COMBINATORIAL IDENTITIES INVOLVING HARMONIC NUMBERS

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Abstract

We present a new combinatorial identity with two parameters, and applying it, we establish several combinatorial identities involving harmonic numbers; some of them have already been considered previously, and the others are new. For example, we prove that

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}H_{n-k}=H_n^2+\sum_{k=1}^{n}\frac{(-1)^k}{k^2\binom{n}{k}}$$

and $$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}H_{n-k}=\frac{H_n[H_n^2+H_n^{(2)}]}{2}-\sum_{k=0}^{n-1}\frac{(-1)^k[H_n-H_k]}{(k+1)(n-k)\binom{n}{k}}.$$

1. Introduction

Let $s\in\mathbb{C}$. Then the generalized harmonic numbers $H_n^{(s)}$ of order s are defined by

$$H_n^{(s)}=\sum_{k=1}^{n}\frac{1}{k^s},\quad H_0^{(s)}=0\quad\text{and}\quad H_n^{(1)}=H_n; \tag{1}$$

see [1, 13]. These numbers have various applications in number theory, combinatorics, analysis, computer science and differential equations. Recently, they have found applications in evaluating Feynman diagrams contributions of perturbed quantum field theory; see [7, 8]. Throughout this paper we let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{Z}^-=\{-1,-2,-3,\cdots\}$ and $\mathbb{Z}_0^-=\mathbb{Z}^-\cup\{0\}$. The polygamma functions $\psi^{(n)}(s)$ $(s\in\mathbb{C}\backslash\mathbb{Z}^-)$ are defined by

$$\psi^{(n)}(s)=\frac{d^{n+1}}{ds^{n+1}}\log\Gamma(s)=\frac{d^n}{ds^n}\psi(s),\ n=0,1,2,\cdots, \tag{2}$$

where $\Gamma(s)$ is the classical Euler gamma function, and $\psi^{(0)}(s) = \psi(s)$ is the digamma function. Let us recall some basic properties of these functions that will be used frequently in this work. A well-known relationship between the polygamma functions $\psi^{(n)}(s)$ and the generalized harmonic numbers $H_n^{(s)}$ is given by

$$\psi^{(m-1)}(n+1) - \psi^{(m-1)}(1) = (-1)^{m-1}(m-1)!H_n^{(m)}, \tag{3}$$

for $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$; see [6, pg. 22]. The digamma function ψ and harmonic numbers H_n are related by

$$\psi(n+1) = -\gamma + H_n \quad (n \in \mathbb{N}); \tag{4}$$

see [23, pg. 31], where $\gamma = 0.57721\cdots$ is the Euler-Mascheroni constant. The digamma function ψ possesses the following properties:

$$\psi\left(s+\frac{1}{2}\right) = 2\psi(2s) - \psi(s) - 2\log 2 \quad (s \in \mathbb{C}\backslash\mathbb{Z}^-), \tag{5}$$

and

$$\psi(s) - \psi(1-s) = -\pi\cot(\pi s); \tag{6}$$

see [23, pg. 25]. The gamma function satisfies the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad (s \in \mathbb{C}\backslash\mathbb{Z}^-), \tag{7}$$

and the duplication formula

$$\Gamma\left(s+\frac{1}{2}\right) = \frac{\Gamma(2s)\Gamma(1/2)}{2^{2s-1}\Gamma(s)}; \tag{8}$$

see [16, pgs. 346, 349]. The binomial coefficients $\binom{s}{t}$ $(s, t \in \mathbb{C}\backslash\mathbb{Z}^-)$ are defined by

$$\binom{s}{t} = \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)}, \tag{9}$$

and they satisfy the following identities for $n, k \in \mathbb{N}$ with $k \leq n$.

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{and} \quad \frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}. \tag{10}$$

The beta function $B(s,t)$ is defined by

$$B(s,t) = \int_0^1 u^{s-1}(1-u)^{t-1}du \quad (\Re(s) > 0\,, \Re(t) > 0).$$

The gamma and beta functions are related by

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}; \tag{11}$$

see [9, p.251]. In this paper, we shall frequently use the following form of the binomial coefficients:

$$f_n(s) := \binom{s+n}{k} = \frac{\Gamma(s+n+1)}{k!\Gamma(s+n-k+1)}.$$

Taking the logarithm of both sides of this equation, and then differentiating with respect to s, gives

$$f_n'(s) = \binom{s+n}{k}\{\psi(s+n+1) - \psi(s+n-k+1)\}. \tag{12}$$

Let us also define

$$g_n(s) = \binom{s+n}{n}.$$

Then, differentiating with respect to s yields

$$g_n'(s) = \binom{s+n}{n}[\psi(s+n+1) - \psi(s+1)]. \tag{13}$$

In the literature, there are many interesting papers dealing with finite sums involving the binomial coefficients and harmonic numbers. For example, we have

$$\sum_{k=0}^{n}\binom{n}{k}^2 H_k = \binom{2n}{n}\{2H_n - H_{2n}\} \quad \text{(see [3])},$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\{H_k - 2H_{2k}\} = \frac{4^n}{n}\binom{2n}{n}^2 \quad \text{(see [22])},$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_{n+k}^2 = \frac{1}{n\binom{2n}{n}}\left\{H_n - H_{2n} - \frac{2}{n}\right\} \quad \text{(see [24])}.$$

