# A NOTE ON APPLICATIONS FROM EXTENDING ABEL'S LEMMA 

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#### Abstract

This note presents new applications from an extension of Abel's lemma in addition to its more general form established by Andrews and Freitas. An interesting relationship is derived between the lemma and series involving integer values of the Riemann zeta function.


## 1. Introduction

In a paper by Andrews and Freitas [4], an extension of Abel's lemma was further generalized and several new $q$-series were established. Recall that Abel's lemma is the simple result that $\lim _{z \rightarrow 1^{-}}(1-z) \sum_{n=0}^{\infty} a_{n} z^{n}=\lim _{n \rightarrow \infty} a_{n}$. We use the shifted factorial notation $(a)_{n}=a(a+1) \cdots(a+n-1)$ in this paper [2]. Their result may be stated as follows.

Proposition 1. ([4, Proposition 1.2]) Let $f(z)=\sum_{n=0}^{\infty} \alpha_{n} z^{n}$ be analytic for $|z|<1$, and assume for some positive integer $M$ and a fixed complex number $\alpha$ that (i) $\sum_{n=0}^{\infty}(n+1)_{M}\left(\alpha_{M+n}-\alpha_{M+n-1}\right)$ converges, and (ii) $\lim _{n \rightarrow \infty}(n+1)_{M}\left(\alpha_{M+n}-\right.$ $\alpha)=0$. Then

$$
\frac{1}{M} \lim _{z \rightarrow 1^{-}}\left(\frac{\partial^{M}}{\partial z^{M}}(1-z) f(z)\right)=\sum_{n=0}^{\infty}(n+1)_{M-1}\left(\alpha-\alpha_{n+M-1}\right)
$$

The formula being generalized here is given in [3, Proposition 2.1], where it was used to find generating functions for special values of certain $L$-functions. A corollary of the extension of Abel's lemma was also given in [7].

In [1] we find a simple formula attributed to Christian Goldbach:

$$
\begin{equation*}
\sum_{n=0}^{\infty}(1-\zeta(n+2))=-1 \tag{1}
\end{equation*}
$$

It does not appear that a relationship between this result and the extension of Abel's lemma has been found previously, but as we shall see, (1) is merely a simple consequence of the lemma. To this end, we shall prove some more general formulas in the next section which we believe are interesting applications of the AndrewsFreitas formula. For this, we will use a result from Hashimoto et al. [6]. For similar identities, the reader is referred to [5] and [8]. The main theorems presented here appear to differ considerably from previous similar examples, such as [8, Equation 2.4] where the sums involving $(n)_{M}$ run over $M$, while ours run over $n$.

## 2. Theorems

This section establishes some interesting theorems, which we hope will add value to the Andrews-Freitas formula. To assist in the proofs of the theorems, we require the following lemma.

Lemma 1. If $f(z)$ has no factor $(1-z)^{-1}$, then

$$
\lim _{z \rightarrow 1^{-}} \frac{\partial^{M}}{\partial z^{M}}(1-z) f(z)=-M f^{(M-1)}(1)
$$

Proof. Put $f_{1}(z)=(1-z)$, and $f_{2}(z)=f(z)$. By Leibniz's rule, we arrive at

$$
\begin{aligned}
\lim _{z \rightarrow 1^{-}} \frac{\partial^{M}}{\partial z^{M}}(1-z) f(z) & =\lim _{z \rightarrow 1^{-}} \sum_{j=0}^{M}\binom{M}{j} f_{1}^{(j)} f_{2}^{(M-j)} \\
& =\lim _{z \rightarrow 1^{-}}\binom{M}{1} f_{1}^{(1)} f_{2}^{(M-1)} \\
& =-M \lim _{z \rightarrow 1^{-}} f_{2}^{(M-1)}(z)
\end{aligned}
$$

We have used the facts that if $j=0$, then $f_{1}^{(0)}$ is zero when $z \rightarrow 1^{-}$, and for $j>1$, $f_{1}^{(j)}=0$.

As usual, we let $\gamma$ denote Euler's constant [2]. We also define the polygamma function [2] to be the $(M+1)$-th derivative of the logarithm of the Gamma function, i.e., $\psi^{(M)}(z)=\frac{\partial^{M+1}}{\partial z^{M+1}}(\log \Gamma(z))$.

