ON BIVARIATE AND TRIVARIATE MIKI-TYPE IDENTITIES FOR BERNOULLI POLYNOMIALS

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Abstract

It is the main purpose of this paper to reconsider some bivariate convolution identities discovered by Pan and Sun (and later by Zagier) for Bernoulli polynomials that involve two different types of sums (precisely, an ordinary sum and a binomial sum) and rephrase them into more concise and convincing forms of identities based on operation methods for the generating function. Furthermore, we extend one of them to a third-order trivariate convolution identity.

1. Introduction

Let B_n and $B_n(x)$, $n = 0, 1, 2, \ldots$, be the Bernoulli numbers and polynomials defined by the generating functions

$$\mathcal{F}(t) := \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} \quad (|t| < 2\pi);$$

$$\mathcal{F}(t, x) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x) t^n}{n!} \quad (|t| < 2\pi),$$

respectively. It is easy to find the values $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}$, and so on. Since $\mathcal{F}(t) + \frac{1}{2}t$ is an even function, if $n \geq 3$ is odd, then $B_n = 0$. Further, we see that $B_n(0) = B_n$ and $B_n(x)$ is represented by

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i} \quad (n \geq 0).$$

Numerous recurrence relations for these numbers and polynomials have been developed over the years (cf., e.g., [29, 26, 20, 16]). Among them, the most basic linear recurrence relations are

$$\begin{aligned}
&\text{(i)} \quad B_0 = 1, \quad \sum_{i=0}^{n} \binom{n}{i} B_i = B_n \quad (n \geq 2); \\
&\text{(ii)} \quad B_0(x) = 1, \quad \sum_{i=0}^{n-1} \binom{n}{i} B_i(x) = nx^{n-1} \quad (n \geq 2),
\end{aligned} \tag{1.1}$$

which can be easily deduced from the functional identities $\mathcal{F}(t)e^t = \mathcal{F}(t) + t$ and $\mathcal{F}(t,x)e^t = \mathcal{F}(t,x) + te^{xt}$, respectively. On the other hand, the most basic quadratic recurrence relations are

$$\begin{aligned}
&\text{(i)} \quad \sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n; \\
&\text{(ii)} \quad \sum_{i=0}^{n} \binom{n}{i} B_i(x) B_{n-i}(y) = n(x+y-1)B_{n-1}(x+y) \\
&\qquad\qquad - (n-1)B_n(x+y).
\end{aligned} \tag{1.2}$$

The first identity (1.2) (i), usually attributed to Euler, reflects a property of the identity $\mathcal{F}(t)^2 = (1-t)\mathcal{F}(t) - t\frac{d}{dt}\mathcal{F}(t)$. The second identity (1.2) (ii) is a bivariate polynomial analogue of the first, due to Nörlund [27] (cf. [20, (50.11.2)]).

Dilcher [15] and later Chen [10] extended (1.2) (ii) to the higher-order identity

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \binom{n}{j_1,\ldots,j_k} B_{j_1}(x_1)\cdots B_{j_k}(x_k) \\
&= k\binom{n}{k} \sum_{r=0}^{k-1} (-1)^r \left\{ \sum_{j=0}^{k-1-r} \binom{r+j}{j} s(k, r+1+j) y^j \right\} \frac{B_{n-k+1+r}(y)}{n-k+1+r},
\end{aligned} \tag{1.3}$$

where $y := x_1 + \cdots + x_k$ for the independent variables $x_1, \ldots, x_k$ and $s(n,k)$ is the Stirling number of the first kind defined by the generating function

$$(z)_n := z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k) z^k. \tag{1.4}$$

Of course (1.3) is very interesting in its own right, but we now pay attention to extraordinary identities that involve two different kinds of convolution sums (an ordinary sum and a binomial sum). Such type of identities was first discovered by Miki [25] for Bernoulli numbers and later it was generalized in many directions. We therefore refer them to as “Miki-type” identities en-bloc for convenience sake.

This paper is to a large extent motivated by Miki-type identities for Bernoulli numbers and polynomials. Thus we begin by giving, in Section 2, a brief overview of the development process of these types of identities. Section 3 is just for some elementary lemmas, which will be needed later in our discussion. In Section 4 we reconsider the bivariate formulas for Bernoulli polynomials discovered by Pan and Sun, and later by Zagier, from our original point of view using the generating function methods. As a result, we rephrase them into more concise and convincing forms of identities. Subsequently, we extend one of them to a third-order trivariate convolution identity and close this paper by asking an open question on the multiple sum having any number of independent variables.

2. A Brief Overview on Miki-Type Identities

In 1978 Miki [25] discovered a remarkable identity that involves both ordinary and binomial convolutions for Bernoulli numbers using a certain congruence modulo p^2 (p an odd prime) for the Fermat quotient $q_p(a) := (a^{p-1}-1)/p$ of base a, where a is a positive integer with $p \nmid a$. His identity can be stated as follows:

$$\sum_{i=2}^{n-2} \frac{B_i}{i}\frac{B_{n-i}}{n-i} = \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i}{i}\frac{B_{n-i}}{n-i} + 2H_n \frac{B_n}{n} \quad (n \geq 4), \tag{2.1}$$

where $H_n := 1+\frac{1}{2}+\cdots+\frac{1}{n}$ is the harmonic number. Shortly afterward, Shiratani and Yokoyama [30] found another proof of (2.1) using p-adic analysis. An elementary proof has been presented in [2] by developing a linear version of (2.1), namely

$$\sum_{i=1}^{n-1} \frac{B_i}{i} k^{n-i} = \sum_{i=1}^{n-1} \binom{n}{i} \frac{B_i}{i} k^{n-i} + \sum_{j=1}^{k} \frac{(k-j)^n}{j} + k^n \left(H_n - H_k\right) \quad (n, k \geq 1),$$

which is actually equivalent to Faulhaber's formula for the power sum of the first k positive integers. In particular, when $k = 1$, this linear one reduces to an identity equivalent to (1.1) (i). For details, see [2, Sections 2 and 3].