Over the decades, combinatorial identities involving harmonic numbers have attracted the interest of many mathematicians, and many remarkable identities have been discovered by using a variety of methods. In [17], the authors computed the following family of sums by using differential operator and Zeilberger's algorithm for definite hypergeometric sums.

$$\sum_{k=0}^{n}\binom{n}{k}^m\{1 + m(n-2k)H_k\} \quad m = 1, 2, 3, 4, 5.$$

Please see [2,3,4,5,12,14,15,17,18,21,22,24,25] and the references cited therein for more identities on this issue. The aim of this paper is to present further combinatorial identities involving harmonic numbers. First, we establish a new general combinatorial identity involving two parameters, and by differentiating and integrating both sides of this identity with respect to these parameters, we obtain several harmonic number identities. Some of them have already been considered previously, and the others are new. Although many other combinatorial identities can be derived by using these identities, for briefness, we have selected here just some of them, and we intend to prepare a separate paper containing many other applications.

Now we are ready to present our main results.

2. Main Results

Theorem 1. *Let $n \in \mathbb{N}$, $s \in \mathbb{C}\backslash\mathbb{Z}^-$ and $x \in \mathbb{C}$. Then the following identity holds:*

$$\sum_{k=0}^{n} \binom{s+n}{k} x^k = (1+x)^n \left[1 + s \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{s+k}{k} \left(\frac{x}{x+1} \right)^{k+1} \right]. \tag{14}$$

Proof. We prove the theorem by mathematical induction. Clearly, (14) is valid for $n = 1$. We assume that it is valid for n, and then we show that it is also valid for $n+1$. We have

$$\sum_{k=0}^{n+1} \binom{s+n+1}{k} x^k = \sum_{k=0}^{n} \binom{s+n+1}{k} x^k + \binom{s+n+1}{n+1} x^{n+1}.$$

Using the first relation in (10), this becomes

$$\sum_{k=0}^{n+1} \binom{s+n+1}{k} x^k = \sum_{k=0}^{n} \binom{s+n}{k} x^k + \sum_{k=1}^{n} \binom{s+n}{k-1} x^k + \binom{s+n+1}{n+1} x^{n+1}.$$

Setting $k-1 = k'$ in the second sum on the right-hand side of this equation, and then dropping the prime on k, we get, after a simple computation,

$$\begin{aligned} \sum_{k=0}^{n+1} \binom{s+n+1}{k} x^k &= \sum_{k=0}^{n} \binom{s+n}{k} x^k + x \sum_{k=0}^{n} \binom{s+n}{k} x^k \\ &\quad + \left[\binom{s+n+1}{n+1} - \binom{s+n}{n} \right] x^{n+1}. \end{aligned}$$

By (10) we have $\binom{s+n+1}{n+1} - \binom{s+n}{n} = \binom{s+n}{1+n}$; thus, we get

$$\sum_{k=0}^{n+1} \binom{s+n+1}{k} x^k = (1+x) \sum_{k=0}^{n} \binom{s+n}{k} x^k + \binom{s+n}{1+n} x^{1+n}. \tag{15}$$

Therefore, by induction hypothesis, we deduce that

$$\sum_{k=0}^{n+1}\binom{s+n+1}{k}x^k=(1+x)^{n+1}+s(1+x)^n\sum_{k=0}^{n-1}\binom{s+k}{k}\frac{x^{k+1}}{(k+1)(1+x)^k}$$
$$+\binom{s+n}{n+1}x^{n+1}=(1+x)^{n+1}+s(1+x)^n\sum_{k=0}^{n}\binom{s+k}{k}\frac{x^{k+1}}{(k+1)(1+x)^k}$$
$$+\left[\binom{s+n}{1+n}-\frac{s}{n+1}\binom{s+n}{n}\right]x^{n+1}.$$

Since $\binom{s+n}{n+1}-\frac{s}{n+1}\binom{s+n}{n}=0$, this proves that (14) is also valid for $n+1$. This completes the proof of Theorem 1. □

