RICHNESS OF ARITHMETIC PROGRESSIONS IN COMMUTATIVE SEMIGROUPS

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Abstract

Furstenberg and Glasner proved that for an arbitrary $k \in \mathbb{N}$, any piecewise syndetic set contains k-term arithmetic progressions and, in a sense to be made precise later, the set of such arithmetic progressions is piecewise syndetic in $\mathbb{Z}^2$. They used the algebraic structure of $\beta\mathbb{N}$. The above result was extended for arbitrary semigroups by Bergelson and Hindman, again using the structure of the Stone-Čech compactification of a general semigroup. Beiglböck provided an elementary proof of the above result and asked whether the combinatorial argument in his proof can be enhanced in a way which makes it applicable to a more abstract setting. In a recent work S. Jana and the second author of this paper provided an affirmative answer to Beiglböck's question for countable commutative semigroups. In this work we will extend the result of Beiglböck in different types of settings.

1. Introduction

A subset S of $\mathbb{Z}$ is called syndetic if there exists $r \in \mathbb{N}$ such that $\bigcup_{i=1}^{r}(S-i) = \mathbb{Z}$ and it is called thick if it contains arbitrary long intervals. Sets which can be expressed as the intersection of thick and syndetic sets are called piecewise syndetic sets.

For a general commutative semigroup $(S,+)$, a set $A \subseteq S$ is said to be syndetic in $(S,+)$, if there exists a finite nonempty set $F \subseteq S$ such that $\bigcup_{t\in F} -t + A = S$ where $-t + A = \{s \in S : t + s \in A\}$. A set $A \subseteq S$ is said to be thick if for every finite nonempty set $E \subseteq S$, there exists an element $x \in S$ such that $E + x \subseteq A$. A set $A \subseteq S$ is said to be piecewise syndetic if there exists a finite nonempty set $F \subseteq S$ such that $\bigcup_{t\in F}(-t + A)$ is thick in S. It can be proved that a piecewise syndetic set is the intersection of a thick set and a syndetic set [10, Theorem 4.49].

One of the famous Ramsey theoretic results is van der Waerden's Theorem [12] which states that at least one cell of any partition $\{C_1, C_2, \ldots, C_r\}$ of $\mathbb{N}$ contains arithmetic progressions of arbitrary length. The following theorem is due to van der Waerden [12].

Theorem 1. *Given any $r, l \in \mathbb{N}$, there exists $N(r,l) \in \mathbb{N}$, such that for any r-partition of $[1, N]$, at least one cell of the partition contains an l-length arithmetic progression.*

It follows from van der Waerden's Theorem that any piecewise syndetic subset A of $\mathbb{N}$ contains arbitrarily long arithmetic progressions. To see this, pick finite $F \subseteq \mathbb{N}$ such that $\bigcup_{t\in F} -t + A$ is thick in $\mathbb{N}$. Let $r = |F|$ and let a length k be given. Pick l as guaranteed for r and k and pick x such that $\{1, 2, \ldots, l\} + x \subseteq \bigcup_{t\in F} -t + A$. For $t \in F$, let $C_t = \{y \in \{1, 2, \ldots, l\} : y + x \in (-t + A)\}$. Pick a and d in $\mathbb{N}$ and $t \in F$ such that $\{a, a + d, a + 2d, \ldots, a + (k-1)\,d\} \subseteq C_t$ and let $a' = a + x + t$. Then $\{a', a' + d, a' + 2d, \ldots, a' + (k-1)\,d\} \subseteq A$.

H. Furstenberg and E. Glasner in [7] algebraically and Beiglböck in [1] combinatorially proved that if S is a piecewise syndetic subset of $\mathbb{Z}$ and $l \in \mathbb{N}$ then the set of all l-length progressions contained in S is also large. The statement is the following.

Theorem 2. *Let $k \in \mathbb{N}$ and assume that $S \subseteq \mathbb{Z}$ is piecewise syndetic. Then $\{(a,d) : \{a, a+d, \ldots, a+kd\} \subseteq S\}$ is piecewise syndetic in $\mathbb{Z}^2$.*

In a recent work [8, Theorem 6], the authors have extended the technique of Beigelböck in general commutative semigroups and proved the following.

Theorem 3. *Let $(S, +)$ be a commutative semigroup and let F be any finite subset of S. Then for any piecewise syndetic set $M \subseteq S$, the collection*

$$\{(a,n) \in S \times \mathbb{N} : \, a + n \cdot F \subseteq M\}$$

is piecewise syndetic in $(S \times \mathbb{N}, +)$.