Inspired by Miki's work, Matiyasevich [24] discovered a different type of formula and announced it on his website without proof. His original formula is

$$\sum_{i=2}^{n-2} B_i B_{n-i} = \frac{2}{n+2} \sum_{i=2}^{n-2} \binom{n+2}{i} B_i B_{n-i} + \frac{n(n+1)}{n+2} B_n \quad (n \geq 4), \tag{2.2}$$

which was later proved and generalized in many directions (see, e.g., [11, 18, 1]).

On the other hand, in 2005 Gessel [19] established a polynomial analogue of (2.1) using the Stirling numbers of the second kind and proved that for $n \geq 1$,

$$\sum_{i=1}^{n-1} \frac{B_i(x)}{i}\frac{B_{n-i}(x)}{n-i} = \frac{2}{n} \sum_{i=0}^{n-1} \binom{n}{i} \frac{B_i(x) B_{n-i}}{n-i} + B_{n-1}(x) + 2H_{n-1}\frac{B_n(x)}{n}. \tag{2.3}$$

We can find a short and intelligible proof of (2.3) in Crabb's paper [14] (see also [1]).

Further, a polynomial analogue of (2.2) can be stated as follows (cf. [1, 28]):

$$\sum_{i=1}^{n-1} B_i(x)B_{n-i}(x) = \frac{2}{n+2}\sum_{i=0}^{n-1}\binom{n+2}{i}B_i(x)B_{n-i} + \frac{n(n+1)}{6}B_{n-1}(x) + (n-1)B_n(x) \quad (n \geq 1). \tag{2.4}$$

In 2006 Pan and Sun [28] generalized both (2.3) and (2.4) to bivariate identities using differences and derivatives of polynomials and proved that if $x \neq y$, then

$$\begin{aligned}
\text{(i)}\quad & \sum_{i=0}^{n} B_i(x)B_{n-i}(y) \\
&= \sum_{i=0}^{n}\binom{n+1}{i+1}\frac{B_i(x-y)B_{n-i}(y)+B_i(y-x)B_{n-i}(x)}{i+2} \\
&\quad + \frac{B_{n+1}(x)+B_{n+1}(y)}{(x-y)^2} - \frac{2\left(B_{n+2}(x)-B_{n+2}(y)\right)}{(n+2)(x-y)^3} \quad (n \geq 0); \\
\text{(ii)}\quad & \sum_{i=1}^{n-1} \frac{B_i(x)}{i}\frac{B_{n-i}(y)}{n-i} \\
&= \sum_{i=1}^{n}\binom{n-1}{i-1}\frac{B_i(x-y)B_{n-i}(y)+B_i(y-x)B_{n-i}(x)}{i^2} \\
&\quad + H_{n-1}\frac{B_n(x)+B_n(y)}{n} + \frac{B_n(x)-B_n(y)}{n(x-y)} \quad (n \geq 1).
\end{aligned} \tag{2.5}$$

As we have seen, these identities involve some rational fraction terms with powers of $x-y$ in the denominator. However, these denominators disappear by virtue of a certain formula for the difference $B_n(x)-B_n(y)$ (see Proposition 4.3 below).

Concerning a product of two Bernoulli polynomials, it is well-known that (cf., e.g., Nielsen [26, p.75], Carlitz [7, 8, 9] and Zagier [34, (A.30)])

$$\frac{B_i(x)}{i}\frac{B_j(x)}{j} = \sum_{m=0}^{\lfloor \frac{i+j}{2}\rfloor}\left\{\frac{1}{i}\binom{i}{2m}+\frac{1}{j}\binom{j}{2m}\right\}\frac{B_{2m}B_{i+j-2m}(x)}{i+j-2m} + (-1)^{i-1}\frac{(i-1)!(j-1)!}{(i+j)!}B_{i+j} \quad (i, j \geq 1). \tag{2.6}$$

Without going into detail, (2.6) has an intimate connection with the integral

$$\int_0^x B_r(t)B_s(t)dt = \frac{r!s!}{(r+s+1)!}\sum_{m=0}^{r}(-1)^m\binom{r+s+1}{r-m}T_m^{(r,s)}(x) \quad (r, s \geq 1),$$

where $T_m^{(r,s)}(x) := B_{r-m}(x)B_{s+1+m}(x) - B_{r-m}B_{s+1+m}$ (see [4, Proposition 1]).

In 2014 Zagier [34, (A.35)] extended (2.6) to a bivariate version by using some significant properties of the periodic extension $\overline{B}_n(x) := B_n(x - \lfloor x \rfloor)$ of $B_n(x)$ defined on the interval $(0, 1]$. Actually, finding the integral formula

$$\int_0^1 \overline{B}_r(x+\alpha)\overline{B}_s(x+\beta)dx = (-1)^{r-1}\frac{r!s!}{(r+s)!}\overline{B}_{r+s}(\alpha-\beta) \quad (r, s \geq 1),$$

where α, β are arbitrary real numbers, he proved that

$$\begin{aligned}\frac{B_i(x)}{i}\frac{B_j(y)}{j} = \sum_{m=0}^{\max\{i,j\}} &\left\{\frac{1}{i}\binom{i}{m}\frac{B_{i+j-m}(y)}{i+j-m} + \frac{(-1)^m}{j}\binom{j}{m}\frac{B_{i+j-m}(x)}{i+j-m}\right\}B_m^+(x-y) \\ &+ (-1)^{j-1}\frac{(i-1)!(j-1)!}{(i+j)!}B_{i+j}^+(x-y) \quad (i, j \geq 1),\end{aligned}$$

where, using his notation, $B_k^+(x)$ denotes

$$B_k^+(x) := \frac{B_k(x) + (-1)^k B_k(-x)}{2} = \frac{B_k(x) + B_k(x+1)}{2} = B_k(x) + \frac{k}{2}x^{k-1}.$$