Theorem 2. *Let $n\in\mathbb{N}$, $s\in\mathbb{C}\backslash\mathbb{Z}^-$ and $x\in\mathbb{C}$. Then we have*

$$\sum_{k=1}^{n}\binom{s+n}{k}\{\psi(s+n+1)-\psi(s+n-k+1)\}x^k$$
$$=(1+x)^n\sum_{k=0}^{n-1}\frac{\binom{s+k}{k}}{k+1}\{1+s[\psi(s+k+1)-\psi(s+1)]\}\left(\frac{x}{x+1}\right)^{k+1}. \tag{16}$$

Proof. The proof follows from differentiating both sides of (14) with respect to s, and then using formulas (12) and (13). □

Corollary 1. *For $n\in\mathbb{N}$ and $s\in\mathbb{C}\backslash\mathbb{Z}^-$ we have*

$$\sum_{k=0}^{n}(-1)^k\binom{s+n}{k}[\psi(s+n+1)-\psi(s+n-k+1)]$$
$$=\frac{(-1)^n}{n}\binom{s+n-1}{n-1}[1+s(\psi(s+n)-\psi(s+1))]. \tag{17}$$

Proof. The proof immediately follows from (16) by taking $x=-1$. □

Theorem 3. *Let $n\in\mathbb{N}$, $s\in\mathbb{C}\backslash\mathbb{Z}^-$ and $x\in\mathbb{C}$. Then, we have*

$$\sum_{k=0}^{n}\binom{s+n}{k}\Big\{(\psi(s+n+1)-\psi(s+n-k+1))^2$$
$$+\psi'(s+n+1)-\psi'(s+n-k+1)\Big\}x^k$$
$$=(1+x)^n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{s+k}{k}\Big\{2(\psi(s+k+1)-\psi(s+1))$$
$$+s\Big[(\psi(s+k+1)-\psi(s+1))^2+\psi'(s+k+1)-\psi'(s+1)\Big]\Big\}\left(\frac{x}{x+1}\right)^{k+1}. \tag{18}$$

Proof. Differentiating both sides of (16) with respect to s, and then using (12) and (13), we conclude that (18) is valid. The next corollary follows from (18) by setting $x = -1$. $\square$

Corollary 2. *For $n \in \mathbb{N}$ and $s \in \mathbb{C}\backslash\mathbb{Z}^-$ we have*

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{s+n}{k}\Big\{(\psi(s+n+1)-\psi(s+n-k+1))^2 \\
&+\psi'(s+n+1)-\psi'(s+n-k+1)\Big\} \\
&=\frac{(-1)^n}{n}\binom{s+n-1}{n-1}\Big\{2[\psi(s+n)-\psi(s+1)] \\
&+s\left[(\psi(s+n)-\psi(s+1))^2+\psi'(s+n)-\psi'(s+1)\right]\Big\}. \qquad (19)
\end{aligned}$$

Proof. If we write equation (18) at $x = -1$, the proof is completed. $\square$

Theorem 4. *Let $n \in \mathbb{N}$ and $s \in \mathbb{C}\backslash\mathbb{Z}^-$. Then, we have*

$$\sum_{k=1}^{n}\binom{s+n}{k}\frac{(-1)^{k-1}}{k} = H_n + s\sum_{k=0}^{n-1}\frac{(-1)^k\binom{s+k}{k}}{(k+1)^2\binom{n}{k+1}}. \qquad (20)$$

Proof. If we take the first term of the sum on the left-hand side of (14) to the right, and then divide both sides by x, and finally replace x by $-x$ in the resulting equation, we obtain the following equality:

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}x^{k-1} = -\frac{(1-x)^n-1}{(1-x)-1} + s\sum_{k=0}^{n-1}\frac{(-1)^k\binom{s+k}{k}}{k+1}(1-x)^{n-k-1}x^k. \qquad (21)$$

Integrating both sides of (21) from $x = 0$ to $x = 1$ yields

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{s+n}{k}\frac{(-1)^k}{k} \\
&= -\int_0^1\frac{(1-x)^n-1}{(1-x)-1}\,dx + s\sum_{k=0}^{n-1}\frac{(-1)^k\binom{s+k}{k}}{k+1}\int_0^1(1-x)^{n-k-1}x^k\,dx. \qquad (22)
\end{aligned}$$

It is well-known that

$$\int_0^1\frac{(1-x)^n-1}{(1-x)-1}\,dx = H_n;$$

see [19], and

$$\int_0^1(1-x)^{n-k-1}x^k\,dx = \frac{(n-k-1)!k!}{n!} = \frac{1}{(k+1)\binom{n}{k+1}}.$$

Substituting these expressions in (22), we see that (20) is valid. $\square$

Theorem 5. *Let $n \in \mathbb{N}$ and $s \in \mathbb{C}\backslash\mathbb{Z}^-$. Then we have*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{s+n}{k} [\psi(s+n+1) - \psi(s+n-k+1)]$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^k \binom{s+k}{k}}{(k+1)^2 \binom{n}{k+1}} + s \sum_{k=0}^{n-1} \frac{(-1)^k \binom{s+k}{k}}{(k+1)^2 \binom{n}{k+1}} [\psi(s+k+1) - \psi(s+1)]. \tag{23}$$

