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# INFINITE SUMMATION FORMULAS INVOLVING THE RIEMANN–ZETA FUNCTION

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**Abstract**

By some hypergeometric summation theorems, the authors establish a series of new infinite summation formulas involving generalized harmonic numbers related to the Riemann-Zeta function, with three different patterns.

## 1. Introduction

Following Slater [13], the generalized hypergeometric series is defined by

$$ {}_{p+1}F_q\left[\begin{matrix}\alpha_0, \alpha_1, \ldots, \alpha_p \\ \beta_1, \quad \ldots, \beta_q\end{matrix} \,\Big|\, z\right] := \sum_{n=0}^{\infty} \frac{(\alpha_0)(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, $$

where the shifted factorial is defined by

$$ (x)_0 :\equiv 1 \quad \text{and} \quad (x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1), \quad n = 1, 2, \cdots . $$

Here and in what follows, all hypergeometric series are convergent. For more details of hypergeometric series, we refer the readers to [13]. For the $\Gamma$-function, the Weierstrass product expression [5] holds:

$$ \Gamma(z) = z^{-1} \prod_{n=1}^{\infty} (1+1/n)^z/(1+z/n), $$

and the logarithmic differentiation with the Euler constant is being given by

$$ \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \frac{z-1}{(n+1)(n+z)}, \quad \gamma = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} \frac{1}{k} - \ln n\right\}. $$

Also, the following expansions of $\Gamma$-function are found in [17]:

$$\Gamma(1-z) = \exp\left\{\sum_{k=1}^{\infty} \frac{\sigma_k}{k} z^k\right\}; \tag{1}$$

$$\Gamma(\frac{1}{2}-z) = \sqrt{\pi}\exp\left\{\sum_{k=1}^{\infty} \frac{\tau_k}{k} z^k\right\}, \tag{2}$$

where the Riemann-Zeta sequences $\{\sigma_k, \tau_k\}$ are defined by

$$\sigma_1 = \gamma, \qquad \sigma_m = \zeta(m) = \sum_{k=1}^{\infty} \frac{1}{k^m}, \quad m = 2, 3, \cdots;$$

$$\tau_1 = \gamma + 2\ln 2, \qquad \tau_m = (2^m - 1)\zeta(m), \quad m = 2, 3, \cdots.$$

Throughout the paper, we use the Euler summation formulas [9] to express $\zeta(2)$ and $\zeta(4)$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

The generalized harmonic numbers which are defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \qquad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}, \quad r = 2, 3, \cdots,$$

$$O_n = \sum_{k=1}^{n} \frac{1}{2k-1}, \qquad O_n^{(r)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^r}, \quad r = 2, 3, \cdots,$$

can be used to express the following finite products [11] through the symmetric functions:

$$\prod_{k=1}^{n}(1+\frac{x}{k}) = 1 + xH_n + \frac{x^2}{2}(H_n^2 - H_n^{(2)}) + \frac{x^3}{6}(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}) + \cdots; \tag{3}$$

$$\prod_{k=1}^{n}(1-\frac{x}{k})^{-1} = 1 + xH_n + \frac{x^2}{2}(H_n^2 + H_n^{(2)}) + \frac{x^3}{6}(H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}) + \cdots; \tag{4}$$

$$\prod_{k=1}^{n}(1+\frac{y}{2k-1}) = 1 + yO_n + \frac{y^2}{2}(O_n^2 - O_n^{(2)}) + \frac{y^3}{6}(O_n^3 - 3O_nO_n^{(2)} + 2O_n^{(3)}) + \cdots; \tag{5}$$

$$\prod_{k=1}^{n}(1-\frac{y}{2k-1})^{-1} = 1 + yO_n + \frac{y^2}{2}(O_n^2 + O_n^{(2)}) + \frac{y^3}{6}(O_n^3 + 3O_nO_n^{(2)} + 2O_n^{(3)}) + \cdots. \tag{6}$$

The identities involving generalized harmonic numbers related to the Riemann-Zeta function can be established by many different methods. For example, de Doelder

[10] applied the irregular integral and Borwein [7] used the Parseval identity on Fourier series to get some of the identities. Also, the hypergeometric method [12] was used to obtain infinite summation formulas involving generalized harmonic numbers related to the Riemann-Zeta function. Recently, some results about the Riemann-Zeta function have been established by Ablinger [2, 3] who viewed the series as specializations of generating series and obtained integral representations. More interesting infinite summation formulas related to the Riemann-Zeta function have been found, such as, in [8, 9, 14, 16].