Taking here $i + j = n \geq 2$ and summing up over $i = 1, 2, \ldots, n-1$, he finally established the Miki-type bivarite identity such that

$$\begin{aligned}&\sum_{i=1}^{n-1}\frac{B_i(x)}{i}\frac{B_{n-i}(y)}{n-i} - H_{n-1}\frac{B_n(x)+B_n(y)}{n} \\ &= \sum_{m=1}^{n-1}\binom{n-1}{m}\left(B_{n-m}(y) + (-1)^m B_{n-m}(x)\right)\frac{B_m^+(x-y)}{m} \\ &\quad + (1+(-1)^n)\frac{B_n^+(x-y)}{n^2} \quad \text{(see [34, (A.36)])}.\end{aligned} \tag{2.7}$$

As has been confirmed by Zagier himself, (2.7) is just the same as (2.5) (ii) if $x \neq y$.

Although all the above formulas are quadratic, Gessel [19, (4)] found a third-order analogue of Miki's (2.1) for Bernoulli numbers, namely

$$\begin{aligned}&\sum_{\substack{i+j+k=n \\ i,j,k\geq 2}}\frac{B_iB_jB_k}{ijk} - \sum_{\substack{i+j+k=n \\ i,j,k\geq 2}}\binom{n}{i,j,k}\frac{B_iB_jB_k}{ijk} - 3H_n\sum_{i=2}^{n-2}\binom{n}{i}\frac{B_iB_{n-i}}{i(n-i)} \\ &= 6H_{n,2}\frac{B_n}{n} - \frac{n^2-3n+5}{4}\frac{B_{n-2}}{n-2} \quad (n \geq 4),\end{aligned} \tag{2.8}$$

where $H_{n,2} := \sum_{1\leq i<j\leq n}\frac{1}{ij}$ $(n \geq 2)$ is the generalized harmonic number.

It is worth mentioning that a slightly modified version of (2.8) has been derived by Dunne and Schubert [18, Theorem 5.2] from a quite different point of view using some basic tools from perturbative quantum field theory and string theory.

Besides the above identities, there are, as a matter of course, many interesting new approaches and generalizations of Miki-type identities in various directions. For instance, see [31, 13, 11, 12, 21, 22, 33, 17] and the references therein.

3. Some Lemmas

To begin with, we present the following lemma related to the multivariate beta integral, which will be needed later in our discussion.

Lemma 3.1. *Let $s \geq 2$ and $j_1, \ldots, j_s \geq 0$ be arbitrary integers. With the variables of integration $u_1, u_2, \ldots, u_{s-1}$, let us set $w_s := 1 - \sum_{i=1}^{s-1} u_i$. Then we have*

$$\begin{aligned}\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{s-2}} & \left(u_1^{j_1} \cdots u_{s-1}^{j_{s-1}} w_s^{j_s}\right) du_{s-1} \cdots du_1 \\ &= \frac{j_1! j_2! \cdots j_s!}{(j_1 + \cdots + j_s + s - 1)!}.\end{aligned} \tag{3.1}$$

Proof. We will give an elementary proof by induction on s. In the case when $s = 2$, several methods of proof are available, but perhaps the simplest being to use the partial fraction decomposition of the reciprocal, namely

$$\prod_{r=0}^{n} \frac{1}{X+r} = \frac{1}{n!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{X+r} \quad (n \geq 0).$$

Actually, putting here $X = j_1 + 1$ and $n = j_2$, we have

$$\begin{aligned}\int_0^1 u_1^{j_1} w_2^{j_2} du_1 &= \int_0^1 u_1^{j_1} (1-u_1)^{j_2} du_1 = \sum_{r=0}^{j_2} (-1)^r \binom{j_2}{r} \int_0^1 u_1^{j_1+r} du_1 \\ &= \sum_{r=0}^{j_2} (-1)^r \binom{j_2}{r} \frac{1}{j_1+1+r} = j_2! \prod_{r=0}^{j_2} \frac{1}{j_1+1+r} = \frac{j_1! j_2!}{(j_1+j_2+1)!},\end{aligned}$$

and hence (3.1) holds true for $s = 2$. For the next induction step, assuming that (3.1) is true for $s = k \geq 3$, we have

$$\begin{aligned}&\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{k-1}} \left(u_1^{j_1} \cdots u_k^{j_k} w_{k+1}^{j_{k+1}}\right) du_k \cdots du_1 \\ &\quad = \frac{j_2! \cdots j_{k+1}!}{(j_2 + \cdots + j_{k+1} + k - 1)!} \int_0^1 \left(u_1^{j_1} w_2^{j_2+j_3+\cdots+j_{k+1}+k-1}\right) du_1 \\ &\quad = \frac{j_2! \cdots j_{k+1}!}{(j_2 + \cdots + j_{k+1} + k - 1)!} \cdot \frac{j_1! (j_2 + \cdots + j_{k+1} + k - 1)!}{(j_1 + \cdots + j_{k+1} + k)!} \\ &\quad = \frac{j_1! j_2! \cdots j_{k+1}!}{(j_1 + \cdots + j_{k+1} + k)!},\end{aligned}$$

which shows that (3.1) is true again for $s = k+1$; so the proof is now complete. □

Another proof of (3.1) can be found in [32, (1.2)], where it was further generalized.

We will utilize only the special cases of (3.1) for $s = 2, 3$ in the next Section 4, namely, changing the notation of variables in integrals,

$$\int_0^1 u^i (1-u)^j du = \frac{i!j!}{(i+j+1)!}; \tag{3.2}$$

$$\int_0^1 \int_0^{1-u} u^i v^j (1-u-v)^k dv du = \frac{i!j!k!}{(i+j+k+2)!}. \tag{3.3}$$

The following lemma is essentially the same as stated in [5, Lemma 1]. However, for the sake of completeness we wish to give its full proof once again.