Proof. If we differentiate both sides of (20) with respect to s, and use (12) and (13), the desired conclusion follows. □

Theorem 6. *For $n \in \mathbb{N}$ and $s \in \mathbb{C}\backslash\mathbb{Z}^-$ we have*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{s+n}{k} \Big\{ (\psi(s+n+1) - \psi(s+n-k+1))^2$$
$$+ \psi'(s+n+1) - \psi'(s+n-k+1) \Big\}$$
$$= 2 \sum_{k=0}^{n-1} \frac{(-1)^k \binom{s+k}{k}}{(k+1)^2 \binom{n}{k+1}} [\psi(s+k+1) - \psi(s+1)]$$
$$+ s \sum_{k=0}^{n-1} \frac{(-1)^k \binom{s+k}{k}}{(k+1)^2 \binom{n}{k+1}} \Big\{ (\psi(s+k+1) - \psi(s+1))^2 + \psi'(s+k+1) - \psi'(s+1) \Big\}. \tag{24}$$

Proof. If we differentiate both sides of (23) with respect to s, the proof is follows. □

Theorem 7. *For $n \in \mathbb{N}$ and $s \in \mathbb{C}\backslash\mathbb{Z}^-$ the following identity holds:*

$$\sum_{k=1}^{n} \binom{s+n}{k} \frac{(-1)^{k-1}}{k^2} = \frac{H_n^2 + H_n^{(2)}}{2} + s \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{s+k}{k} \frac{H_n - H_k}{(n-k)\binom{n}{k}}. \tag{25}$$

Proof. Integrating both sides of (21) from $x = 0$ to $x = u$, we get

$$\sum_{k=1}^{n} (-1)^k \binom{s+n}{k} \frac{u^k}{k} = \int_0^u \frac{(1-x)^n - 1}{x} dx$$
$$+ s \sum_{k=0}^{n-1} \frac{(-1)^k \binom{s+k}{k}}{k+1} \int_0^u (1-x)^{n-k-1} x^k dx.$$

Divide both sides of this equation by u, and then integrate each side from $u = 0$ to $u = 1$ to obtain

$$\sum_{k=1}^{n} (-1)^k \binom{s+n}{k} \frac{1}{k^2} = \int_0^1 \frac{1}{u} \int_0^u \frac{(1-x)^n - 1}{x} dx du$$
$$+ s \sum_{k=0}^{n-1} \frac{(-1)^k \binom{s+k}{k}}{k+1} \int_0^1 \frac{1}{u} \int_0^u (1-x)^{n-k-1} x^k dx du. \tag{26}$$

Integration by parts gives

$$\int_0^1 \frac{1}{u}\int_0^u \frac{(1-x)^n-1}{x}dxdu = \log u \int_0^u \frac{(1-x)^n-1}{x}dx\bigg|_{u=0}^{u=1} - \int_0^1 \frac{\log u}{u}[(1-u)^n-1]du. \tag{27}$$

The first term on the right-hand side of (27) is equal to zero. Therefore, applying integration by parts, we get

$$\int_0^1 \frac{1}{u}\int_0^u \frac{(1-x)^n-1}{x}dxdu = -\frac{1}{2}\log^2 u[(1-u)^n-1]\bigg|_{u=0}^{u=1} - \frac{n}{2}\int_0^1 \log^2 u(1-u)^{n-1}du. \tag{28}$$

The first term on the right-hand side of (28) is also equal to zero, hence, we get by (11) and (3)

$$\begin{aligned}\int_0^1 \frac{1}{u}\int_0^u \frac{(1-x)^n-1}{x}dxdu &= -\frac{n}{2}\int_0^1 \log^2 u(1-u)^{n-1}du \\ &= -\frac{n}{2}\frac{d^2}{dt^2}\int_0^1 u^t(1-u)^{n-1}du\bigg|_{t=0} \\ &= -\frac{n}{2}\frac{d^2}{dt^2}\frac{\Gamma(t+1)\Gamma(n)}{\Gamma(t+n+1)}\bigg|_{t=0} = -\frac{H_n^2+H_n^{(2)}}{2}. \end{aligned} \tag{29}$$

An application of integration by parts to the second integral on the right-hand side of (26) leads to

$$\int_0^1 \frac{1}{u}\int_0^u (1-x)^{n-k-1}x^k dxdu = \log u\int_0^u (1-x)^{n-k-1}x^k dx\bigg|_{u=0}^{u=1} - \int_0^1 \log u(1-u)^{n-k-1}u^k du.$$