Applying the hypergeometric method, we will establish some infinite summation formulas involving generalized harmonic numbers related to the Riemann-Zeta function from the famous summation theorems due to Gauss, Watson and Bailey. In fact, all the results can be computed automatically using the computer algebra package HarmonicSums [1]. Here, we will present the following three different patterns which have not been found before,

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{k^i 2^{2k}} P_k, \qquad i = 1, 2, 3;$$
$$\sum_{k=1}^{\infty} \frac{P_k}{k^i 2^k}, \quad i = 1, 2;$$
$$\sum_{k=1}^{\infty} \frac{3^k}{k^2 \binom{2k}{k}} P_k,$$

where $P_k$ is a polynomial in $H_k^{(r)}$ or $O_k^{(r)}$ $(k, r \in Z^+)$.

## 2. Summation Formulas Related to the Riemann-Zeta Function From Gauss and Watson Summation Theorems

In this section, we will derive some infinite summation formulas related to the Riemann-Zeta function from the Gauss summation theorem (7) and Watson summation theorem (40) with the pattern as follows:

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{k^i 2^{2k}} P_k, \qquad i = 1, 2, 3,$$

where $P_k$ is a polynomial in $H_k^{(r)}$ or $O_k^{(r)}$ $(k, r \in Z^+)$.

**Theorem 2.1 (Gauss, [13] ).** *The following summation formula holds for complex parameters $a, b, c$ with $\mathcal{R}(c - a - b) > 0$:*

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{7}$$

Performing the replacements $a \to a + \frac{1}{2}$ and $c \to c+1$ in (7), we attain the following expression:

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}+a, b\\ 1+c\end{matrix}\,\Big|\, 1\right] = \frac{\Gamma(1+c)\Gamma(\frac{1}{2}+c-a-b)}{\Gamma(\frac{1}{2}+c-a)\Gamma(1+c-b)}. \tag{8} $$

Recalling the definition of hypergeometric series and applying (1) and (2), we can write (8) as

$$ 1+b\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\prod_{i=1}^{k}(1+\frac{2a}{2i-1})\prod_{m=1}^{k-1}(1+\frac{b}{m})}{k2^{2k}\prod_{j=1}^{k}(1+\frac{c}{j})} \tag{9} $$

$$ \begin{aligned} &= \exp\Big\{\sum_{k=1}^{\infty}\frac{(-1)^k\tau_k}{k}\big[(c-a-b)^k-(c-a)^k\big]+\sum_{k=1}^{\infty}\frac{(-1)^k\sigma_k}{k}\big[c^k-(c-b)^k\big]\Big\}\\ &= \exp\Big\{\tau_1 b+\tau_2(ab-bc+\frac{1}{2}b^2)+\tau_3(a^2b+ab^2-2abc+\frac{1}{3}b^3-b^2c+bc^2)\\ &+\tau_4(a^3b+\frac{3}{2}a^2b^2-3a^2bc+ab^3-3ab^2c+3abc^2+\frac{1}{4}b^4-b^3c+\frac{3}{2}b^2c^2-bc^3)\\ &+\tau_5(a^4b+2a^3b^2-4a^3bc+2a^2b^3-6a^2b^2c+6a^2bc^2+ab^4-4ab^3c+6ab^2c^2\\ &-4abc^3+\frac{1}{5}b^5-b^4c+2b^3c^2-2b^2c^3+bc^4)+\cdots\Big\}\times\exp\Big\{-\sigma_1 b+\sigma_2(-\frac{1}{2}b^2\\ &+bc)+\sigma_3(-\frac{1}{3}b^3+b^2c-bc^2)+\sigma_4(-\frac{1}{4}b^4+b^3c-\frac{3}{2}b^2c^2+bc^3)+\sigma_5(-\frac{1}{5}b^5\\ &+b^4c-2b^3c^2+2b^2c^3-bc^4)+\cdots\Big\}. \end{aligned} \tag{10} $$

Applying the formulas (3)–(5), we can expand the infinite summation over products in (9) to obtain another alternative multivariate series expansion with generalized harmonic number. Applying the relations (5) and (6), we can obtain a compact representation of a multivariate series by the power series expansions in (10) with the Riemann-Zeta function. Therefore, comparing the coefficients of the above two multivariate series expansions term-by-term, we get a number of infinite summation formulas involving generalized harmonic numbers related to the Riemann-Zeta function. Here, we use a self-explanatory notation $[a^ib^jc^k]$ to show the process of extracting the coefficients of the monomial $a^ib^jc^k$ from multivariate power series expansions.

