcal{H}$.

Given $m > 0$, suppose that for each finite set of positive real numbers S, we have $[\![S]\!]_\lambda \in \mathcal{H}$, provided that $\#S < m$. Take an arbitrary finite set of real numbers S having precisely $\#S = m$ elements. Denote ν_0, ν_1, ν_2,..., ν_{r-1} the elements of $S \cup \lambda S$ written in increasing order. Consider the word $[\![S]\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{r-1}$ as given in Definition 5.

The inequality $\lambda > 1$ implies that there exists at least one integer i satisfying $u_i \neq a$ and $1 \le i \le r-1$. Define $j := \min\{i : u_i \neq a \text{ and } 1 \le i \le r-1\}$.

Suppose that $u_j = b$. Setting $S' := S\backslash\{\nu_0\}$, we have

$$[\![S']\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{j-2}\, \widehat{u}_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1},$$

where the hat indicates that the corresponding letter is suppressed. Indeed, $\lambda\nu_0 = \nu_j$ and $\nu_0 = \nu_1 = \ldots. = \nu_{j-1} = a$.

By the induction hypothesis, $[\![S']\!]_\lambda \in \mathcal{H}$. Hence, $[\![S]\!]_\lambda \in \mathcal{H}$, because it can be transformed into $[\![S']\!]_\lambda \in \mathcal{H}$ using the relation $a\, b = \varepsilon$ from $\mathcal{C}$.

Suppose that $u_j = c$ and $u_{j+1} = b$. Setting $S' := S\backslash\{\nu_0\}$, we have

$$[\![S']\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1}.$$

By the induction hypothesis, $[\![S']\!]_\lambda \in \mathcal{H}$. Hence, $[\![S]\!]_\lambda \in \mathcal{H}$, because it can be transformed into $[\![S']\!]_\lambda \in \mathcal{H}$ using the relation $a\, c\, b = a\, b$.

Suppose that $u_j = c$ and $u_{j+1} = c$. Setting $S' := S\backslash\{\nu_0\}$, we have

$$[\![S']\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1}.$$

By the induction hypothesis, $[\![S']\!]_\lambda \in \mathcal{H}$. Hence, $[\![S]\!]_\lambda \in \mathcal{H}$, because it can be transformed into $[\![S']\!]_\lambda \in \mathcal{H}$ using the relation $c\, c = c$.

Finally, suppose that $u_j = c$ and $u_{j+1} = a$. Setting $S' := S\backslash\{\nu_0\}$, we have

$$[\![S']\!]_\lambda = u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1}.$$

By the induction hypothesis, $[\![S']\!]_\lambda \in \mathcal{H}$. Then using the rewriting rules from $\mathcal{C}$, the word

$$u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{j+1} \ldots u_{r-1}$$

can be reduced to

$$u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, \widehat{u}_j\, u_{i_1}\;\, u_{i_2} \ldots u_{i_h},$$

where $u_{i_1} = b$, and the word obtained after the reduction $u_{j-1}\, u_{i_1} = \varepsilon$,

$$u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{i_2} \ldots u_{i_h},$$

can be reduced to the empty word using the rewriting rules from $\mathcal{C}$. So, using the rewriting rules from $\mathcal{C}$, the original word $[\![S]\!]_\lambda$ can be reduced to

$$u_0\, u_1\, u_2 \ldots u_{j-2}\, u_{j-1}\, u_j\, u_{i_1}\;\, u_{i_2} \ldots u_{i_h},$$

and the word obtained after the reduction $u_{j-1}\, u_j\, u_{i_1} = a\, c\, b = a\, b = \varepsilon$, can be reduced to the empty word as we mentioned above. Hence, $[\![S]\!]_\lambda \in \mathcal{H}$.

By induction, we conclude that $[\![S]\!]_\lambda \in \mathcal{H}$ for any finite set of positive real numbers S. $\square$

Lemma 10. *Consider a finite set of positive real numbers S. For any real number $\lambda > 1$, we have $\gamma\left(\llbracket S \rrbracket_\lambda\right) = \langle\!\langle S \rangle\!\rangle_\lambda$, where γ is the morphism defined in Lemma 3.*

Proof. In virtue of the identity $(S \cup \lambda S) \backslash (S \cap \lambda S) = S \triangle \lambda S$, the result follows just combining Definition 3 and Definition 5. □

Example 7. Lemma 10 can be illustrated by means of Fig 1 and Fig 2.