Theorem 1. For positive integers $M$, the following result applies

$$
\begin{gathered}
\sum_{n=0}^{\infty}(n+1)_{M-1}(1-\zeta(n+M+1)) \\
=(-1)^{M-1} \sum_{j=0}^{M-1}\binom{M-1}{j} j!\psi^{(M-j-1)}(1)+(M-1)!(-1)^{M}+\gamma(-1)^{M-1}(M-1)!
\end{gathered}
$$

Proof. First we write down the well-known Taylor expansion of the digamma function [1, 2], for $|z|<1$,

$$
\begin{equation*}
\psi^{(0)}(z+1)=-\gamma-\sum_{k=1}^{\infty} \zeta(k+1)(-z)^{k} \tag{2}
\end{equation*}
$$

Alternatively, (2) can be expressed as

$$
\begin{equation*}
-z^{-1} \psi^{(0)}(1-z)-z^{-1} \gamma=\sum_{k=0}^{\infty} \zeta(k+2) z^{k} \tag{3}
\end{equation*}
$$

Inserting the functional equation for $\psi^{(0)}(z)$, given by [1, 2]

$$
\begin{equation*}
\psi^{(0)}(z+1)=\psi^{(0)}(z)+\frac{1}{z} \tag{4}
\end{equation*}
$$

into (3) and multiplying by $(1-z)$ gives

$$
\begin{equation*}
-z^{-1}(1-z)\left(\psi^{(0)}(2-z)-(1-z)^{-1}\right)-z^{-1}(1-z) \gamma=(1-z) \sum_{k=0}^{\infty} \zeta(k+2) z^{k} \tag{5}
\end{equation*}
$$

Now apply the operator $M^{-1} \lim _{z \rightarrow 1^{-}} \frac{\partial^{M}}{\partial z^{M}}$ to (5). Next apply Lemma 1 to the term involving the digamma function on the left-hand side and Proposition 1 to the right-hand side. After a little algebra, one arrives at the desired result, thereby completing the proof.

For $M=1$, Theorem 1 reduces to (1).
Theorem 2. For positive integers $M$ and $N$,

$$
\begin{gathered}
\sum_{n=0}^{\infty}(n+1)_{M-1}(n+M+1)^{N}(1-\zeta(n+M+1))=\sum_{l=1}^{N} S(N+1, l+1)(-1)^{l+1} g_{M, l} \\
+(-1)^{M-1} \sum_{j=0}^{M-1}\binom{M-1}{j} j!\psi^{(M-j-1)}(1)+\gamma(-1)^{M-1}(M-1)!
\end{gathered}
$$

where for $l \geq 0$,

$$
g_{M, l}:=-\sum_{j=0}^{M-1}\binom{M-1}{j}(-1)^{M-1-j} \psi^{(l+M-1-j)}(1) \frac{(l-1)!}{(l-1-j)!},
$$

and $S(n, l)$ represents the Stirling numbers of the second kind.

Proof. In [6], the following formula is given for integers $N \geq 1$ and $\Re(a)>0$ :

$$
\begin{equation*}
\sum_{k=2}^{\infty} k^{N} z^{k} \zeta(k, a)=\sum_{l=1}^{N} S(N+1, l+1) l!\zeta(l+1, a-z) z^{l+1}-z\left(\psi^{(0)}(a-z)-\psi^{(0)}(a)\right) \tag{6}
\end{equation*}
$$

for $|z|<|a|$, where $\zeta(s, a)$ is the Hurwitz zeta function [2]. Note that the above result has been corrected for $N \geq 1$. We have also shifted the sum by replacing $l$ by $l+1$ for our convenience. Now $\lim _{n \rightarrow \infty} \zeta(n, a)$ equals 0 if $a>1$, equals 1 if $a=1$, and equals $\infty$ if $0<a<1$. Hence (6) is of interest to us only if $\mathrm{a}=1$. Therefore, we put $\mathrm{a}=1$, which gives

$$
\begin{equation*}
\sum_{k=2}^{\infty} k^{N} z^{k} \zeta(k)=\sum_{l=1}^{N} S(N+1, l+1) l!\zeta(l+1,1-z) z^{l+1}-z\left(\psi^{(0)}(1-z)-\psi^{(0)}(1)\right) \tag{7}
\end{equation*}
$$

Differentiating (4) $l$ times, we obtain

$$
\begin{equation*}
\psi^{(l)}(2-z)=\psi^{(l)}(1-z)+(1-z)^{-l-1}(-1)^{l} l!. \tag{8}
\end{equation*}
$$