**Proposition 2.2.** *The following infinite summation formulas related to* $\zeta(2)$ *hold* *true:*

$$[ab] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k}{k 2^{2k}} = \frac{\pi^2}{4}; \tag{11}$$

$$[bc] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k}{k 2^{2k}} = \frac{\pi^2}{3}; \tag{12}$$

$$[b^2] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1}}{k 2^{2k}} = \frac{\pi^2}{6} + 2\ln^2 2. \tag{13}$$

**Proposition 2.3.** *The following infinite summation formulas related to $\zeta(3)$ hold true:*

$$[a^2b] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (O_k^2 - O_k^{(2)})}{k 2^{2k}} = \frac{7\zeta(3)}{2}; \tag{14}$$

$$[bc^2] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (H_k^2 + H_k^{(2)})}{k 2^{2k}} = 12\zeta(3); \tag{15}$$

$$[abc] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k O_k}{k 2^{2k}} = 7\zeta(3). \tag{16}$$

**Proposition 2.4.** *The following infinite summation formulas related to $\zeta(2)$ and $\zeta(3)$ hold true:*

$$[ab^2] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1} O_k}{k 2^{2k}} = \frac{7}{2}\zeta(3) + \frac{\pi^2}{2}(\ln 2); \tag{17}$$

$$[b^2c] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1} H_k}{k 2^{2k}} = 6\zeta(3) + \frac{2\pi^2}{3}(\ln 2); \tag{18}$$

$$[b^3] \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (H_{k-1}^2 - H_{k-1}^{(2)})}{k 2^{2k}} = 4\zeta(3) + \frac{2\pi^2}{3}(\ln 2) + \frac{8}{3}\ln^3 2. \tag{19}$$

Now we can obtain the following infinite summation formulas $\sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k}{k^i 2^{2k}}$ with $i = 1, 2$:

$$(11) \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k}{k 2^{2k}} = \frac{\pi^2}{4};$$

$$(16) - (17) \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k}{k^2 2^{2k}} = \frac{7\pi^2}{12} - \frac{\pi^2}{2}(\ln 2).$$

**Proposition 2.5.** *The following infinite summation formulas related to $\zeta(4)$ hold true:*

$$[a^2bc] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k (O_k^2 - O_k^{(2)})}{k 2^{2k}} = \frac{\pi^4}{4}; \tag{20}$$

$$[abc^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k (H_k^2 + H_k^{(2)})}{k 2^{2k}} = \frac{\pi^4}{2}; \tag{21}$$

$$[a^3b] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (O_k^3 - 3O_k O_k^{(2)} + 2O_k^{(3)})}{k 2^{2k}} = \frac{\pi^4}{8}; \tag{22}$$

$$[bc^3] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (H_k^3 + 3H_k H_k^{(2)} + 2H_k^{(3)})}{k 2^{2k}} = \frac{14\pi^4}{15}. \tag{23}$$

**Proposition 2.6.** *The following infinite summation formulas related to $\zeta(2)$, $\zeta(3)$ and $\zeta(4)$ hold true:*

$$[a^2b^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1} (O_k^2 - O_k^{(2)})}{k 2^{2k}} = \frac{3\pi^4}{16} + 7(\ln 2)\zeta(3); \tag{24}$$

$$[b^2c^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1} (H_k^2 + H_k^{(2)})}{k 2^{2k}} = \frac{26\pi^4}{45} + 24(\ln 2)\zeta(3); \tag{25}$$

$$[ab^2c] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1} H_k O_k}{k 2^{2k}} = \frac{\pi^4}{3} + 14(\ln 2)\zeta(3); \tag{26}$$

$$[ab^3] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k (H_{k-1}^2 - H_{k-1}^{(2)})}{k 2^{2k}} = \frac{\pi^4}{4} + 14(\ln 2)\zeta(3) + \pi^2(\ln^2 2); \tag{27}$$

$$[b^3c] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k (H_{k-1}^2 - H_{k-1}^{(2)})}{k 2^{2k}} = \frac{19\pi^4}{45} + 24(\ln 2)\zeta(3) + \frac{4\pi^2}{3}(\ln^2 2). \tag{28}$$

**Proposition 2.7.** *The following infinite summation formulas related to $\zeta(5)$ hold true:*

$$[a^3bc] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k (O_k^3 - 3O_k O_k^{(2)} + 2O_k^{(3)})}{k 2^{2k}} = 93\zeta(5); \tag{29}$$

$$[abc^3] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k (H_k^3 + 3H_k H_k^{(2)} + 2H_k^{(3)})}{k 2^{2k}} = 372\zeta(5); \tag{30}$$

$$[a^2bc^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (O_k^2 - O_k^{(2)})(H_k^2 + H_k^{(2)})}{k 2^{2k}} = 186\zeta(5). \tag{31}$$