Lemma 11. *Consider a finite set of positive real numbers S. For any real number $\lambda > 1$, the equality $\alpha\left(\llbracket S \rrbracket_\lambda\right) = \langle\!\langle S \rangle\!\rangle_{\lambda'}$ holds for all $\lambda' \in]\lambda, +\infty[$ near enough to λ, where α is the morphism defined in Lemma 6.*

Proof. For any $\lambda' \in]\lambda, +\infty[$, the change from $S \cup \lambda\, S$ to $S \cup \lambda'\, S$ keeps fixed the points in S and it displaces the points in $\lambda\, S$ to the right. This displacement to the right can be made as small as we want just setting λ' near enough to λ. In particular, any point in $S \cap \lambda S$, after this transformation, becomes a pair of different points, one stays at the original position and the other one displaces to the right an arbitrary small distance. Notice that $S \cap \lambda'\, S = \emptyset$ for all $\lambda' \in]\lambda, +\infty[$ near enough to λ (this guarantees that λ' will be regular). Combining Definition 3 and Definition 5, we conclude that $\alpha\left(\llbracket S \rrbracket_\lambda\right) = \langle\!\langle S \rangle\!\rangle_{\lambda'}$ provided that $\lambda' \in]\lambda, +\infty[$ is near enough to λ. □

Example 8. Lemma 11 can be illustrated by means of Fig 2 and Fig 3.

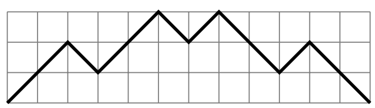

Figure 1: Representation of $\langle\!\langle 126 \rangle\!\rangle_2 = aabaababbabb$.

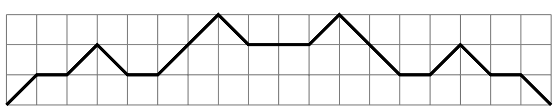

Figure 2: Representation of $\llbracket 126 \rrbracket_2 = acabcaabccabbcabcb$.

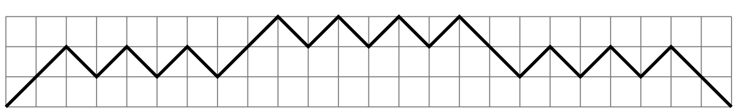

Figure 3: Representation of $\llbracket 126 \rrbracket_{2.001} = \langle\!\langle 126 \rangle\!\rangle_{2.001} = aabababaabababbababb$.

Lemma 12. *Let S be a finite set of positive real numbers. The step function $]1,+\infty[\ \longrightarrow \mathbb{N}$, given by $\lambda \mapsto \Omega\left(\langle\langle S\rangle\rangle_\lambda\right)$, is continuous from the right, i.e., given a real number $\lambda > 1$, for each real number $\lambda' \in\ ]\lambda,+\infty[$, we have $\Omega\left(\langle\langle S\rangle\rangle_\lambda\right) = \Omega\left(\langle\langle S\rangle\rangle_{\lambda'}\right)$, provided that λ' is near enough to λ.*

Proof. By Lemma 9, $[\![S]\!]_\lambda \in \mathcal{H}$. By Lemma 10, $\gamma\left([\![S]\!]_\lambda\right) = \langle\langle S\rangle\rangle_\lambda$, where γ is the morphism defined in Lemma 3. Using Lemma 5 we obtain $\Theta\left([\![S]\!]_\lambda\right) = \Omega\left(\langle\langle S\rangle\rangle_\lambda\right)$. By Lemma 11, $\alpha\left([\![S]\!]_\lambda\right) = \langle\langle S\rangle\rangle_{\lambda'}$ for all $\lambda' \in\ ]\lambda,+\infty[$ near enough to λ, where α is the morphism defined in Lemma 6. Using Lemma 8 we obtain $\Theta\left([\![S]\!]_\lambda\right) = \Omega\left(\langle\langle S\rangle\rangle_{\lambda'}\right)$ for all $\lambda' \in\ ]\lambda,+\infty[$ near enough to λ. Therefore, $\Omega\left(\langle\langle S\rangle\rangle_\lambda\right) = \Omega\left(\langle\langle S\rangle\rangle_{\lambda'}\right)$ for all $\lambda' \in\ ]\lambda,+\infty[$ near enough to λ. □