Lemma 3.2. *Let $m \geq 2$ be an integer and $f_\alpha(Y)$ be the function of Y defined by*

$$f_\alpha(Y) := \frac{1}{Y^\alpha - 1} \quad \textit{for} \quad \alpha \in \mathbb{C} \backslash \{0\}.$$

Then for $\alpha_1, \dots, \alpha_m \in \mathbb{C} \backslash \{0\}$ satisfying $\kappa_m := \alpha_1 + \cdots + \alpha_m \neq 0$ we have

$$\prod_{i=1}^m f_{\alpha_i}(Y) = f_{\kappa_m}(Y) \left\{ 1 + \sum_{r=1}^{m-1} \sum_{1 \leq i_1 < \cdots < i_r \leq m} \prod_{j=1}^r f_{\alpha_{i_j}}(Y) \right\}. \tag{3.4}$$

Proof. Letting $y_1, \dots, y_m$ be arbitrary non-zero numbers or polynomials, we first consider the regular expansion

$$\prod_{i=1}^m (y_i + 1) - 1 = \prod_{i=1}^m y_i \left\{ 1 + \sum_{r=1}^{m-1} \sum_{1 \leq i_1 < \cdots < i_r \leq m} \prod_{j=1}^r \frac{1}{y_{i_j}} \right\}.$$

Dividing both sides of this by the left-hand side and then multiplying by $\prod_{i=1}^m y_i^{-1}$, we have

$$\prod_{i=1}^m y_i^{-1} = \left(\prod_{i=1}^m (y_i + 1) - 1 \right)^{-1} \left\{ 1 + \sum_{r=1}^{m-1} \sum_{1 \leq i_1 < \cdots < i_r \leq m} \prod_{j=1}^r \frac{1}{y_{i_j}} \right\}. \tag{3.5}$$

Setting here $y_i = 1/f_{\alpha_i}(Y) = Y^{\alpha_i} - 1$ for $i = 1, 2, \dots, m$, it can be shown that

$$\left(\prod_{i=1}^m (y_i + 1) - 1 \right)^{-1} = (Y^{\kappa_m} - 1)^{-1} = f_{\kappa_m}(Y),$$

and thereby, we see that (3.5) implies (3.4). $\square$

In addition, we often use the obvious integral representations of H_n such that

$$H_n = \int_0^1 \frac{1-u^n}{1-u} du = \int_0^1 \frac{1-(1-u)^n}{u} du \quad (n \geq 1). \tag{3.6}$$

4. Main Results

We first prove the following Miki-type bivariate identities that involve two different kinds of sums by manipulating the generating function $\mathcal{F}(t,x)$.

Theorem 4.1. *For all integers $n \geq 1$ we have*

$$\begin{aligned}(n+2)\sum_{i=0}^{n} B_i(x)B_{n-i}(y) &= \sum_{i=1}^{n} i\binom{n+2}{i+2}(x-y)^{i-1}B_{n-i}(y) \\ &\quad + \sum_{i=0}^{n}\binom{n+2}{i}\left(B_i(x)B_{n-i}(y-x)+B_i(y)B_{n-i}(x-y)\right);\end{aligned} \tag{4.1}$$

$$\begin{aligned}(n+1)\sum_{i=1}^{n}\frac{B_i(x)B_{n-i}(y)}{i} &= \sum_{i=1}^{n}\binom{n+1}{i+1}(x-y)^{i-1}B_{n-i}(y) \\ &\quad + \sum_{i=0}^{n-1}\binom{n+1}{i}\left(B_i(x)B_{n-i}(y-x)+\frac{B_i(y)B_{n-i}(x-y)}{n-i}\right) \\ &\quad + (n+1)(H_n-1)B_n(y)+(n+1)B_n(x);\end{aligned} \tag{4.2}$$

$$\begin{aligned}(n+1)\sum_{i=0}^{n-1}\frac{B_i(x)B_{n-i}(y)}{n-i} &= \sum_{i=1}^{n} i\binom{n+1}{i+1}(x-y)^{i-1}B_{n-i}(y) \\ &\quad + \sum_{i=0}^{n-1}\binom{n+1}{i}\left(B_i(y)B_{n-i}(x-y)+\frac{B_i(x)B_{n-i}(y-x)}{n-i}\right) \\ &\quad + (n+1)(H_n-1)B_n(x)+(n+1)B_n(y)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}n\sum_{i=1}^{n-1}\frac{B_i(x)}{i}\frac{B_{n-i}(y)}{n-i} &= \sum_{i=1}^{n}\binom{n}{i}(x-y)^{i-1}B_{n-i}(y) \\ &\quad + \sum_{i=1}^{n}\frac{1}{i}\binom{n}{i}\left(B_{n-i}(x)B_i(y-x)+B_{n-i}(y)B_i(x-y)\right) \\ &\quad + H_{n-1}\left(B_n(x)+B_n(y)\right).\end{aligned} \tag{4.4}$$

Proof. Let u be an arbitrary real or complex number with $u \neq 0, 1$. Applying (3.4) with $m = 2$ to $\alpha_1 = u$ and $\alpha_2 = 1-u$, we obtain the identity

$$f_u(Y)f_{1-u}(Y) = f_1(Y)\left(1+f_u(Y)+f_{1-u}(Y)\right),$$

that is, more concretely,

$$\frac{1}{Y^u-1}\cdot\frac{1}{Y^{1-u}-1} = \frac{1}{Y-1}\left(1+\frac{1}{Y^u-1}+\frac{1}{Y^{1-u}-1}\right).$$