Let $(S, +)$ be a commutative semigroup and let M be a piecewise syndetic set in S. For any $d' \in S$, take $F = \{d', 2d', ..., (l+1)d'\}$. Then from [8, Theorem 6] it follows that there exist $n \in \mathbb{N}$ and $a \in S$ such that:

$$a + n \cdot F = \{a + nd', a + 2nd', ..., a + (l+1)nd'\} \subseteq M.$$

So, $\{(c,d) : \{c, c+d, ..., c+ld\} \subseteq M\}$ is non-empty.

Parallelly the following problem comes from Theorem 2.

Problem 4. Let S be a countable commutative semigroup and let A be any piecewise syndetic subset of S. Is it true that for any $l \in \mathbb{N}$,

$$\{(s,t) \in S \times S : \, \{s, s+t, s+2t, \ldots, s+lt\} \subseteq A\}$$

is piecewise syndetic in $S \times S$?

At this time we are unable to give complete answer to this question but we have a proof of a weaker version of the theorem for countable commutative semigroups. We will also give an answer to Problem 4 for some special kinds of semigroups.

2. Proofs of Our Results

The following lemma was proved in [3] for general semigroups using the algebraic structure of the Stone-Čech compactification of an arbitrary semigroup and in [8] for commutative semigroups combinatorially.

Lemma 5. *Let $(S,+)$ and $(T,+)$ be commutative semigroups, let $\varphi : S \to T$ be a homomorphism and let $A \subseteq S$. If A is piecewise syndetic in S and $\varphi(S)$ is piecewise syndetic in T, then $\varphi(A)$ is piecewise syndetic in T.*

Now we need the following useful lemma.

Lemma 6. *Let $(S,+)$ be a countable commutative semigroup and let $A \subseteq S \times S$ be piecewise syndetic. Then for any $c \in S$ and $a \in \mathbb{N}$,*

$$\{(s + at + c, t) : (s,t) \in A\}$$

is piecewise syndetic in $S \times S$.

Proof. Let $c \in S$ and consider the following homomorphism $\psi_c : S \times S \longrightarrow S \times S$ defined by $\psi_c(s,t) = (s + c, t)$. We claim that this map preserves piecewise syndeticity.

As $A \subseteq S \times S$ is a piecewise syndetic set, there exists a finite subset $E_1 = \{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$ of $S \times S$ such that $\bigcup_{i=1}^{r}(-(a_i, b_i) + A)$ is thick and since

$$\bigcup_{i=1}^{r}(-(a_i, b_i) + A) \subseteq \bigcup_{i=1}^{r}(-(a_i + c, b_i) + \psi_c(A)),$$

the set $\bigcup_{i=1}^{r}(-(a_i + c, b_i) + \psi_c(A))$ is thick. So we have $\psi_c(A)$ is piecewise syndetic.

Now, for any $a \in \mathbb{N}$, suppose the semigroup homomorphism $\varphi_a : S \times S \to S \times S$ is defined by $\varphi_a(s,t) = (s + at, t)$. Now we will show that $\varphi_a(S \times S)$ is thick in $S \times S$ and hence piecewise syndetic. Let $F = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$ be a finite nonempty subset of $S \times S$. Fix $y \in S$ and let $x = ay + \sum_{i=1}^{n} av_i$. To see that $F + (x,y) \subseteq \varphi_a(S \times S)$, let $j \in \{1, 2, \ldots, n\}$. Let $t = v_j + y$ and let $s = u_j + \sum_{i \in \{1,2,\ldots,n\} \setminus \{j\}} av_i$. Then $(u_j, v_j) + (x,y) = \varphi_a(s,t)$. So from Lemma 5, φ_a map preserves piecewise syndeticity.

Therefore, $\{(s + at + c, t) : (s,t) \in A\} = \varphi_a \circ \psi_c(A)$ is piecewise syndetic in $S \times S$. □

The following is a weaker version of Problem 4. We would like to thank the referee for correcting and simplifying the proof of the following theorem.

Theorem 7. *Let S be a countable commutative semigroup and let A be piecewise syndetic in S. Then for $l \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that*

$$\{(s,t) \in S \times S : \{s, s+dt, \ldots, s+ldt\} \subseteq A\}$$

is piecewise syndetic in $S \times S$.