Lemma 13. *Let L be a finite set of real numbers. Consider the step function $f :]0,+\infty[\longrightarrow \mathbb{N}$ such that $f_L(t)$ is the number of connected components of $\mathcal{T}(L;t)$. The function $f_L(t)$ is continuous from the right, i.e., given a real number $t > 0$ we have $f_L(t') = f_L(t)$ for all $t' \in\ ]t,+\infty[$ near enough to t.*

Proof. Let $\ell_0, \ell_1, \ell_2, ..., \ell_{k-1}$ be the elements of L written in increasing order, i.e.,

$$L = \{\ell_0 < \ell_1 < \ell_2 < ... < \ell_{k-1}\}.$$

Define $c := f_L(t)$. In virtue of (3), we can write $\mathcal{T}(L;t)$ as the union

$$\mathcal{T}(L,t) = [\ell_{i_1}, \ell_{i_2} + t] \cup [\ell_{i_3}, \ell_{i_4} + t] \cup ... \cup \left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t\right]$$

of the pairwise disjoint sets $[\ell_{i_1}, \ell_{i_2} + t]$, $[\ell_{i_3}, \ell_{i_4} + t]$, ..., $\left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t\right]$, for some subsequence $i_1 < i_2 < i_3 < i_4 < ... < i_{2c-1} < i_{2c}$ of $0, 1, 2, ..., k-1$. So, for all $t' \in\ ]t,+\infty[$, the set $\mathcal{T}(L;t')$ can be expressed as the union

$$\mathcal{T}(L,t') = [\ell_{i_1}, \ell_{i_2} + t'] \cup [\ell_{i_3}, \ell_{i_4} + t'] \cup ... \cup \left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t'\right],$$

where some of sets in the list $[\ell_{i_1}, \ell_{i_2} + t']$, $[\ell_{i_3}, \ell_{i_4} + t']$, ..., $\left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t'\right]$ may overlap among them. Assuming that t' is near enough to t, we guarantee that the sets $[\ell_{i_1}, \ell_{i_2} + t']$, $[\ell_{i_3}, \ell_{i_4} + t']$, ..., $\left[\ell_{i_{2c-1}}, \ell_{i_{2c}} + t'\right]$ are pairwise disjoint. Hence, $f_L(t) = f_L(t')$ for all $t' \in\ ]t,+\infty[$ near enough to t. Therefore, $f_L(t)$ is continuous from the right. □

Using the previous auxiliary results, we can prove Proposition 1.

Proof. (Proposition 1) By Lemma 12, the step function $]1,+\infty[\ \longrightarrow \mathbb{N}$, given by $\lambda \mapsto \Omega\left(\langle\langle S\rangle\rangle_\lambda\right)$, is continuous from the right. By Lemma 13, the step function $f_L :]0,+\infty[\ \longrightarrow \mathbb{N}$ is continuous from the right, where $f_L(t)$ is the number of connected components of $\mathcal{T}(L;t)$. Notice that the step function $]1,+\infty[\ \longrightarrow \mathbb{N}$, given by $\lambda \mapsto f_L(\ln\lambda) - \Omega\left(\langle\langle S\rangle\rangle_\lambda\right)$, is continuous from the right, because the natural logarithm is continuous on $]0,+\infty[$. By Lemma 2, $f_L(\ln\lambda') - \Omega\left(\langle\langle S\rangle\rangle_{\lambda'}\right) = 0$ for all $\lambda' \in \left]\lambda, +\infty\right[$ near enough to λ (this guarantees that λ' is regular). Hence, $f_L(\ln\lambda) - \Omega(\langle\langle S\rangle\rangle_\lambda) = 0$ follows by continuity from the right. Therefore, the space $\mathcal{T}(L;t)$ has precisely $\Omega(\langle\langle S\rangle\rangle_\lambda)$ connected components. □

Proposition 3. *Given a real number $\lambda > 1$, an integer $n \geq 1$ is λ-densely divisible if and only if $\mathcal{T}_\lambda(n)$ is connected.*

Proof. Suppose that n is λ-densely divisible and $\mathcal{T}_\lambda(n)$ is disconnected. In virtue of (3), there are two divisors of n, denoted $d < d'$, satisfying

$$\ln d + \ln\lambda < \ln d'$$

and there is no divisor of n on the interval $\left]d, d'\right[$. Using the fact that n is λ-densely divisible, there is a divisor of n on the interval $[\lambda^{-1}R, R]$, with $1 \leq R := \lambda\,(d+\epsilon) < d' \leq n$, for all $\epsilon > 0$ small enough. Notice that $[\lambda^{-1}R, R] \subseteq \left]d, d'\right[$. So, there is a divisor of n on the interval $\left]d, d'\right[$. By *reductio ad absurdum*, if n is λ-densely divisible then $\mathcal{T}_\lambda(n)$ is connected.