We put here $Y = e^t$ and then multiply both sides by $u(1-u)t^2e^{(y+u(x-y))t}$. Then, noting that

$$e^{(y+u(x-y))t} = e^{xt}e^{(y-x)(1-u)t} = e^{yt}e^{(x-y)ut},$$

one can find the functional identity

$$\begin{aligned}&\mathcal{F}(ut,x)\mathcal{F}((1-u)t,y)\\ =&u(1-u)te^{u(x-y)t}\mathcal{F}(t,y) + u\mathcal{F}(t,x)\mathcal{F}((1-u)t,y-x)\\ &+(1-u)\mathcal{F}(t,y)\mathcal{F}(ut,x-y).\end{aligned} \tag{4.5}$$

Differentiating both sides of (4.5) n times with respect to t and then setting $t = 0$, we are able to derive the following identity:

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}u^i(1-u)^{n-i}B_i(x)B_{n-i}(y)\\ =&n\sum_{i=0}^{n-1}\binom{n-1}{i}u^{i+1}(1-u)(x-y)^iB_{n-1-i}(y)\\ &+\sum_{i=0}^{n}\binom{n}{i}u(1-u)^{n-i}B_i(x)B_{n-i}(y-x)\\ &+\sum_{i=0}^{n}\binom{n}{i}u^{n-i}(1-u)B_i(y)B_{n-i}(x-y).\end{aligned} \tag{4.6}$$

All the proofs of (4.1)–(4.4) we will give below rely on this identity.

(a) In order to deduce (4.1), we integrate both sides of (4.6) from 0 to 1 with respect to u based on (3.2) and use the identities $n\binom{n-1}{i}\frac{(i+1)!1!}{(i+3)!} = \frac{i+1}{(n+2)(n+1)}\binom{n+2}{i+3}$ and $\binom{n}{i}\frac{(n-i)!1!}{(n+2-i)!} = \frac{1}{(n+2)(n+1)}\binom{n+2}{i}$. Then we get (4.1) immediately by multiplying the whole by $(n+2)(n+1)$.

(b) For (4.2), we gather the terms in (4.6) that involve $B_n(y)$ in one place and then divide the whole by u to obtain the identity

$$\begin{aligned}&\sum_{i=1}^{n}\binom{n}{i}u^{i-1}(1-u)^{n-i}B_i(x)B_{n-i}(y) + \frac{(1-u)^n-(1-u)}{u}B_n(y)\\ =&n\sum_{i=0}^{n-1}\binom{n-1}{i}u^i(1-u)(x-y)^iB_{n-1-i}(y)\\ &+\sum_{i=0}^{n}\binom{n}{i}(1-u)^{n-i}B_i(x)B_{n-i}(y-x)\\ &+\sum_{i=0}^{n-1}\binom{n}{i}(1-u)u^{n-1-i}B_i(y)B_{n-i}(x-y).\end{aligned}$$

We integrate both sides of this identity from 0 to 1 with respect to u based on (3.2). Then using the identities $n\binom{n-1}{i}\frac{i!1!}{(i+2)!} = \frac{1}{n+1}\binom{n+1}{i+2}$, $\binom{n}{i}\frac{0!(n-i)!}{(n+1-i)!} = \frac{1}{n+1}\binom{n+1}{i}$ and $\binom{n}{i}\frac{1!(n-1-i)!}{(n+1-i)!} = \frac{1}{(n+1)(n-i)}\binom{n+1}{i}$, from (3.6) we obtain (4.2) after multiplying the whole by $n+1$.

(c) Similar to the above, gathering the terms in (4.6) that involve $B_n(x)$ in one place and dividing the whole by $1-u$, we get the identity

$$\begin{aligned}
&\sum_{i=0}^{n-1}\binom{n}{i}u^i(1-u)^{n-1-i}B_i(x)B_{n-i}(y)+\frac{u^n-u}{1-u}B_n(x)\\
&\qquad =n\sum_{i=0}^{n-1}\binom{n-1}{i}u^{i+1}(x-y)^iB_{n-1-i}(y)\\
&\qquad\quad +\sum_{i=0}^{n-1}\binom{n}{i}u(1-u)^{n-1-i}B_i(x)B_{n-i}(y-x)\\
&\qquad\quad +\sum_{i=0}^{n}\binom{n}{i}u^{n-i}B_i(y)B_{n-i}(x-y).
\end{aligned}$$

We now integrate both sides of this from 0 to 1 with respect to u. By applying (3.2), (3.6) and the same binomial identities as used in (b) we can deduce (4.3) after multiplying the whole by $n+1$.

(d) For (4.4), we next gather the terms in (4.6) that involve $B_n(x)$ and $B_n(y)$ in one place and then divide the whole by $u(1-u)$ to obtain the identity

$$\begin{aligned}
&\sum_{i=1}^{n-1}\binom{n}{i}u^{i-1}(1-u)^{n-1-i}B_i(x)B_{n-i}(y)+\frac{u^{n-1}-1}{1-u}B_n(x)\\
&\qquad +\frac{(1-u)^{n-1}-1}{u}B_n(y)\\
&\qquad =n\sum_{i=0}^{n-1}\binom{n-1}{i}u^{i}(x-y)^iB_{n-1-i}(y)\\
&\qquad\quad +\sum_{i=0}^{n-1}\binom{n}{i}(1-u)^{n-1-i}B_i(x)B_{n-i}(y-x)\\
&\qquad\quad +\sum_{i=0}^{n-1}\binom{n}{i}u^{n-1-i}B_i(y)B_{n-i}(x-y).
\end{aligned}$$

In the same way as before, by integrating both sides of this from 0 to 1 with respect to u based on (3.2) and using (3.6) we can deduce (4.4), as desired. □

Here note that we can deduce (4.4) with no discussion at all of (d). Indeed, adding two identities (4.2) and (4.3) side-by-side without changing the original array of the sums and using the trivial identities $\frac{1}{i}+\frac{1}{n-i}=\frac{n}{i(n-i)}$, $(i+1)\binom{n+1}{i+1}=(n+1)\binom{n}{i}$ and $H_n-\frac{1}{n}=H_{n-1}$ for $n\geq 1$, one can get (4.4).