The first term on the right-hand side of this equation is equal to zero. So, from (11) and (3) we get

$$\begin{aligned}&\int_0^1 \frac{1}{u}\int_0^u (1-x)^{n-k-1}x^k dxdu = -\int_0^1 \log u(1-u)^{n-k-1}u^k du \\ &= -\int_0^1 \frac{d}{dt}u^t(1-u)^{n-k-1}du\bigg|_{t=k} = -\frac{d}{dt}\int_0^1 u^t(1-u)^{n-k-1}du\bigg|_{t=k} \\ &= -\frac{d}{dt}\frac{\Gamma(t+1)\Gamma(n-k)}{\Gamma(n+t-k+1)}\bigg|_{t=k} = \frac{H_n-H_k}{(n-k)\binom{n}{k}}. \end{aligned} \tag{30}$$

Inserting the values of the integrals given in (29) and (30) into (26), and using (3), we complete the proof of Theorem 7. □

Theorem 8. *For $n \in \mathbb{N}$ and $s \in \mathbb{C} \backslash \mathbb{Z}^-$ we have*

$$\begin{aligned}
&\sum_{k=1}^{n} \binom{s+n}{k} \frac{(-1)^{k-1}}{k^2} [\psi(s+n+1) - \psi(s+n-k+1)] \\
&= \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{s+k}{k} \frac{H_n - H_k}{(n-k)\binom{n}{k}} \\
&+ s \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{s+k}{k} \frac{H_n - H_k}{(n-k)\binom{n}{k}} [\psi(s+k+1) - \psi(s+1)]. \qquad (31)
\end{aligned}$$

Proof. The proof follows from differentiating both sides of (25) with respect to s, and then using (12) and (13). □

Theorem 9. *For all $n \in \mathbb{N}$ and $s \in \mathbb{C} \backslash \mathbb{Z}^-$ we have*

$$\begin{aligned}
&\sum_{k=1}^{n} \binom{s+n}{k} \frac{(-1)^{k-1}}{k^2} \Big\{ (\psi(s+n+1) - \psi(s+n-k+1))^2 \\
&+ \psi'(s+n+1) - \psi'(s+n-k+1) \Big\} \\
&= 2 \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{s+k}{k} \frac{H_n - H_k}{(n-k)\binom{n}{k}} [\psi(s+k+1) - \psi(s+1)] \\
&+ s \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{s+k}{k} \frac{H_n - H_k}{(n-k)\binom{n}{k}} \Big\{ (\psi(s+k+1) - \psi(s+1))^2 \\
&+ \psi'(s+k+1) - \psi'(s+1) \Big\}. \qquad (32)
\end{aligned}$$

Proof. The proof follows from differentiating both sides of (31) with respect to s, and using (12) and (13). □

3. Applications

In this section, we present several applications of our main results, which are derived by taking particular values for the parameters s and x.

Identity 1. Let $n \in \mathbb{N}$. Then we have

$$\sum_{k=1}^{n} \frac{1}{2^{2k}} \binom{2k}{k} (2H_{2k} - H_k) = \frac{2n+1}{2^{2n}} \binom{2n}{n} [2H_{2n+1} - H_n - 2].$$

Proof. Putting $x = -1$ and performing the replacement $s \to s - n$ in (14), we get

$$\sum_{k=0}^{n}(-1)^k \binom{s}{k} = (-1)^n \binom{s-1}{n}. \tag{33}$$

If we differentiate both sides of (33) with respect to s, and then let $s = -1/2$, the proof follows from (4)-(8). □

Identity 2. For all $n \in \mathbb{N}$ the following identity holds:

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = H_n. \tag{34}$$

Proof. The proof follows from (20) by setting $s = 0$. □

Remark 1. Identity 2 is well-known and due to Euler (see [10], [5] and [2]).

Remark 2. Theorem 4 provides a generalization of (34).

Identity 3. Let $n \in \mathbb{N}$. Then it holds that

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} H_{n-k} = H_n^2 + \sum_{k=1}^{n} \frac{(-1)^k}{k^2 \binom{n}{k}}. \tag{35}$$

Proof. The proof follows from (23) by setting $s = 0$, and using (34) and (3). □

Remark 3. Identity 3 can be compared with the following formula (see [19]):

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} H_{n+k} = H_n^2 + \sum_{k=1}^{n} \frac{1}{k^2 \binom{n+k}{k}}.$$

Identity 4. Let $n \in \mathbb{N}$ and $x \in \mathbb{C}$. Then we have

$$\sum_{k=0}^{n} \binom{n}{k} H_k x^k = (1+x)^n \left[H_n - \sum_{k=1}^{n} \frac{1}{k(x+1)^k} \right]. \tag{36}$$

