TWO NEW EXPLICIT FORMULAS FOR THE BERNOULLI NUMBERS

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Abstract

We give two explicit formulas for the Bernoulli numbers, one in terms of the Stirling numbers of the second kind, and the other in terms of the Eulerian numbers. To the best of our knowledge, these formulas are new. We also derive two additional formulas that are likely already known.

1. Main Results

Definition 1. The *Bernoulli numbers* B_n can be defined by the generating function

$$\frac{t}{e^t-1} = \sum_{n\geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

There are many explicit formulas known for the Bernoulli numbers [1, 2]. Discussions on Bernoulli and Euler polynomials, and their generalized versions, can be found in [3], [4], and [5]. Here we prove the following formulas.

Theorem 1.1. *We have*

$$B_{r+1} = \frac{(-1)^r \cdot (r+1) \cdot 2^r}{2^{r+1}-1} \sum_{k=1}^{r} \frac{S(r,k)}{k+1} (-1)^k 2^{-2k} \frac{(2k-1)!}{(k-1)!}, \tag{1}$$

$$B_{r+1} = \frac{(-1)^r (r+1)}{2^r (2^{r+1}-1)} \binom{2r}{r-1} \sum_{l=1}^{r} (-1)^l \left\langle {r \atop r-l} \right\rangle \frac{\binom{r-1}{l-1}}{\binom{2r}{2l-1}}, \tag{2}$$

$$(-1)^{r-1} B_r = \sum_{k=1}^{r} (-1)^k \frac{S(r,k)}{k+1} \cdot (k-1)!, \tag{3}$$

and

$$(-1)^{r-1} B_r = \sum_{l=1}^{r} \left\langle {r \atop r-l} \right\rangle \frac{(-1)^l}{l \cdot \binom{r+1}{l}}, \tag{4}$$

where B_{r+1} is the Bernoulli number, $S(r,k)$ denotes the Stirling number of the second kind, and the $\left\langle {r \atop r-l} \right\rangle$ represent the Eulerian numbers.

Proof. Our proof of (1) and (2) relies on the following integral representation for the Riemann Zeta function

$$\zeta(s) = \frac{1}{\pi(2-2^s)} \int_0^\infty \frac{x^{-1/2}\mathbf{Li}_s(-x)}{1+x}\,dx, \tag{5}$$

which is valid for all $s \in \mathbb{C} \setminus \{1\}$. Here $\mathbf{Li}_s(-x)$ is the *polylogarithm function.* The above result is derived using the Ramanujan's master theorem in the appendix. The same can also be obtained from formula 3.2.1.6 in the book [6].

The integral representation (5) can be used to obtain

$$\zeta(0) = \frac{1}{\pi} \int_0^\infty \frac{x^{-1/2}\mathbf{Li}_0(-x)}{1+x}\,dx = \frac{-1}{2}.$$

Also,

$$\zeta(-r) = \frac{1}{\pi(2-2^{-r})} \int_0^\infty \frac{x^{-1/2}\mathbf{Li}_{-r}(-x)}{1+x}\,dx.$$

We use the following representation from [7]

$$\mathbf{Li}_{-r}(-x) = \sum_{k=1}^{r} k! S(r,k) \left(\frac{1}{1+x}\right)^{k+1} (-x)^k, \tag{6}$$

which can be easily proved using induction on r.

Now, we can deduce equation (1) from the following steps:

$$\begin{aligned}
\zeta(-r) &= \frac{1}{\pi(2-2^{-r})} \int_0^\infty \frac{x^{-1/2}}{1+x}\mathbf{Li}_{-r}(-x)\,dx \\
&= \frac{1}{\pi(2-2^{-r})} \int_0^\infty \frac{x^{-1/2}}{1+x} \sum_{k=1}^{r} k! S(r,k) \left(\frac{1}{1+x}\right)^{k+1} (-x)^k\,dx \\
&= \frac{1}{\pi(2-2^{-r})} \sum_{k=1}^{r} k! \cdot S(r,k) \cdot (-1)^k \cdot \beta\left(k+\frac{1}{2}, \frac{3}{2}\right) \\
&= \frac{1}{\pi(2-2^{-r})} \sum_{k=1}^{r} \frac{1}{k+1} \cdot S(r,k) \cdot (-1)^k \cdot \Gamma(k+1/2) \cdot \Gamma(3/2) \\
&= \frac{1}{\pi(2-2^{-r})} \sum_{k=1}^{r} \frac{1}{k+1} \cdot S(r,k) \cdot (-1)^k \cdot \frac{\Gamma(2k)}{\Gamma(k)} \cdot 2^{1-2k} \cdot \frac{\pi}{2} \\
&= \frac{2^r}{2^{r+1}-1} \sum_{k=1}^{r} \frac{S(r,k)}{k+1} (-1)^k 2^{-2k} \frac{(2k-1)!}{(k-1)!},
\end{aligned}$$

and the fact that $\zeta(-r) = (-1)^r \frac{B_{r+1}}{r+1}$. Here, $\Gamma(\cdot)$ and $\beta(\cdot,\cdot)$ are the Gamma and Beta functions respectively.