Now, suppose that $\mathcal{T}_\lambda(n)$ is connected and n is not λ-densely divisible. Then there is some $R \in [1, n]$ such that there is no divisor of n on the interval $[\lambda^{-1}R, R]$. It follows that $R > \lambda > 1$, because 1 is a divisor of n. Let d be the largest divisor of n satisfying $d \leq \lambda^{-1}R$. It follows that $d < n$, because $\lambda^{-1}R \leq \lambda^{-1}n < n$. Let d' be the smallest divisor of n satisfying $\lambda^{-1}R < d'$. Notice that $\lambda^{-1}R < d'$, $\lambda\,d \leq R$ and there is no divisor of n on the interval $\left]d, d'\right[$.

Using the fact that $\mathcal{T}_\lambda(n)$ is connected, we have that

$$[\ln d, \ln d + \ln\lambda] \cap [\ln d', \ln d' + \ln\lambda] \neq \emptyset.$$

It follows that $\ln d' \leq \ln d + \ln\lambda$, i.e., $d' \leq \lambda\,d$. So, $\lambda^{-1}R < d' \leq \lambda\,d \leq R$. In particular, $d' \in [\lambda^{-1}R, R]$. By *reductio ad absurdum*, if $\mathcal{T}_\lambda(n)$ is connected then n is λ-densely divisible. □

We proceed now with the proof of the main result of this paper.

Proof. (Theorem 4) Statement (i) follows by Proposition 1 taking S to be the set of divisors of n.

Take an integer $n \geq 1$. By Proposition 3, n is λ-densely divisible if and only if $\mathcal{T}_\lambda(n)$ is connected. By Proposition 1, the space $\mathcal{T}_\lambda(n)$ is connected if and only if $\langle\langle n\rangle\rangle_\lambda$ is irreducible. Hence, n is λ-densely divisible if and only if $\langle\langle n\rangle\rangle_\lambda$ is irreducible. Therefore, statement (ii) holds. □

6. Proof of the Main Result

We proceed to prove our main result.

Proof of Theorem 2. In virtue of (2), the sequence of coefficients of $(q-1)P_n(q)$, read from left to right in the traditional degree-decreasing expansion, can be identified[10] with the Dyck word $\langle\langle n \rangle\rangle_2$ via the correspondence[11] $+1 \mapsto a$, $-1 \mapsto b$ and $0 \mapsto \varepsilon$.

Suppose that n is 2-densely divisible. In virtue of Theorem 4 (ii), the polynomial $(q-1)P_n(q)$ corresponds to the irreducible Dyck word $\langle\langle n \rangle\rangle_2$. For example, the polynomial

$$(q-1)P_6(q) = \underbrace{q^{11}}_{(} + \underbrace{q^{6}}_{(} - \underbrace{q^{5}}_{)} - \underbrace{1}_{)}$$

corresponds to the Dyck word $aabb$, associated to the well-matched parentheses $(\,(\,)\,)$. So, all the coefficients of $P_n(q) - \frac{q^{2n-1}-1}{q-1}$ are non-negatives. It follows that all the coefficients of $P_n(q)$ are non-zero.

Now, suppose that n is not 2-densely divisible. In virtue of Theorem 4 (ii), the polynomial $(q-1)P_n(q)$ corresponds to the reducible Dyck word $\langle\langle n \rangle\rangle_2$. For example, the polynomial

$$(q-1)P_{14}(q) = \underbrace{q^{27}}_{(} - \underbrace{q^{15}}_{)} + \underbrace{q^{12}}_{(} - \underbrace{1}_{)}$$

corresponds to the Dyck word $abab$, associated to the well-matched parentheses $(\,)\,(\,)$. So, $(q-1)P_n(q) = U_n(q) + V_n(q)$, for two polynomials $U_n(q)$ and $V_n(q)$ corresponding to some non-empty Dyck words u and v, respectively, such that $\langle\langle n \rangle\rangle_2 = uv$. We assume that the degree of every term in $U_n(q)$ is greater than the degree of $V_n(q)$. In our example, $U_{14}(q) = q^{27} - q^{15}$ and $V_{14}(q) = q^{12} - 1$. Let k be the degree of $V_n(q)$. It follows that the coefficient of q^k in $P_n(q)$ is equal to zero. □

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