Now exchanging x for y in (4.2) and comparing it with (4.3), one is able to deduce the following very practical identity that can be used to connect two sums for Bernoulli polynomials with different variables. However, we wish to introduce below a straightforward proof of this not relying on (4.2) and (4.3), which was personally communicated by Karl Dilcher to the present author (April, 2019).

Corollary 4.2. *For an integer $n\geq 1$ we have*

$$\sum_{i=1}^{n}\binom{n+1}{i+1}(y-x)^{i-1}B_{n-i}(x)=\sum_{i=1}^{n}i\binom{n+1}{i+1}(x-y)^{i-1}B_{n-i}(y). \tag{4.7}$$

Proof. Since (4.7) is trivial if $x=y$, let us assume that $x\neq y$. As is easily seen, the functional identity $(e^{(x-y)t}-1)\mathcal{F}(t,y)=\mathcal{F}(t,x)-\mathcal{F}(t,y)$ implies that

$$\sum_{i=1}^{n}\binom{n}{i}(x-y)^{i}B_{n-i}(y)=B_n(x)-B_n(y). \tag{4.8}$$

Multiplying both sides of (4.7) by $(x-y)^2$, we shall prove the identity

$$\sum_{i=1}^{n}\binom{n+1}{i+1}(y-x)^{i+1}B_{n-i}(x)=\sum_{i=1}^{n}i\binom{n+1}{i+1}(x-y)^{i+1}B_{n-i}(y). \tag{4.9}$$

For brevity denoting the left-hand side of (4.9) by $P(n)$, we split up the right-hand side of (4.9) as follows:

$$\sum_{i=1}^{n}i\binom{n+1}{i+1}(x-y)^{i+1}B_{n-i}(y)=Q_1(n)-Q_2(n),$$

where

$$Q_1(n):=\sum_{i=1}^{n}(i+1)\binom{n+1}{i+1}(x-y)^{i+1}B_{n-i}(y);$$
$$Q_2(n):=\sum_{i=1}^{n}\binom{n+1}{i+1}(x-y)^{i+1}B_{n-i}(y).$$

Considering (4.8) with $n+1$ instead of n and separating the first term corresponding to $i=1$ in the sum on the left-hand side, we have

$$Q_2(n)+(n+1)(x-y)B_n(y)=B_{n+1}(x)-B_{n+1}(y).$$

This identity also yields, by symmetry in x and y,

$$P(n)+(n+1)(y-x)B_n(x)=B_{n+1}(y)-B_{n+1}(x).$$

So adding up these, it follows that

$$Q_2(n) + P(n) = (n+1)(x-y)(B_n(x) - B_n(y)). \tag{4.10}$$

On the other hand, using (4.8) and the obvious identity $(i+1)\binom{n+1}{i+1} = (n+1)\binom{n}{i}$, we see that $Q_1(n)$ can be expressed as follows:

$$\begin{aligned}
Q_1(n) &= \sum_{i=1}^{n} (i+1)\binom{n+1}{i+1}(x-y)^{i+1} B_{n-i}(y) \\
&= (n+1)(x-y)\sum_{i=1}^{n}\binom{n}{i}(x-y)^{i} B_{n-i}(y) \\
&= (n+1)(x-y)(B_n(x) - B_n(y)).
\end{aligned}$$

Thus we have only to compare this with (4.10) to show $P(n) = Q_1(n) - Q_2(n)$. $\square$

From (4.7) we see that (4.3) can be obtained from (4.2) by interchanging x and y. It should not be surprising that (4.7) is effective not only for Bernoulli polynomials, but also for any Appell polynomials (for example, such as Euler, Genocchi, monic Hermite and modified Laguerre polynomials) with different variables (cf. [3]).

Note that the following proposition provides relationships between Pan-Sun's identities (2.5) (i)-(ii) and some of the identities in Theorem 4.1.

Proposition 4.3. *In the case when $x \neq y$, (2.5) (i) (resp. (2.5) (ii)) is equivalent to (4.1) (resp. (4.4)).*

Proof. We now recall (4.8) and for brevity set

$$U_n(x,y) := \sum_{i=1}^{n}\binom{n}{i}(x-y)^{i} B_{n-i}(y) = B_n(x) - B_n(y). \tag{4.11}$$

Using the identity $i\binom{n+2}{i+2} = (n+2)\binom{n+1}{i+1} - 2\binom{n+2}{i+2}$, the first sum on the right-hand side of (4.1) can be written as, assuming that $x \neq y$,

$$\begin{aligned}
&\sum_{i=1}^{n} i\binom{n+2}{i+2}(x-y)^{i-1} B_{n-i}(y) \\
&= (n+2)\sum_{i=1}^{n}\binom{n+1}{i+1}(x-y)^{i-1} B_{n-i}(y) - 2\sum_{i=1}^{n}\binom{n+2}{i+2}(x-y)^{i-1} B_{n-i}(y) \\
&= (n+2)\left\{\frac{U_{n+1}(x,y)}{(x-y)^2} - \binom{n+1}{1}\frac{B_n(y)}{x-y}\right\} \\
&\qquad - 2\left\{\frac{U_{n+2}(x,y)}{(x-y)^3} - \binom{n+2}{1}\frac{B_{n+1}(y)}{(x-y)^2} - \binom{n+2}{2}\frac{B_n(y)}{x-y}\right\}
\end{aligned}$$

$$= \frac{(n+2)\left(U_{n+1}(x,y)+2B_{n+1}(y)\right)}{(x-y)^2} - \frac{2U_{n+2}(x,y)}{(x-y)^3}$$
$$= \frac{(n+2)\left(B_{n+1}(x)+B_{n+1}(y)\right)}{(x-y)^2} - \frac{2\left(B_{n+2}(x)-B_{n+2}(y)\right)}{(x-y)^3}.$$