Another representation for $\mathbf{Li}_{-r}(-x)$ is the following from [8]:

$$\mathbf{Li}_{-r}(-x) = \frac{1}{(1+x)^{r+1}} \cdot \sum_{j=0}^{r-1} \left\langle {r \atop j} \right\rangle \cdot (-x)^{r-j}. \tag{7}$$

Now, to derive equation (2) we follow the steps below

$$\begin{aligned}
\zeta(-r) &= \frac{1}{\pi(2-2^{-r})} \int_0^\infty \frac{x^{-1/2}}{1+x} \mathbf{Li}_{-r}(-x)\, dx \\
&= \frac{1}{\pi(2-2^{-r})} \int_0^\infty \frac{x^{-1/2}}{1+x} \cdot \frac{1}{(1+x)^{r+1}} \cdot \sum_{j=0}^{r-1} \left\langle {r \atop j} \right\rangle \cdot (-x)^{r-j}\, dx \\
&= \frac{1}{\pi(2-2^{-r})} \sum_{j=0}^{r-1} (-1)^{r-j} \left\langle {r \atop j} \right\rangle \cdot \beta\left(r-j+\frac{1}{2}, j+\frac{3}{2}\right) \\
&= \frac{1}{\pi(2-2^{-r})} \sum_{j=0}^{r-1} (-1)^{r-j} \left\langle {r \atop j} \right\rangle \cdot \frac{\Gamma(r-j+1/2)\,\Gamma(j+3/2)}{\Gamma(r+2)} \\
&= \frac{1}{\pi(2-2^{-r})} \sum_{l=1}^{r} (-1)^{l} \left\langle {r \atop r-l} \right\rangle \cdot \frac{\Gamma(l+1/2)\,\Gamma(r-l+3/2)}{\Gamma(r+2)} \\
&= \frac{1}{\pi(2-2^{-r})} \sum_{l=1}^{r} (-1)^{l} \left\langle {r \atop r-l} \right\rangle \cdot \frac{2^{1-2l}\sqrt{\pi}\,\Gamma(2l)}{\Gamma(l)\Gamma(r+2)} \cdot \frac{2^{-1-2(r-l)}\sqrt{\pi}\,\Gamma(2(r-l+1)}{\Gamma(r-l+1)} \\
&= \frac{1}{2^r(2^{r+1}-1)\cdot(r+1)!} \sum_{l=1}^{r} (-1)^{l} \left\langle {r \atop r-l} \right\rangle \frac{(2l-1)!}{(l-1)!} \cdot \frac{(2r-2l+1)!}{(r-l)!} \\
&= \frac{1}{2^r(2^{r+1}-1)\cdot(r+1)!} \sum_{l=1}^{r} (-1)^{l} \left\langle {r \atop r-l} \right\rangle \frac{\binom{r-1}{l-1}}{\binom{2r}{2l-1}} \cdot \frac{(2r)!}{(r-1)!},
\end{aligned}$$

and use the fact that $\zeta(-r) = (-1)^r \frac{B_{r+1}}{r+1}$.

To prove (3) and (4), we use the integral representation

$$\zeta(s+1) = \frac{-1}{s} \int_0^\infty \frac{\mathbf{Li}_s(-x)}{(x)(1+x)}\, dx, \tag{8}$$

and the representations (6) and (7). The above equation can be derived by the result in the appendix. □

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Appendix

Theorem 1.2. For all $s \in \mathbb{C} \setminus \{1\}$, and $0 < n < 1$, we have

$$\int_0^\infty x^{n-1} \frac{\mathbf{Li}_s(-x)}{1+x} dx = \frac{\pi}{\sin n\pi}(\zeta(s) - \zeta(s, 1-n)), \tag{9}$$

where $\zeta(s, 1-n)$ represents the Hurwitz zeta function.

Proof. Let

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}$$

be the generalized harmonic number. Then, we have the following generating function from [9]:

$$\frac{\mathbf{Li}_s(x)}{1-x} = \sum_{n=1}^{\infty} H_n^{(s)} x^n,$$

for $|x| < 1$. We can also write

$$\frac{\mathbf{Li}_s(-x)}{(1+x)} = \sum_{n=1}^{\infty} H_n^{(s)} (-x)^n. \tag{10}$$

We have the following explicit form:

$$H_n^{(s)} = \zeta(s) - \zeta(s, n+1). \tag{11}$$

Next, we use Ramanujan's master theorem from [10], which is,

$$\int_0^\infty x^{n-1}\{\phi(0) - x\phi(1) + x^2\phi(2) - \cdots\}\, dx = \frac{\pi}{\sin n\pi}\phi(-n),$$

where the integral is convergent for $0 < \mathbf{Re}(n) < 1$, and after certain conditions are satisfied by ϕ. Now, using Ramanujan's master theorem with equations (10) and (11) gives us required equation (9). $\square$