Proof. Taking $s = 0$ in (16), and using (4), we get

$$\sum_{k=0}^{n} \binom{n}{k} [H_n - H_{n-k}] x^k = (1+x)^n \sum_{k=1}^{n} \frac{1}{k} \left(\frac{x}{1+x} \right)^k. \tag{37}$$

If we replace x by $1/x$ and then multiply both sides of (37) by x^n, we get

$$\sum_{k=0}^{n} \binom{n}{k} [H_n - H_{n-k}] x^{n-k} = (1+x)^n \sum_{k=1}^{n} \frac{1}{k(x+1)^k}, \tag{38}$$

which is equivalent with (36), since $\sum_{k=0}^{n} \binom{n}{k} H_{n-k} x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} H_k x^k$. □

If we set $x = 1$ in (36), we get the following known result (see [6, 12, 21]).

Identity 5. Let $n \in \mathbb{N}$. Then, it holds that

$$\sum_{k=1}^{n} \binom{n}{k} H_k = 2^n \left[H_n - \sum_{k=1}^{n} \frac{1}{k2^k} \right].$$

Identity 6. For $n \in \mathbb{N}$ and $x \in \mathbb{C}$ we have

$$\sum_{k=1}^{n} \binom{n}{k} \left[H_k^2 + H_k^{(2)} \right] x^k = (1+x)^n \left[H_n^2 + H_n^{(2)} + 2 \sum_{k=1}^{n} \frac{H_{k-1} - H_n}{k(1+x)^k} \right]. \tag{39}$$

Proof. By setting $s = 0$ in (18), and using (3) and (4), we can readily deduce that

$$\sum_{k=0}^{n} \binom{n}{k} \left[(H_n - H_{n-k})^2 - H_n^{(2)} + H_{n-k}^{(2)} \right] x^k = 2(1+x)^n \sum_{k=1}^{n} \frac{H_{k-1}}{k} \left(\frac{x}{1+x} \right)^k. \tag{40}$$

Expanding the quadratic term on the left-hand side of (40) and using (36), after some simplifications, we find that

$$\sum_{k=0}^{n} \binom{n}{k} \left[H_{n-k}^2 + H_{n-k}^{(2)} \right] x^k - (1+x)^n \left[H_n^2 + H_n^{(2)} - 2H_n \sum_{k=1}^{n} \frac{1}{k} \left(\frac{x}{1+x} \right)^k \right]$$
$$= 2(1+x)^n \sum_{k=1}^{n} \frac{H_{k-1}}{k} \left(\frac{x}{1+x} \right)^k. \tag{41}$$

If we replace x by $1/x$ here, and then multiply both sides of (41) by x^n, after some simplifications, we get the desired identity (39). $\square$

If we set $x = -1$ in (39), we get the following known result; see [20].

Identity 7. For $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} \left\{ H_k^2 + H_k^{(2)} \right\} = -\frac{2}{n^2}.$$

The following identity is known and computer program *Mathematica* recognizes it.

Identity 8. For all $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \frac{(-1)^k}{k\binom{n}{k}} = \frac{(-1)^n - 1}{n+1}. \tag{42}$$

Proof. Setting $s = 1$ in (20), we get

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} = H_n + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k\binom{n}{k}}. \tag{43}$$

Since

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} - \frac{(-1)^n}{n+1},$$

if we use (34), it follows that

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} = H_{n+1} - \frac{(-1)^n}{n+1}.$$

This completes the proof of (42) by the help of (43). □

Identity 9. Let $n \in \mathbb{N}$. Then we have

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \left\{ H_{n-k}^2 + H_{n-k}^{(2)} \right\} = H_n^3 + H_n H_n^{(2)} + 2 \sum_{k=1}^{n} \frac{(-1)^k [H_n - H_{k-1}]}{k^2 \binom{n}{k}}.$$

Proof. Putting $s = 0$ in (24), we obtain that

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \left\{ (H_n - H_{n-k})^2 - H_n^{(2)} + H_{n-k}^{(2)} \right\} = 2 \sum_{k=1}^{n-1} \frac{(-1)^k H_k}{(k+1)^2 \binom{n}{k+1}}.$$

Expanding the quadratic term here, it simplifies to

$$\begin{aligned} \left[H_n^2 - H_n^{(2)} \right] \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} - 2H_n \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} H_{n-k} \\ + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \left\{ H_{n-k}^2 + H_{n-k}^{(2)} \right\} = 2 \sum_{k=1}^{n-1} \frac{(-1)^k H_k}{(k+1)^2 \binom{n}{k+1}}. \end{aligned}$$

Using (34) and (35) here and simplifying the resulting equation, the desired conclussion follows. □