**Proposition 2.8.** *The following infinite summation formulas related to* $\zeta(2)$, $\zeta(3)$, $\zeta(4)$ *and* $\zeta(5)$ *hold true:*

$$[a^2b^2c] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k H_{k-1}(O_k^2 - O_k^{(2)})}{k2^{2k}} = 93\zeta(5) + \frac{\pi^4}{2}(\ln 2) + \frac{14\pi^2}{3}\zeta(3); \quad (32)$$

$$[ab^2c^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k H_{k-1}(H_k^2 + H_k^{(2)})}{k2^{2k}} = 186\zeta(5) + \pi^4(\ln 2) + \frac{23\pi^2}{3}\zeta(3); \quad (33)$$

$$[a^3b^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1}(O_k^3 - 3O_kO_k^{(2)} + 2O_k^{(3)})}{k2^{2k}} = \frac{93}{2}\zeta(5) + \frac{\pi^4}{4}(\ln 2) + \frac{21\pi^2}{8}\zeta(3); \quad (34)$$

$$[b^2c^3] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1}(H_k^3 + 3H_kH_k^{(2)} + 2H_k^{(3)})}{k2^{2k}} = 360\zeta(5) + \frac{28\pi^4}{15}(\ln 2) + 12\pi^2\zeta(3); \quad (35)$$

$$[a^2b^3] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (O_k^2 - O_k^{(2)})(H_{k-1}^2 - H_{k-1}^{(2)})}{k2^{2k}} = 62\zeta(5) + \frac{3\pi^4}{4}(\ln 2) + \frac{14\pi^2}{3}\zeta(3) + 14(\ln^2 2)\zeta(3); \quad (36)$$

$$[b^3c^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (H_k^2 + H_k^{(2)})(H_{k-1}^2 - H_{k-1}^{(2)})}{k2^{2k}} = 240\zeta(5) + \frac{104\pi^4}{45}(\ln 2) + 12\pi^2\zeta(3) + 48(\ln^2 2)\zeta(3); \quad (37)$$

$$[ab^3c] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k O_k (H_{k-1}^2 - H_{k-1}^{(2)})}{k2^{2k}} = 124\zeta(5) + \frac{4\pi^4}{3}(\ln 2) + \frac{23\pi^2}{3}\zeta(3) + 28(\ln^2 2)\zeta(3); \quad (38)$$

$$[ab^4] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} O_k (H_{k-1}^3 - 3H_{k-1}H_{k-1}^{(2)} + 2H_{k-1}^{(3)})}{k2^{2k}} = 93\zeta(5) + \frac{3\pi^4}{2}(\ln 2) + \frac{13\pi^2}{2}\zeta(3) + 42(\ln^2 2)\zeta(3) + 2\pi^2(\ln^3 2). \quad (39)$$

Therefore, we get the following infinite summation formula:

$$(29) - (34) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} (O_k^3 - 3O_kO_k^{(2)} + 2O_k^{(3)})}{k^2 2^{2k}} = \frac{93\zeta(5)}{2} - \frac{\pi^4}{4}(\ln 2) - \frac{21\pi^2}{8}\zeta(3).$$

Applying the formula (32), we obtain the following infinite summation formulas of the pattern $\sum_{k=1}^{\infty} \frac{\binom{2k}{k}(O_k^2 - O_k^{(2)})}{k^i 2^{2k}}$ with $i = 1, 2, 3$:

$$(45) \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(O_k^2 - O_k^{(2)})}{k 2^{2k}} = \frac{7\zeta(3)}{2};$$

$$(20) - (24) \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(O_k^2 - O_k^{(2)})}{k^2 2^{2k}} = \frac{\pi^4}{8} - 7(\ln 2)\zeta(3) - \frac{3\pi^2}{8};$$

$$(31) + (36) \qquad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(O_k^2 - O_k^{(2)})}{k^3 2^{2k}} = 31\zeta(5) - \frac{\pi^4}{8}(\ln 2) - \frac{7\pi^2}{3}\zeta(3) + 7(\ln^2 2)\zeta(3).$$

Next, we will deduce some infinite summation formulas related to the Riemann-Zeta function from the Watson summation theorem.

**Theorem 2.9 (Watson [13]).** *For the complex parameters $a, b, c$ with $\mathcal{R}(1 - a - b + 2c) > 0$, the following summation formula holds true:*

$$ {}_3F_2 \left[ \begin{matrix} a, b, c \\ \frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b, 2c \end{matrix} \,\Big|\, 1 \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2} + c)\Gamma(\frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b)\Gamma(\frac{1}{2} - \frac{1}{2}a - \frac{1}{2}b + c)}{\Gamma(\frac{1}{2} + \frac{1}{2}a)\Gamma(\frac{1}{2} + \frac{1}{2}b)\Gamma(\frac{1}{2} - \frac{1}{2}a + c)\Gamma(\frac{1}{2} - \frac{1}{2}b + c)}. \quad (40)$$