Proof. Since A is piecewise syndetic in S, there exists a finite subset E of S, such that $\bigcup_{c \in E} -c + A$ is thick in S.

Let $|E| = r$, say $E = \{c_1, c_2, \ldots, c_r\}$, and let $N(r, l+1) = N$ be the van der Waerden number.

The set of all possible $(l+1)$-length arithmetic progressions in $[1, N]$ is finite as $[1, N]$ is finite. Let $H = \{h_1, h_2, \ldots, h_n\}$ be the set of such progressions with $|H| = n$ (say).

Then, for any $(s_1, t_1) \in S \times S$, if the set $\{s_1 + t_1, s_1 + 2t_1, \ldots, s_1 + Nt_1\}$ is partitioned into r cells, one of the cells contains an l-length arithmetic progression.

Consider the set $B = \left\{(s,t) \in S \times S : \ s + [1, N]\, t \subseteq \bigcup_{c \in E} -c + A\right\}$. It is easy to verify that B is thick in $S \times S$.

Define $\varphi : B \to [1, n] \times [1, r]$ as follows. Given $(s,t) \in B$, $s + [1, N]\, t \subseteq \bigcup_{c \in E} (-c + A)$ so there exist $i \in [1, n]$ and $c \in E$ such that $s + h_i t \subseteq -c + A$. Pick the least $i \in [1, n]$ such that there exists $c \in E$ with $s + h_i t \subseteq -c + A$ and then pick the least $j \in [1, r]$ such that $s + h_i t \subseteq -c_j + A$ and define $\varphi(s,t) = (i, j)$. Since B is thick, there is some $(i, j) \in [1, n] \times [1, r]$ such that $\{(s,t) \in B : \varphi(s,t) = (i,j)\}$ is piecewise syndetic. (We are using the elementary fact that if the union of finitely many sets is piecewise syndetic, then one of them is.) Pick $a, d \in S$ such that

$$h_i = \{a, a+d, \ldots, a+ld\}.$$

Then

$$Q = \{(s,t) \in B : \ \{s+at, s+at+dt, \ldots, s+at+ldt\} \subseteq -c_j + A\}$$

is piecewise syndetic.

Thus, the set $\tilde{Q} = \{(s + at + c_j, t) : (s,t) \in Q\}$ is piecewise syndetic by Lemma 6 and this proves the theorem. □

It is not necessary that for any commutative semigroup G and for any $n \in \mathbb{N}$, $n \cdot G$, the collection of n times added elements of G, is piecewise syndetic in G. e.g. take any $n \in \mathbb{N} \setminus \{1\}$ and let $A = n \cdot \mathbb{Z}[x]$. Suppose that A is piecewise syndetic in $\mathbb{Z}[x]$ and pick $F = \{f_1, f_2, \ldots, f_m\} \subseteq \mathbb{Z}[x]$ such that $\bigcup_{i=1}^{m} -f_i + A$ is thick. Let $r = \max\{\deg(f_i) : 1 \leq i \leq m\}$ and let $G = \left\{x^{r+1}, 2x^{r+1}\right\}$. Pick $f \in \mathbb{Z}[x]$ such that

$G+f\subseteq\bigcup_{i=1}^{m}-f_i+A$. Pick i and j in $\{1,2,\ldots,m\}$ and g and h in $\mathbb{Z}[x]$ such that $f_i+x^{r+1}+f=ng$ and $f_j+2x^{r+1}+f=nh$. Let a, b, and c be the coefficients of x^{r+1} in f, g, and h respectively. Then $1+a=nb$ and $2+a=nc$, which is impossible. Thus A is not piecewise syndetic in $(\mathbb{Z}[x],+)$.

Now we are taking $\mathcal{A}$ to be the collection of all those countable commutative semigroups $(S,+)$ with the property that for any $d\in\mathbb{N}$, $d\cdot S=\{dx: x\in S\}$ is piecewise syndetic in S. Clearly $\mathcal{A}$ includes all the divisible semigroups such as $(\mathbb{Q},+),(\mathbb{Q}^{+},+)$, $(\mathbb{Q}/\mathbb{Z},+)$ etc. and others like $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}[i]$ etc. We will say a semigroup $(S,+)$ is a semigroup of class $\mathcal{A}$ if $S\in\mathcal{A}$.

Lemma 8. *Let S be a countable commutative semigroup of class $\mathcal{A}$ and assume that A is a piecewise syndetic subset of $S\times S$. Then for any $d\in\mathbb{N}$,*

$$\{(s,dt):(s,t)\in A\}$$

is piecewise syndetic in $S\times S$.