Identity 10. Let $m, n \in \mathbb{N}$. Then we have

$$\sum_{k=0}^{n} (-1)^k \binom{mn}{k} H_{mn-k} = \frac{(-1)^n}{m} \binom{mn}{n} \left[(m-1) H_{(m-1)n} - \frac{1}{mn} \right]. \tag{44}$$

Proof. If we write equation (33) at $s = mn$, we get

$$\sum_{k=0}^{n} (-1)^k \binom{mn}{k} = (-1)^n \binom{mn-1}{n}. \tag{45}$$

If we differentiate both sides of (33) with respect to s, and then set $s = mn$ ($m \in \mathbb{N}$), we, in view of (3), get

$$\sum_{k=0}^{n}(-1)^k\binom{mn}{k}\{H_{mn}-H_{mn-k}\} = (-1)^n\binom{mn-1}{n}\{H_{mn-1}-H_{mn-n-1}\}. \quad (46)$$

But if we use (45), this becomes

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{mn}{k}\{H_{mn}-H_{mn-k}\}\\ &= H_{mn}\sum_{k=0}^{n}(-1)^k\binom{mn}{k} - \sum_{k=0}^{n}(-1)^k\binom{mn}{k}H_{mn-k}\\ &= (-1)^n H_{mn}\binom{mn-1}{n} - \sum_{k=0}^{n}(-1)^k\binom{mn}{k}H_{mn-k}.\end{aligned}$$

Using this identity in (46) and simplifying the result, we complete the proof of (44). $\square$

The following identity is an immediate consequence of Identity 10 with $m = 2$ and $m = 3$.

Identity 11. For $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}H_{2n-k} = \frac{(-1)^n}{2}\binom{2n}{n}\left(H_n - \frac{1}{2n}\right)$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{3n}{k}H_{3n-k} = \frac{(-1)^n}{3}\binom{3n}{n}\left(2H_n - \frac{1}{3n}\right).$$

Identity 12. For $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n}\frac{(-1)^k H_k}{k\binom{n}{k}} = \frac{(-1)^n H_{n+1}}{n+1} + \sum_{k=1}^{n+1}\frac{(-1)^k}{k^2\binom{n+1}{k}}.$$

Proof. Setting $s = 1$ in (23), and using (3) and $\psi(2) = 1 + \psi(1)$, we obtain

$$\begin{aligned}&\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n+1}{k}(H_{n+1}-H_{n-k+1})\\ &= \sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)\binom{n}{k+1}} + \sum_{k=0}^{n-1}\frac{(-1)^k(H_{k+1}-1)}{(k+1)\binom{n}{k+1}}\\ &= \sum_{k=1}^{n}\frac{(-1)^{k-1}H_k}{k\binom{n}{k}}. \quad (47)\end{aligned}$$

On the other hand, we have

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n+1}{k}(H_{n+1}-H_{n-k+1}) = H_{n+1}\left[\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n+1}{k} - \frac{(-1)^n}{n+1}\right] - \sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n+1}{k}H_{n+1-k}.$$

Using (34) and (35), we conclude from this equation that

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n+1}{k}(H_{n+1}-H_{n-k+1}) = \frac{(-1)^{n+1}H_{n+1}}{n+1} + \sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k^2\binom{n+1}{k}}. \qquad (48)$$

Equating the right-hand sides of (47) and (48), we see that Identity 12 is valid. □

Identity 13. For $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n}\frac{(-1)^{k-1}4^k\binom{n}{k}}{\binom{2k}{k}} = \frac{1}{2n-1}.$$

Proof. Setting $s = -\frac{1}{2}$ and $x = -1$ in (14), and using (8) and (9), the proof is completed. □

Remark 4. Identity 13 is known and can be found in [22, Theorem 4.5] .

If we set $s = 0$ in (25), we get the following known result (see [2]).

Identity 14. Let $n \in \mathbb{N}$. Then we have

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k^2} = \frac{H_n^2 + H_n^{(2)}}{2}. \qquad (49)$$

Identity 15. For $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n}\frac{(-1)^k H_{n-k}}{k\binom{n}{k}} = \frac{1-(-1)^n}{(n+1)^2} - \frac{H_n}{n+1}.$$

Proof. If we take $s = 1$ in (25), we get

$$\sum_{k=1}^{n}\binom{n+1}{k}\frac{(-1)^{k-1}}{k^2} = \frac{H_n^2 + H_n^{(2)}}{2} + \sum_{k=0}^{n-1}\frac{(-1)^k(H_n - H_k)}{(n-k)\binom{n}{k}}. \qquad (50)$$

By (49), we arrive at

$$\begin{aligned}\sum_{k=1}^{n}\binom{n+1}{k}\frac{(-1)^{k-1}}{k^2} &= \sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^{k-1}}{k^2} - \frac{(-1)^n}{(n+1)^2}\\ &= \frac{H_{n+1}^2 + H_{n+1}^{(2)}}{2} - \frac{(-1)^n}{(n+1)^2}. \qquad (51)\end{aligned}$$