Making the substitutions $a \to a + 1$ and $c \to c + \frac{1}{2}$ in (40), we obtain

$$ {}_3F_2 \left[ \begin{matrix} 1 + a, b, \frac{1}{2} + c \\ 1 + \frac{1}{2}a + \frac{1}{2}b, 1 + 2c \end{matrix} \,\Big|\, 1 \right] = \frac{\Gamma(\frac{1}{2})\Gamma(1 + c)\Gamma(1 + \frac{1}{2}a + \frac{1}{2}b)\Gamma(\frac{1}{2} - \frac{1}{2}a - \frac{1}{2}b + c)}{\Gamma(1 + \frac{1}{2}a)\Gamma(\frac{1}{2} + \frac{1}{2}b)\Gamma(\frac{1}{2} - \frac{1}{2}a + c)\Gamma(1 - \frac{1}{2}b + c)}. \quad (41)$$

By the definition of hypergeometric series and applying (1) and (2), we can restate (41) as:

$$1 + b \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{k 2^{2k}} \frac{\prod_{i=1}^{k}(1 + \frac{a}{i})(1 + \frac{2c}{2i-1}) \prod_{m=1}^{k-1}(1 + \frac{b}{m})}{\prod_{j=1}^{k}(1 + \frac{a+b}{2j})(1 + \frac{2c}{j})} \quad (42)$$

$$= \exp\Big\{ \sum_{k=1}^{\infty} (-1)^k \frac{\tau_k}{k} \big[(c - \frac{a}{2} - \frac{b}{2})^k - (\frac{b}{2})^k - (c - \frac{a}{2})^k\big]$$

$$+ \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} \big[c^k + (\frac{a}{2} + \frac{b}{2})^k - (\frac{a}{2})^k - (c - \frac{b}{2})^k\big] \Big\} \quad (43)$$

$$= \exp\Big\{ \tau_1 b + \tau_2(\frac{1}{4}ab - \frac{1}{2}bc) + \tau_3(\frac{1}{8}a^2 b + \frac{1}{8}ab^2 - \frac{1}{2}abc - \frac{1}{4}b^2 c + \frac{1}{2}bc^2 + \frac{1}{12}b^3)$$

$$+ \tau_4(\frac{1}{16}a^3 b + \frac{3}{32}a^2 b^2 - \frac{3}{8}a^2 bc + \frac{1}{16}ab^3 + \frac{3}{4}abc^2 - \frac{3}{8}ab^2 c - \frac{1}{8}b^3 c + \frac{3}{8}b^2 c^2 - \frac{1}{2}bc^3)$$

$$+\,\tau_5(\frac{1}{32}a^4b+\frac{1}{16}a^3b^2-\frac{1}{4}a^3bc-\frac{3}{8}a^2b^2c+\frac{1}{16}a^2b^3+\frac{3}{4}a^2bc^2+\frac{3}{4}ab^2c^2-abc^3$$
$$-\frac{1}{4}ab^3c+\frac{1}{32}ab^4+\frac{1}{80}b^5-\frac{1}{16}b^4c+\frac{1}{4}b^3c^2-\frac{1}{2}b^2c^3+\frac{1}{2}bc^4)+\cdots\Big\}$$
$$\times\ \exp\Big\{-\sigma_1 b+\sigma_2(\frac{1}{4}ab+\frac{1}{2}bc)+\sigma_3(-\frac{1}{8}a^2b-\frac{1}{8}ab^2-\frac{1}{12}b^3+\frac{1}{4}b^2c-\frac{1}{2}bc^2)$$
$$+\,\sigma_4(\frac{1}{16}a^3b+\frac{3}{32}a^2b^2+\frac{1}{16}ab^3-\frac{3}{8}b^2c^2+\frac{1}{2}bc^3+\frac{1}{8}b^3c)+\ \sigma_5(-\frac{1}{32}a^4b-\frac{1}{16}a^3b^2$$
$$-\frac{1}{16}a^2b^3-\frac{1}{32}ab^4+\frac{1}{16}b^4c-\frac{1}{4}b^3c^2+\frac{1}{2}b^2c^3-\frac{1}{2}bc^4-\frac{1}{80}b^5)+\cdots\Big\}. \tag{44}$$

Its power series expansion via (3)–(6) leads us to the infinite summation formula involving generalized harmonic numbers related to the Riemann-Zeta function.