Proof. Let $d\in\mathbb{N}$ and define $\chi_d: S\times S\longrightarrow S\times S$ as $\chi_d(s,t)=(s,dt)$. Now, it is easy to check that $\chi_d(S\times S)$ is piecewise syndetic in $S\times S$ (we leave the routine verification to the reader). Then χ_d preserves piecewise syndeticity by Lemma 5. □

So we have the following result.

Proposition 9. *Let S be a countable commutative semigroup of class $\mathcal{A}$ and let A be piecewise syndetic in S. Then for $l\in\mathbb{N}$,*

$$\{(s,t)\in S\times S:\{s,s+t,s+2t,\ldots,s+lt\}\subseteq A\}$$

is piecewise syndetic in $S\times S$.

At this moment we are unable to derive the above proposition for a general commutative semigroup which would give an affirmative answer of Problem 4. Thus the question remains open.

3. Applications

The set $AP^{l+1}=\{(a,a+b,a+2b,\ldots,a+lb): a,b\in S\}$ is a commutative subsemigroup of S^{l+1}. Using a result deduced in [4, Theorem 3.7(a)] it is easy to see that for any piecewise syndetic set $A\subseteq S$, $A^{l+1}\cap AP^{l+1}$ is piecewise syndetic in AP^{l+1}. Now, as a consequence of Proposition 9 we will derive this result not for all but for a large class of semigroups in the following result.

Corollary 10. *Let $S \in \mathcal{A}$ be a countable commutative semigroup. Then for any piecewise syndetic set $A \subseteq S$, $A^{l+1} \cap AP^{l+1}$ is piecewise syndetic in AP^{l+1}.*

Proof. Let us take a surjective homomorphism $\varphi : S \times S \to AP^{l+1}$ by $\varphi(a,b) = (a, a+b, a+2b, \ldots, a+lb)$. Then by Lemma 5 the map φ preserves piecewise syndeticity.

Let $B = \{(s,t) \in S \times S : \{s, s+t, s+2t, \ldots, s+lt\} \subseteq A\}$. Then by Proposition 9, $\varphi(B)$ is piecewise syndetic in AP^{l+1}.

Now clearly, $\varphi(B) \subseteq A^{l+1} \cap AP^{l+1}$, so by Proposition 9 we get our required result. □

Now we will give a combinatorial proof of Proposition 9 replacing the condition of piecewise syndeticity by quasi-centrality which is another notion of largeness and is very close to the more familiar notion of central set.

A quasi-central set is generally defined in terms of the algebraic structure of βS. But it has a combinatorial characterization which will be needed for our purpose, stated below.

Theorem 11. [9, Theorem 3.7] *For a countable semigroup $(S, \cdot)$, $A \subseteq S$ is quasi-central if and only if there is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of A such that:*

1. *for each $n \in \mathbb{N}$ and each $x \in C_n$, there exists $m \in \mathbb{N}$ with $C_m \subseteq x^{-1}C_n$; and*
2. *for each $n \in \mathbb{N}$, C_n is piecewise syndetic.*

The following lemma is essential for our result.

Lemma 12. *The notion of quasi-central is preserved under surjective semigroup homomorphism.*

Proof. Let $\varphi : S_1 \to S_2$ be a surjective semigroup homomorphism. Let A be quasi-central in S_1 and let $\left\{\langle A_i \rangle_{i\in\mathbb{N}} : A_i \subseteq A\right\}$ be a chain of piecewise syndetic sets satisfying the properties of Theorem 11:

$$A \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$$

Now in S_2 consider the following sequence:

$$\varphi(A) \supseteq \varphi(A_1) \supseteq \varphi(A_2) \supseteq \ldots \supseteq \varphi(A_n) \supseteq \ldots$$

Due to the surjectivity of φ, $\varphi(A)$ and $\varphi(A_i)$ for $i \in \mathbb{N}$ are piecewise syndetic. Hence property (2) of Theorem 11 is verified.

Let $m \in \mathbb{N}$ and let $y \in \varphi(A_m)$. Then there exists some $x \in A_m$ such that $\varphi(x) = y$. Consider the set $y^{-1}\varphi(A_m)$. Now as $x^{-1}A_m \supseteq A_n$ for some n, we have for any $z \in A_n$, $xz \in A_m$ and then $y\varphi(z) \in \varphi(A_m)$ and so $\varphi(z) \in y^{-1}\varphi(A_m)$. Thus $y^{-1}\varphi(A_m) \supseteq \varphi(A_n)$. Hence we have verified property (1) of Theorem 11 as required. □

Now we will deduce Proposition 9 for quasi-central sets.

