SUMS OF POWERS OVER EQUALLY SPACED FIBONACCI NUMBERS

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Received: 7/18/19, Accepted: 4/5/20, Published: 5/8/20

Abstract

Recent results about sums of cubes of Fibonacci numbers are extended to arbitrary powers.

1. Introduction

Frontczak [2] evaluates

$$\sum_{k=0}^{n} F_{mk}^3, \quad \sum_{k=0}^{n} (-1)^k F_{mk}^3, \quad \sum_{k=0}^{n} L_{mk}^3, \quad \sum_{k=0}^{n} (-1)^k L_{mk}^3,$$

with Fibonacci and Lucas numbers and m being an odd integer.

We show here how to deal with general integer exponents (not just 3), and drop the restriction that m must be odd.

Note that several papers about the evaluation of

$$\sum_{k=0}^{n} F_{2k}^{2l}$$

and similar sums have been written over the past few years; see, for example [1, 4]. The instance where 2 is replaced by m is somewhat more delicate but quite instructive.

For later use, we mention the Binet formulae: if

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2},$$

then $\sqrt{5}F_n = \alpha^n - \beta^n$ and $L_n = \alpha^n + \beta^n$.

2. The Summation of Shifted Fibonacci Numbers

To deal with a sum over F_{mn} (fixed m), we first consider a generating function:

$$\sum_{n\ge 0} F_{nm} z^n = \frac{zF_m}{1-(F_{m+1}+F_{m-1})z+(-1)^m z^2} = \frac{zF_m}{1-L_m z+(-1)^m z^2}.$$

For a proof, rewrite it as

$$[1-(F_{m+1}+F_{m-1})z+(-1)^m z^2]\sum_{n\ge 0} F_{nm} z^n = zF_m$$

and compare coefficients:

$$F_{nm}-(F_{m+1}+F_{m-1})F_{(n-1)m}+(-1)^m F_{(n-2)m} = [\![n=1]\!]F_m.$$

This can be proved by the Binet formula or otherwise and is classical.

Consequently,

$$\begin{aligned}&\frac{1}{1-z}\sum_{n\ge 0} F_{nm} z^n\\ &= \frac{F_m}{1-F_{m-1}-F_{m+1}+(-1)^m}\left[\frac{1}{1-z}-\frac{1-z(-1)^m}{1-(F_{m+1}+F_{m-1})z+(-1)^m z^2}\right].\end{aligned}$$

Now we read off the coefficient of z^n:

$$\begin{aligned}\sum_{k=0}^{n} F_{km} &= \frac{F_m}{1-F_{m-1}-F_{m+1}+(-1)^m}\\ &\quad -[z^n]\frac{F_m}{1-F_{m-1}-F_{m+1}+(-1)^m}\left[\frac{1-z(-1)^m}{1-(F_{m+1}+F_{m-1})z+(-1)^m z^2}\right]\\ &= \frac{F_m}{1-F_{m-1}-F_{m+1}+(-1)^m}\\ &\quad -\frac{1}{1-F_{m-1}-F_{m+1}+(-1)^m}[z^n]\frac{F_m}{1-(F_{m+1}+F_{m-1})z+(-1)^m z^2}\\ &\quad +\frac{1}{1-F_{m-1}-F_{m+1}+(-1)^m}[z^n]\frac{-z(-1)^m F_m}{1-(F_{m+1}+F_{m-1})z+(-1)^m z^2}\\ &= \frac{F_m-F_{(n+1)m}+(-1)^m F_{nm}}{1-F_{m-1}-F_{m+1}+(-1)^m}.\end{aligned}$$

So we notice that we have a closed formula for each fixed integer m.

Let us also do the analogous computation for alternating sums:

$$\sum_{n\ge 0} F_{nm}(-1)^n z^n = \frac{-zF_m}{1+(F_{m+1}+F_{m-1})z+(-1)^m z^2}$$

and

$$\frac{1}{1-z}\sum_{n\geq 0}F_{nm}(-1)^n z^n = -\frac{F_m}{1+F_{m-1}+F_{m+1}+(-1)^m}\frac{1}{1-z}$$
$$+\frac{F_m}{1+F_{m-1}+F_{m+1}+(-1)^m}\frac{1-(-1)^m z}{1+z(F_{m-1}+F_{m+1})+(-1)^m z^2}.$$

Reading off the coefficient of z^n on both sides leads to

$$\sum_{k=0}^{n}(-1)^k F_{mk} = \frac{-F_m+(-1)^n F_{(n+1)m}+(-1)^{n+m}F_{nm}}{1+F_{m-1}+F_{m+1}+(-1)^m}.$$

3. Summing Shifted Lucas Numbers

First, we need the generating function

$$\frac{2-zL_m}{1-zL_m+z^2(-1)^m} = \sum_{k\geq 0}L_{mk}z^k,$$

which holds for $m \geq 0$.

It can again be checked by writing it as

$$2-zL_m = (1-zL_m+z^2(-1)^m)\sum_{k\geq 0}L_{mk}z^k,$$

comparing coefficients and prove that the resulting coefficients are zero for $k \geq 2$, either by the Binet formula or by using classical identities for Lucas numbers.