**Proposition 2.10.** *The following infinite summation formulas related to $\zeta(3)$ hold true:*

$$[a^2b]\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}(H_k^2-3H_k^{(2)})}{k2^{2k}}=6\zeta(3); \tag{45}$$

$$[bc^2]\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}[(H_k-O_k)^2+(H_k^{(2)}-O_k^{(2)})]}{k2^{2k}}=\frac{3}{2}\zeta(3); \tag{46}$$

$$[abc]\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}H_k(H_k-O_k)}{k2^{2k}}=\frac{7}{2}\zeta(3); \tag{47}$$

$$[b^3]\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}[(2H_{k-1}-H_k)^2+(H_k^{(2)}-4H_{k-1}^{(2)})]}{k2^{2k}}=4\zeta(3)+\frac{32}{3}\ln^3 2. \tag{48}$$

Also, we obtain the following infinite summation formulas:

$$(18)-(16)\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}H_k}{k^2 2^{2k}}=\frac{9}{2}\zeta(3)-\frac{2\pi^2}{3}\ln 2; \tag{49}$$

$$(18)+(49)\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}H_k^2}{k2^{2k}}=\frac{21}{2}\zeta(3);$$

$$(15)+(19)\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}H_{k-1}}{k^2 2^{2k}}=\frac{5}{2}\zeta(3)-\frac{\pi^2}{3}\ln 2-\frac{4}{3}\ln^3 2; \tag{50}$$

$$(18)-(50)\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}H_{k-1}^2}{k2^{2k}}=\frac{7}{2}\zeta(3)+\pi^2(\ln 2)+\frac{4}{3}\ln^3 2.$$

**Proposition 2.11.** *The following infinite summation formulas related to $\zeta(4)$ hold true:*

$$[a^2bc]\quad \sum_{k=1}^{\infty}\frac{\binom{2k}{k}(H_k-O_k)(H_k^2-3H_k^{(2)})}{k2^{2k}}=\frac{\pi^4}{4}; \tag{51}$$

$$[abc^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k[(H_k - O_k)^2 + (H_k^{(2)} - O_k^{(2)})]}{k2^{2k}} = \frac{\pi^4}{8}; \tag{52}$$

$$[a^3 b] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(H_k^3 - 9H_k H_k^{(2)} + 14H_k^{(3)})}{k2^{2k}} = \frac{8\pi^4}{15}. \tag{53}$$

We also get the following infinite summation formulas related to the Riemann-zeta functions:

$$3 \times (23) + (53) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k^3 + 5H_k^{(3)}}{k2^{2k}} = \frac{5\pi^4}{6};$$

$$7 \times (23) - (53) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k^3 + 5H_k H_k^{(2)}}{k2^{2k}} = \pi^4;$$

$$(23) - (53) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_k H_k^{(2)} - H_k^{(3)}}{k2^{2k}} = \frac{\pi^4}{30}.$$

**Proposition 2.12.** *The following infinite summation formulas related to $\zeta(2)$ and $\zeta(3)$ hold true:*

$$[ab^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(H_k^{(2)} + 2H_k H_{k-1} - H_k^2)}{k2^{2k}} = 3\zeta(3) + \frac{4\pi^2}{3}(\ln 2); \tag{54}$$

$$[b^2 c] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(H_k - O_k)(2H_{k-1} - H_k)}{k2^{2k}} = \frac{3}{2}\zeta(3) + \frac{\pi^2}{3}(\ln 2). \tag{55}$$

From the above formulas, we can obtain

$$(12) - (11) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(H_k - O_k)}{k2^{2k}} = \frac{\pi^2}{12};$$

$$(55) - (47) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(H_k - O_k)}{k^2 2^{2k}} = \zeta(3) - \frac{\pi^2}{6}(\ln 2).$$

**Proposition 2.13.** *The following infinite summation formulas related to $\zeta(2)$, $\zeta(3)$ and $\zeta(4)$ hold true:*

$$[a^2 b^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}[H_k^2 H_{k-1} + H_k^{(2)}(4H_k - 3H_{k-1})]}{k2^{2k}} = \frac{32\pi^4}{45} + 12(\ln 2)\zeta(3); \tag{56}$$

$$[b^2 c^2] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(2H_{k-1} - H_k)[(H_k - O_k)^2 + (H_k^{(2)} - O_k^{(2)})]}{k2^{2k}} = \frac{13\pi^4}{180} + 6(\ln 2)\zeta(3); \tag{57}$$

$$[ab^2c] \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}(H_k - O_k)(H_k^{(2)} + 2H_k H_{k-1} - H_k^2)}{k2^{2k}} = \frac{13\pi^4}{72} + 14(\ln 2)\zeta(3). \tag{58}$$

Also, we arrive at the following infinite summation formulas:

$$(52)+(57) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k} H_{k-1}[(H_k - O_k)^2 + (H_k^{(2)} - O_k^{(2)})]}{k2^{2k}} = \frac{71\pi^4}{720} + 3(\ln 2)\zeta(3);$$

$$(52)-(57) \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}[(H_k - O_k)^2 + (H_k^{(2)} - O_k^{(2)})]}{k^2 2^{2k}} = \frac{19\pi^4}{720} - 3(\ln 2)\zeta(3).$$