From this identity we see that (2.5) (i) is equivalent to (4.1). On the other hand, the equivalence between (2.5) (ii) and (4.4) can be easily shown by direct use of (4.11). Indeed, for (2.5) (ii) we just have to apply the identity, divided both sides of (4.11) by $x-y \neq 0$, to the first sum on the right-hand side of (4.4) and use the identity $\frac{1}{i}\binom{n}{i} = \frac{n}{i^2}\binom{n-1}{i-1}$. Of course the converse is also true. □

As a consequence of Proposition 4.3, we state that Pan-Sun's (2.5) (i) and (2.5) (ii) can be supplanted by (4.1) and (4.4) valid even if $x = y$, respectively. In addition, the equivalence between Zagier's (2.7) and (4.4) can be verified using the identity

$$x^{i-1} = -\frac{1}{i}\left(B_i(x) - (-1)^i B_i(-x)\right) = \frac{2}{i}\left(B_i^+(x) - B_i(x)\right) \quad (i \geq 1).$$

Indeed, by replacing here x with $x-y$ and substituting it into each term of the first sum on the right-hand side of (4.4) we get (2.7), and the vice versa is also true.

Next, we extend (4.1) to a third-order trivariate convolution in the same way as the proof of Theorem 4.1 and establish the following identity.

Theorem 4.4. *For an integer $n \geq 2$ we have*

$$\begin{aligned}(n+3) &\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} B_i(x)B_j(y)B_k(z)\\ &= \sum_{\substack{i+j+k=n-2\\ i,j,k\geq 0}} (i+1)(j+1)\binom{n+3}{k}(y-x)^i(z-x)^j B_k(x)\\ &\quad + \sum_{\substack{i+j+k=n-1\\ i,j,k\geq 0}} (i+1)\binom{n+3}{j} P_{i,j,k}(x,y,z)\\ &\quad + \sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} \binom{n+3}{i} Q_{i,j,k}(x,y,z),\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}P_{i,j,k}(x,y,z) := (z-y)^i B_j(y)B_k(x-y) + (x-z)^i B_j(z)B_k(y-z)\\ + (y-x)^i B_j(x)B_k(z-x)\end{aligned}$$

and

$$\begin{aligned}Q_{i,j,k}(x,y,z) := B_i(z)B_j(x-z)B_k(y-z) + B_i(x)B_j(y-x)B_k(z-x)\\ + B_i(y)B_j(z-y)B_k(x-y).\end{aligned}$$

Proof. Letting u, v and w be real or complex numbers satisfying $u, v, w \neq 0, 1$ and $u+v+w=1$, we consider the special case of (3.4) for $m=3$, namely

$$\begin{aligned} f_u(Y)f_v(Y)f_w(Y) = f_1(Y)\,\{1 &+ f_u(Y) + f_v(Y) + f_w(Y) \\ &+ f_u(Y)f_v(Y) + f_v(Y)f_w(Y) + f_w(Y)f_u(Y)\}\,. \end{aligned}$$

We now put $Y = e^t$ and multiply both sides by $uvwt^3 e^{(ux+vy+wz)t}$. Then, since

$$e^{(ux+vy+wz)t} = e^{zt}e^{(x-z)ut}e^{(y-z)vt} = e^{xt}e^{(y-x)vt}e^{(z-x)wt} = e^{yt}e^{(z-y)wt}e^{(x-y)ut},$$

we can find the functional identity

$$A_1 = A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8, \tag{4.13}$$

where each function $A_i = A_i(t; u, v, w; x, y, z)$ $(1 \leq i \leq 8)$ is exactly given by

$$\begin{aligned} &A_1 := \mathcal{F}(ut, x)\mathcal{F}(vt, y)\mathcal{F}(wt, z); && A_2 := uvwt^2 e^{(y-x)vt}e^{(z-x)wt}\mathcal{F}(t, x); \\ &A_3 := vwte^{(z-y)wt}\mathcal{F}(t, y)\mathcal{F}(ut, x-y); && A_4 := wute^{(x-z)ut}\mathcal{F}(t, z)\mathcal{F}(vt, y-z); \\ &A_5 := uvte^{(y-x)ut}\mathcal{F}(t, x)\mathcal{F}(wt, z-x); && A_6 := w\mathcal{F}(t, z)\mathcal{F}(ut, x-z)\mathcal{F}(vt, y-z); \\ &A_7 := u\mathcal{F}(t, x)\mathcal{F}(vt, y-x)\mathcal{F}(wt, z-x); && A_8 := v\mathcal{F}(t, y)\mathcal{F}(wt, z-y)\mathcal{F}(ut, x-y). \end{aligned}$$

Let us differentiate partially both sides of (4.13) n times with respect to t and set $t=0$. Then, denoting

$$J_i = J_i(u, v, w; x, y, z) := \frac{\partial^n}{\partial t^n} A_i \Big|_{t=0} \quad (1 \leq i \leq 8),$$

we see that (4.13) yields

$$J_1 = J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8, \tag{4.14}$$

where each J_i $(1 \leq i \leq 8)$ can be formulated as follows:

$$J_1 = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i, j, k} u^i v^j w^k B_i(x)B_j(y)B_k(z);$$

$$J_2 = uvwn(n-1) \sum_{\substack{i+j+k=n-2 \\ i,j,k \geq 0}} \binom{n-2}{i, j, k} v^i w^j (y-x)^i (z-x)^j B_k(x);$$

$$J_3 = vwn \sum_{\substack{i+j+k=n-1 \\ i,j,k \geq 0}} \binom{n-1}{i, j, k} w^i u^k (z-y)^i B_j(y)B_k(x-y);$$

$$J_4 = wun \sum_{\substack{i+j+k=n-1 \\ i,j,k \geq 0}} \binom{n-1}{i, j, k} u^i v^k (x-z)^i B_j(z)B_k(y-z);$$

$$J_5 = uvn \sum_{\substack{i+j+k=n-1 \\ i,j,k \geq 0}} \binom{n-1}{i,j,k} u^i w^k (y-x)^i B_j(x) B_k(z-x);$$