By using (2.15) in [1], we arrive at

$$
\begin{equation*}
\sum_{k=0}^{\infty} k^{N} z^{k}=\sum_{k=1}^{\infty} k^{N} z^{k}=\sum_{l=0}^{N} S(N+1, l+1) l!(1-z)^{-l-1} z^{l+1} \tag{9}
\end{equation*}
$$

Now $S(n, 1)=1$ for all non-negative integers $n$, so we may write (9) for $N \geq 1$ as

$$
\begin{equation*}
\sum_{k=0}^{\infty} k^{N} z^{k}=z(1-z)^{-1}+\sum_{l=1}^{N} S(N+1, l+1) l!(1-z)^{-l-1} z^{l+1} \tag{10}
\end{equation*}
$$

With the aid of $\psi^{(l)}(z)=(-1)^{l+1} l!\zeta(l+1, z)$, and (10), we re-write (7) as

$$
\begin{align*}
\sum_{k=2}^{\infty} k^{N} z^{k} \zeta(k)= & \sum_{l=1}^{N} S(N+1, l+1)\left((-1)^{l+1} \psi^{(l)}(2-z)+(1-z)^{-l-1} l!\right) z^{l+1} \\
& -z\left(\psi^{(0)}(1-z)-\psi^{(0)}(1)\right) \tag{11}
\end{align*}
$$

Comparing (10) with (11) and noting that $\psi^{(0)}(1)=-\gamma$,

$$
\begin{align*}
\sum_{k=2}^{\infty} k^{N} z^{k}(\zeta(k)-1) & =\sum_{l=1}^{N} S(N+1, l+1)(-1)^{l+1} \psi^{(l)}(2-z) z^{l+1}+z-z(1-z)^{-1} \\
& -z\left(\psi^{(0)}(1-z)+\gamma\right) \tag{12}
\end{align*}
$$

Next we choose $\alpha_{n}=(n+2)^{N}(\zeta(n+2)-1)$. Note that since the first term of the Dirichlet series for $\zeta(s)$ has been removed and exponential growth dominates
polynomial growth, $\lim _{n \rightarrow \infty}(n+2)^{N}(\zeta(n+2)-1)=0$. The far right side of Equation (12) may be construed as (3). Multiplying both sides by $z^{-2}$, and applying Proposition 1, we arrive at the following result:

$$
\begin{gather*}
\lim _{z \rightarrow 1^{-}} \frac{\partial^{M}}{\partial z^{M}}\left((1-z) \psi^{(l)}(2-z) z^{l-1}\right)  \tag{13}\\
=\lim _{z \rightarrow 1^{-}} \frac{\partial^{M-1}}{\partial z^{M-1}}\left(\psi^{(l)}(2-z) z^{(l-1)}\right) \\
=-M \sum_{j=0}^{M-1}\binom{M-1}{j}(-1)^{M-1-j} \psi^{(l+M-1-j)}(1) \frac{(l-1)!}{(l-1-j)!} .
\end{gather*}
$$

In obtaining this result, we have used the formula $\lim _{z \rightarrow 1^{-}} \frac{\partial^{M}}{\partial z^{M}}\left(z^{l}\right)=l!/(l-M)!$. This proves the theorem after noting that the $M$-th derivative of $(1-z) z^{-1}-z^{-1}=$ -1 is 0 .

Note that since $N \geq 1$, Theorem 2 is not a generalization of Theorem 1. Hence Theorem 1 is not redundant. Further, for integers $N \geq 1$, we have that $\psi^{(N)}(1)=$ $(-1)^{N+1} N!\zeta(N+1)$.

## 3. Conclusion

It has been shown here that the summation formula used in proving $q$-series identities can also be used to prove identities for series with the Riemann zeta function in their summands. The purpose of this work is to illustrate new methods toward finding new expressions for sums of the form

$$
\sum_{n=0}^{\infty} a_{n}(L(n+\sigma+1)-1)
$$

where the $a_{n}$ are appropriately chosen for the series to converge, and $L(s)$ is a Dirichlet series, whose first term is assumed to be unity and converges when $\Re(s)>$ $\sigma$. It is important to emphasize that our results show that there is a deeper structure to the types of series identities formed through the use of [4, Proposition 1.2]. Previously, it appears that no one has exploited [4, Proposition 1.2] outside of its $q$-series forms, which may have implied that its utility was only limited to basic hypergeometric series. This work is a curious case where interesting new results have been obtained by using a result from a completely different field of mathematics.

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