## 3. Summation Formulas Related to the Riemann-Zeta Function From Bailey Summation Theorem

In this section, we shall establish some infinite summation formulas related to the Riemann-Zeta function by Bailey's summation theorem (59) and another summation formula (72) with two new patterns as follows:

$$\sum_{k=1}^{\infty} \frac{P_k}{k^i 2^k}, \quad i = 1, 2, \qquad \sum_{k=1}^{\infty} \frac{3^k}{k^2 \binom{2k}{k}} P_k,$$

where $P_k$ is a polynomial in $H_k^{(r)}$ or $O_k^{(r)}$ $(k, r \in Z^+)$.

**Theorem 3.1 (Bailey, [13]).** *For complex parameters $a, c$ with $\mathcal{R}(c-1) > 0$, the following summation formula holds true:*

$${}_2F_1\left[\begin{matrix} a, 1-a \\ c \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2} + \frac{1}{2}c)}{\Gamma(\frac{1}{2}c + \frac{1}{2}a)\Gamma(\frac{1}{2} + \frac{1}{2}c - \frac{1}{2}a)}. \tag{59}$$

Performing the replacement $c \to c+1$ in (59), we obtain the following expression:

$${}_2F_1\left[\begin{matrix} a, 1-a \\ 1+c \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}c)\Gamma(1 + \frac{1}{2}c)}{\Gamma(\frac{1}{2} + \frac{1}{2}c + \frac{1}{2}a)\Gamma(1 + \frac{1}{2}c - \frac{1}{2}a)}. \tag{60}$$

Similar to the process illustrated in Theorem 2.1, and after some simplification, some summations involving generalized harmonic numbers related to the Riemann-Zeta function can be derived from the identity (60).

**Proposition 3.2.** *The following infinite summation formula related to $\zeta(2)$ holds true:*

$$[ac] \quad \sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12}. \tag{61}$$

**Proposition 3.3.** *The infinite summation formula related to* $\zeta(3)$ *holds true.*

$$[ac^2] \quad \sum_{k=1}^{\infty} \frac{H_k^2 + H_k^{(2)}}{k2^k} = \frac{3}{2}\zeta(3). \tag{62}$$

**Proposition 3.4.** *The following infinite summation formulas related to* $\zeta(2)$ *and* $\zeta(3)$ *hold true:*

$$[a^2c] \quad \sum_{k=1}^{\infty} \frac{H_k}{k^2 2^k} = \zeta(3) - \frac{\pi^2}{12}(\ln 2); \tag{63}$$

$$[a^3] \quad \sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}}{k2^k} = -\frac{1}{4}\zeta(3) + \frac{\pi^2}{12}(\ln 2) - \frac{1}{6}\ln^3 2. \tag{64}$$

**Proposition 3.5.** *The infinite summation formula related to* $\zeta(4)$ *holds true.*

$$[ac^3] \quad \sum_{k=1}^{\infty} \frac{H_k^3 + 3H_kH_k^{(2)} + 2H_k^{(3)}}{k2^k} = \frac{7\pi^4}{120}. \tag{65}$$

**Proposition 3.6.** *The following infinite summation formulas related to* $\zeta(2)$*,* $\zeta(3)$ *and* $\zeta(4)$ *hold true:*

$$[a^2c^2] \quad \sum_{k=1}^{\infty} \frac{H_k^2 + H_k^{(2)}}{k^2 2^k} = \frac{19\pi^4}{720} - \frac{3}{2}(\ln 2)\zeta(3); \tag{66}$$

$$[a^3c] \quad \sum_{k=1}^{\infty} \frac{H_kH_{k-1}^{(2)}}{k2^k} = -\frac{\pi^4}{360} + (\ln 2)\zeta(3) - \frac{\pi^2}{24}(\ln^2 2); \tag{67}$$

$$[a^4] \quad \sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}}{k^2 2^k} = \frac{\pi^4}{1440} + \frac{1}{4}(\ln 2)\zeta(3) - \frac{\pi^2}{24}(\ln^2 2) + \frac{1}{24}(\ln^4 2). \tag{68}$$

Therefore, we get the following infinite summation formula:

$$(67) - (68) \quad \sum_{k=1}^{\infty} \frac{H_{k-1}H_{k-1}^{(2)}}{k2^k} = -\frac{\pi^4}{288} + \frac{3}{4}(\ln 2)\zeta(3) - \frac{\pi^2}{24}(\ln^2 2).$$