On the other hand, we have

$$\sum_{k=0}^{n-1}\frac{(-1)^k(H_n-H_k)}{(n-k)\binom{n}{k}}=H_n\sum_{k=0}^{n-1}\frac{(-1)^k}{(n-k)\binom{n}{k}}-\sum_{k=0}^{n-1}\frac{(-1)^kH_k}{(n-k)\binom{n}{k}}.$$

If we substitute $n-k=k'$ in the sums on the right-hand side of this equation and then drop the prime on k, we get

$$\sum_{k=0}^{n-1}\frac{(-1)^k(H_n-H_k)}{(n-k)\binom{n}{k}}=H_n\sum_{k=1}^{n}\frac{(-1)^{n-k}}{k\binom{n}{k}}-\sum_{k=1}^{n}\frac{(-1)^{n-k}H_{n-k}}{k\binom{n}{k}}.$$

Using (42) gives

$$\sum_{k=0}^{n-1}\frac{(-1)^k(H_n-H_k)}{(n-k)\binom{n}{k}}=\frac{H_n(1-(-1)^n)}{n+1}-(-1)^n\sum_{k=1}^{n}\frac{(-1)^kH_{n-k}}{k\binom{n}{k}}. \tag{52}$$

Using (51) and (52) in (50), we conculde that Identity 15 is valid. □

Identity 16. For all $n\in\mathbb{N}$ we have

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}H_{n-k}=\frac{H_n\left(H_n^2+H_n^{(2)}\right)}{2}-\sum_{k=0}^{n-1}\frac{(-1)^k(H_n-H_k)}{(k+1)(n-k)\binom{n}{k}}. \tag{53}$$

Proof. Setting $s=0$ in (31) yields

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}(H_n-H_{n-k})=\sum_{k=0}^{n-1}\frac{(-1)^k(H_n-H_k)}{(k+1)(n-k)\binom{n}{k}}. \tag{54}$$

By (49), we have

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}(H_n-H_{n-k})=\frac{H_n\left((H_n)^2+H_n^{(2)}\right)}{2}-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}H_{n-k}. \tag{55}$$

Equating the right-hand sides of (54) and (55), The proof of (53) follows. □

Identity 17. For all $n\in\mathbb{N}$ the following identity holds:

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}\left\{H_{n-k}^2+H_{n-k}^{(2)}\right\}=\frac{\left(H_n^2+H_n^{(2)}\right)^2}{2}-2\sum_{k=0}^{n-1}\frac{(-1)^k(H_n-H_k)^2}{(k+1)(n-k)\binom{n}{k}}.$$

Proof. Setting $s=0$ in (32), we get

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}\left\{(H_n-H_{n-k})^2+H_{n-k}^{(2)}-H_n^{(2)}\right\}=2\sum_{k=0}^{n-1}\frac{(-1)^k(H_n-H_k)H_k}{(k+1)(n-k)\binom{n}{k}}. \tag{56}$$

Clearly, we have

$$\begin{aligned}&\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}\left\{(H_n-H_{n-k})^2+H_{n-k}^{(2)}-H_n^{(2)}\right\}\\&=\left[H_n^2-H_n^{(2)}\right]\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}-2H_n\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}H_{n-k}\\&+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}\left\{H_{n-k}^2+H_{n-k}^{(2)}\right\}.\end{aligned}$$

Thus, by the help of (49) and (53), after some algebraic manipulations, we find that

$$\begin{aligned}&\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}\left\{(H_n-H_{n-k})^2+H_{n-k}^{(2)}-H_n^{(2)}\right\}=-\frac{\left(H_n^2+H_n^{(2)}\right)^2}{2}\\&-2H_n\sum_{k=0}^{n-1}\frac{(-1)^{k-1}(H_n-H_k)}{(k+1)(n-k)\binom{n}{k}}+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}\left\{H_{n-k}^2+H_{n-k}^{(2)}\right\}.\end{aligned} \tag{57}$$

If we equate the right-hand sides of (56) and (57), the conclusion follows. □

Remark 5. We want to provide some comments about how we discovered the core identity (14) of this work. When we started writing this paper, our intention was to generalize the binomial theorem $\sum_{k=0}^{n}\binom{n}{k}x^k=(1+x)^n$. For this purpose, we replace n by $n+s$ in the summand of this sum and define $a_n=\sum_{k=0}^{n}\binom{n+s}{k}x^k$. First, we showed that a_n satisfies the following recurrence relation:

$$a_0=1 \quad \text{and} \quad a_{n+1}=(1+x)a_n+\binom{n+s}{n+1}x^{n+1} \quad \text{for} \quad n\geq 1.$$

Using this relation, formula (14) was found.

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