$$J_6 = w \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i,j,k} w^j v^k B_i(z) B_j(x-z) B_k(y-z);$$

$$J_7 = u \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i,j,k} v^j w^k B_i(x) B_j(y-x) B_k(z-x);$$

$$J_8 = v \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i,j,k} w^j u^k B_i(y) B_j(z-y) B_k(x-y),$$

where $\binom{n}{i,j,k} := \frac{n!}{i!j!k!}$ $(n,i,j,k \geq 0)$ is the multinomial coefficient. Further, letting

$$I_i = I_i(x,y,z) := \int_0^1 \int_0^{1-u} J_i \, dvdu, \quad (1 \leq i \leq 8)$$

we obtain from (4.14) that

$$I_1 = I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8. \tag{4.15}$$

Here, using the multiple integral (3.3), each I_i can be calculated as follows:

$$I_1 = \frac{1}{(n+2)_2} \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} B_i(x) B_j(y) B_k(z);$$

$$I_2 = \frac{1}{(n+3)_3} \sum_{\substack{i+j+k=n-2 \\ i,j,k \geq 0}} (i+1)(j+1) \binom{n+3}{k} (y-x)^i (z-x)^j B_k(x);$$

$$I_3 = \frac{1}{(n+3)_3} \sum_{\substack{i+j+k=n-1 \\ i,j,k \geq 0}} (i+1) \binom{n+3}{j} (z-y)^i B_j(y) B_k(x-y);$$

$$I_4 = \frac{1}{(n+3)_3} \sum_{\substack{i+j+k=n-1 \\ i,j,k \geq 0}} (i+1) \binom{n+3}{j} (x-z)^i B_j(z) B_k(y-z);$$

$$I_5 = \frac{1}{(n+3)_3} \sum_{\substack{i+j+k=n-1 \\ i,j,k \geq 0}} (i+1) \binom{n+3}{j} (y-x)^i B_j(x) B_k(z-x);$$

$$I_6 = \frac{1}{(n+3)_3} \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n+3}{i} B_i(z) B_j(x-z) B_k(y-z);$$

$$I_7 = \frac{1}{(n+3)_3} \sum_{\substack{i+j+k=n\\ i,j,k\ge 0}} \binom{n+3}{i} B_i(x)B_j(y-x)B_k(z-x);$$

$$I_8 = \frac{1}{(n+3)_3} \sum_{\substack{i+j+k=n\\ i,j,k\ge 0}} \binom{n+3}{i} B_i(y)B_j(z-y)B_k(x-y),$$

where $(z)_n$ is the lower factorial polynomial in z of degree n as stated in (1.4). Multiplying both sides of (4.15) by $(n+3)_3$, we finally obtain (4.12). □

We would like to add that (4.12) is a symmetric identity with respect to x, y and z; so it is invariant under cyclic permutations of these letters.

In particular, letting $y = z = x$ in (4.12), one can deduce the following third-order convolution identity in a single variable.

Corollary 4.5. *For an integer $n \ge 2$ we have*

$$(n+3) \sum_{\substack{j+j+k=n\\ i,j,k\ge 0}} B_i(x)B_j(x)B_k(x) = 3\sum_{j=0}^{n-1} \binom{n+3}{j} B_j(x)B_{n-1-j}$$
$$+3 \sum_{\substack{j+j+k=n\\ i,j,k\ge 0}} \binom{n+3}{i} B_i(x)B_iB_k + \binom{n+3}{5} B_{n-2}(x).$$

This identity is exactly consistent with the case for $m = 3$ of the more general identity proved in [5, Theorem 1], namely

$$(n+m) \sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\ldots,k_m\ge 0}} \prod_{i=1}^{m} B_{k_i}(x) = \sum_{r=0}^{m-1} \binom{m}{r} \sum_{\substack{k_0+k_1+\cdots+k_r=n+1-m+r\\ k_0,k_1,\ldots,k_r\ge 0}} \binom{n+m}{k_0}$$
$$\times B_{k_0}(x)B_{k_1}\cdots B_{k_r},$$

which is valid for all integers $n, m \ge 1$ with $n \ge m-1$.

We cannot say for sure at the moment, but it seems that the following question still remains to be resolved as far as we know.

Open Question. Find a closed-form expression for the multiple sum

$$S(x_1,\ldots,x_m) := \sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\ldots,k_m\ge 1}} \prod_{i=1}^{m} \frac{B_{k_i}(x_i)}{k_i} \quad (n \ge m \ge 3), \tag{4.16}$$

where $x_1, \ldots, x_m$ are independent variables.

A certain formula for (4.16) in the case when $m = 3$ and $x_1 = x_2 = x_3$ has been derived in [5, (6.5)], which is of interest in its own right, but rather complicated to restate here. In addition, it should be mentioned that D. S. Kim and T. Kim [23] studied several kinds of multiple sums similar to (4.16) in a single variable case and expressed them by means of linear combinations of Bernoulli polynomials over $\mathbb{Q}$ based on the fact that $\{B_k(x) \mid k = 0, 1, \ldots, m\}$ forms a basis of the vector space $V_m := \{f(x) \in \mathbb{Q}[x] \mid \deg f \le m\}$ with $\dim V_m = m + 1$. On the other hand, A. Bayad and D. Kim [6, Theorem 4] recently discussed the product of arbitrary number of Bernoulli polynomials such that

$$\Phi_{k_1,\ldots,k_m}(x) := \prod_{i=1}^{m} \frac{B_{k_i}(x)}{k_i!} \quad (m \ge 1;\, k_1, \ldots, k_m \ge 1)$$

and expressed it explicitly as a linear combination of $\{B_k(x)/k! \mid k = 0, 1, \ldots, m\}$. Here note that $B_0(x)/0! = 1$ and this set also forms a basis of V_m over $\mathbb{Q}$.

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