**Proposition 3.7.** *The following infinite summation formulas related to* $\zeta(2)$*,* $\zeta(3)$*,*

*$\zeta(4)$ and $\zeta(5)$ hold true:*

$$[a^2c^3] \quad \sum_{k=1}^{\infty} \frac{H_k^3 + 3H_kH_k^{(2)} + 2H_k^{(3)}}{k^2 2^k} = 12\zeta(5) - \frac{7\pi^4}{120}(\ln 2) - \frac{3\pi^2}{8}\zeta(3); \quad (69)$$

$$[a^3c^2] \quad \sum_{k=1}^{\infty} \frac{(H_k^2 + H_k^{(2)})H_{k-1}^{(2)}}{k2^k} = -\frac{15}{4}\zeta(5) + \frac{19\pi^4}{740}(\ln 2) + \frac{7\pi^2}{24}\zeta(3) - \frac{3}{4}(\ln^2 2)\zeta(3); \quad (70)$$

$$[a^4c] \quad \sum_{k=1}^{\infty} \frac{H_kH_{k-1}^{(2)}}{k^2 2^k} = -\zeta(5) + \frac{\pi^4}{360}(\ln 2) + \frac{5\pi^2}{48}\zeta(3) - \frac{1}{2}(\ln^2 2)\zeta(3) + \frac{\pi^2}{72}(\ln^3 2). \quad (71)$$

Here, the results (62)-(72), which do not include binomials, can be easily rewritten in terms of multiple zeta values [4, 6].

Next, we get some summation formulas related to the Riemann-Zeta function due to another summation theorem, which reads as follows.

**Theorem 3.8 (Summation theorem [15]).** *For the complex parameters $b, d$ with $\mathcal{R}(2b+2d-3) < 0$, the following summation formula holds true:*

$${}_3F_2\left[\begin{matrix} b, d, \frac{b+d}{3} \\ \frac{b+d}{2}, \frac{1+b+d}{2} \end{matrix} \,\Big|\, \frac{3}{4}\right] = \frac{\Gamma(1+b+d)\Gamma(1+\frac{1}{3}b)\Gamma(1+\frac{1}{3}d)}{\Gamma(1+b)\Gamma(1+d)\Gamma(1+\frac{b+d}{3})}. \quad (72)$$

By the definition of hypergeometric series and the identities (1) and (2), the summation formula (72) can be reformulated as

$$1 + \frac{2}{3}bd\sum_{k=1}^{\infty} \frac{3^k \prod_{i=1}^{k-1}(1+\frac{b}{i})(1+\frac{d}{i})(1+\frac{b+d}{3i})}{k^2\binom{2k}{k}\prod_{j=1}^{k-1}(1+\frac{b+d}{2j})\prod_{m=1}^{k}(1+\frac{b+d}{2m-1})} \quad (73)$$

$$= \exp\Big\{\sum_{k=1}^{\infty} \frac{(-1)^k\sigma_k}{k}\big[(b+d)^k + (\frac{b}{3})^k + (\frac{d}{3})^k - b^k - d^k - (\frac{b+d}{3})^k\big]\Big\}$$

$$= \exp\Big\{\sigma_2(\frac{8}{9}bd) + \sigma_3(-\frac{26}{27}b^2d - \frac{26}{27}bd^2) + \sigma_4(\frac{80}{81}b^3d + \frac{80}{81}bd^3 + \frac{40}{81}b^2d^2)$$

$$+\sigma_5(-\frac{242}{243}b^4d - \frac{484}{243}b^3d^2 - \frac{484}{243}b^2d^3 - \frac{242}{243}bd^4) + \cdots\Big\}. \quad (74)$$

Some infinite summation formulas involving generalized harmonic numbers related to the Riemann-Zeta function obtained from its power series expansion are presented as follows.

**Proposition 3.9.** *The following infinite summation formula related to $\zeta(3)$ holds true:*

$$[b^2 d] \qquad \sum_{k=1}^{\infty} \frac{3^k(6O_k - 5H_{k-1})}{k^2\binom{2k}{k}} = \frac{26}{3}\zeta(3).$$

**Proposition 3.10.** *The following infinite summation formula related to $\zeta(2)$ and $\zeta(4)$ holds true:*

$$[b^2 d^2] \qquad \sum_{k=1}^{\infty} \frac{3^k[36(O_k^2 + O_k^{(2)}) + 25H_{k-1}^2 + 5H_{k-1}^{(2)} - 60H_{k-1}O_k]}{k^2\binom{2k}{k}} = \frac{40\pi^4}{27}.$$

There are many other infinite summation formulas involving generalized harmonic numbers related to the Riemann-Zeta function that can be established from these summation formulas. Here, we just present some results for examples, the interested reader can do by themselves.

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