Theorem 13. *Let $(S,+)$ be a countable commutative semigroup of class $\mathcal{A}$. Then for any quasi-central $A \subseteq S$ and any $l \in \mathbb{N}$ the collection*

$$\{(a,b) : \{a, a+b, a+2b, \ldots, a+lb\} \subseteq A\}$$

is quasi-central in $(S \times S, +)$.

Proof. As A is quasi-central, Theorem 11 guarantees that there exists a decreasing sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of piecewise syndetic subsets of S, such that property (1) of Theorem 11 is satisfied.

Let $B = \{(a,b) \in S \times S : \{a, a+b, a+2b, \ldots, a+lb\} \subseteq A\}$ and for each $i \in \mathbb{N}$, let $B_i = \{(a,b) \in S \times S : \{a, a+b, a+2b, \ldots, a+lb\} \subseteq A_i\}$. Then by Proposition 9, B is piecewise syndetic in $S \times S$ and for each $i \in \mathbb{N}$, B_i is piecewise syndetic in $S \times S$.

Now choose $n \in \mathbb{N}$ and $(a,b) \in B_n$. Then $\{a, a+b, a+2b, \ldots, a+lb\} \subseteq A_n$. For $i \in \{0,1,\ldots,l\}$, pick $m(i) \in \mathbb{N}$ such that $A_{m(i)} \subseteq -(a+bi) + A_n$. Let $N = \max\{m(i) : i \in \{0,1,\ldots,l\}\}$, then

$$A_N \subseteq \bigcap_{i=0}^{l} \left(-(a+ib) + A_n\right).$$

As for any $(a_1,b_1) \in B_N$ we have

$$\{a_1, a_1+b_1, a_1+2b_1, \ldots, a_1+lb_1\} \subseteq A_N \subseteq \bigcap_{i=0}^{l} \left(-(a+ib) + A_n\right)$$

and $(a_1+a) + i(b_1+b) \in A_n$ for each $i \in \{0,1,2,\ldots,l\}$, therefore $(a_1,b_1) \in -(a,b) + B_n$. This implies $B_N \subseteq -(a,b) + B_n$, which establishes property (1) of Theorem 11. □

The following is an analogue of Corollary 10.

Corollary 14. *Let $(S,+)$ be a countable commutative semigroup of class $\mathcal{A}$. Then for any quasi-central set $A \subseteq S$, $A^{l+1} \cap AP^{l+1}$ is quasi-central in AP^{l+1}.*

Proof. Let $A \subseteq S$ be quasi-central. Then $B = \{(a,b) : \{a, a+b, \ldots, a+lb\} \subseteq A\}$ is quasi-central by Theorem 13.

Now, $\varphi : S \times S \to AP^{l+1}$ defined by $\varphi(a,b) = (a, a+b, \ldots, a+lb)$ is a surjective homomorphism. We claim that $\varphi(B) \subseteq A^{l+1} \cap AP^{l+1}$.

Let $\bar{b} \in \varphi(B)$. Then there exists some $(x,y) \in B$ such that $\varphi(x,y) = \bar{b} = (x, x+y, \ldots, x+ly) \in AP^{l+1}$ and by the definition of B, $\bar{b} \in A^{l+1} \cap AP^{l+1}$. So, $\varphi(B) \subseteq A^{l+1} \cap AP^{l+1}$. Now as B is quasi-central and φ is a surjective homomorphism, $\varphi(B)$ is quasi-central by Lemma 12. Consequently $A^{l+1} \cap AP^{l+1}$ is quasi-central (since $\varphi(B) \subseteq A^{l+1} \cap AP^{l+1}$).

This proves the claim. □

However, there are other different notions of largeness such as J-sets, C-sets and D sets. The first two have their combinatorial characterizations in [10]. In [6], the authors showed the abundance in J-sets and C-sets for $\mathbb{N}$. In [2], it was shown that D sets satisfy the conclusion of the original Central Sets Theorem and remarked that they are in fact C-sets. Reference [11] has a combinatorial characterization of D sets in $\mathbb{Z}$. This result is extended to countable cancellative abelian semigroups in [5, Theorem 2.11]. But we don't know if it is possible to give an affirmative answer of Problem 4 for D sets.

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