Furthermore, for $m \geq 1$,

$$\frac{1}{1-z}\frac{2-zL_m}{1-zL_m+z^2(-1)^m}$$
$$= \frac{1-L_m}{1-L_m+(-1)^m}\frac{1}{1-z}+\frac{1}{1-L_m+(-1)^m}\frac{(-1)^m(1+z-zL_m)}{1-zL_m+z^2(-1)^m}.$$

Comparing coefficients of z^n, this leads to

$$\sum_{k=0}^{n}L_{mk} = \frac{1-L_m}{1-L_m+(-1)^m}$$
$$+\frac{(-1)^m}{1-L_m+(-1)^m}[z^n]\frac{1}{1-zL_m+z^2(-1)^m}$$
$$+\frac{(-1)^m(1-L_m)}{1-L_m+(-1)^m}[z^n]\frac{z}{1-zL_m+z^2(-1)^m}$$
$$= \frac{1-L_m}{1-L_m+(-1)^m}+\frac{(-1)^m}{1-L_m+(-1)^m}\frac{F_{(n+1)m}}{F_m}+\frac{(-1)^m(1-L_m)}{1-L_m+(-1)^m}\frac{F_{nm}}{F_m}.$$

We can deal with an alternating version by small modifications:

$$\frac{2+zL_m}{1+zL_m+z^2(-1)^m}=\sum_{k\ge 0}(-1)^k L_{mk}z^k,$$

and

$$\frac{1}{1-z}\frac{2+zL_m}{1+zL_m+z^2(-1)^m}=\frac{2+L_m}{1+L_m+(-1)^m}\frac{1}{1-z}\\
+\frac{2(-1)^m+L_m+2z(-1)^m+zL_m(-1)^m}{1+L_m+(-1)^m}\frac{1}{1+zL_m+z^2(-1)^m}.$$

Reading off coefficients,

$$\sum_{k=0}^{n}(-1)^k L_{mk}=\frac{2+L_m}{1+L_m+(-1)^m}\\
+\frac{2(-1)^m+L_m}{1+L_m+(-1)^m}(-1)^n\frac{F_{(n+1)m}}{F_m}+\frac{2(-1)^m+L_m(-1)^m}{1+L_m+(-1)^m}(-1)^{n-1}\frac{F_{nm}}{F_m}.$$

4. Expanding Powers of Fibonacci and Lucas Numbers

Our goal is here to expand F_n^j in terms of F_{mn}, and likewise for Lucas numbers. To clarify, we start with a list of such expansions:

$$\begin{aligned}
F_n^2&=\frac{2}{5}F_{2(n+1)}-\frac{3}{5}F_{2n}-\frac{2}{5}(-1)^n\\
F_n^3&=\frac{1}{5}F_{3n}-\frac{3}{5}(-1)^nF_n\\
F_n^4&=\frac{2}{75}F_{4(n+1)}-\frac{7}{75}F_{4n}-\frac{8}{25}(-1)^nF_{2(n+1)}+\frac{12}{25}(-1)^nF_{2n}+\frac{6}{25}\\
F_n^5&=\frac{1}{25}F_{5n}-\frac{1}{5}(-1)^nF_{3n}+\frac{2}{5}F_n\\
F_n^6&=\frac{1}{500}F_{6(n+1)}-\frac{9}{500}F_{6n}-\frac{4}{125}(-1)^nF_{4(n+1)}+\frac{14}{125}(-1)^nF_{4n}\\
&\quad+\frac{6}{25}F_{2(n+1)}-\frac{9}{25}F_{2n}-\frac{4}{25}(-1)^n\\
F_n^7&=\frac{1}{125}F_{7n}-\frac{7}{125}(-1)^nF_{5n}+\frac{21}{125}F_{3n}-\frac{7}{25}(-1)^nF_n.
\end{aligned}$$

The formula for F_n^j, in which j is odd, is easier to guess:

$$F_n^j=\frac{1}{5^{(j-1)/2}}\sum_{0\le s<\frac{j}{2}}F_{(j-2s)n}\binom{j}{s}(-1)^{sn}.$$

The instance j being odd is harder, but here is the result:

$$F_n^j = \frac{1}{5^{j/2}} \sum_{1\le s\le j/2} \frac{2}{F_{2s}} (-1)^{(n+1)(\frac{j}{2}+s)} \binom{j}{j/2+s} F_{2s(n+1)}$$
$$- \frac{1}{5^{j/2}} \sum_{1\le s\le j/2} \frac{L_{2s}}{F_{2s}} (-1)^{(n+1)(\frac{j}{2}+s)} \binom{j}{j/2+s} F_{2sn}$$
$$+ \frac{1}{2\cdot 5^{j/2}} \binom{j}{j/2} \big(1-(-1)^n + (-1)^{j/2} + (-1)^{n+j/2}\big).$$

The results for Lucas numbers are somewhat simpler:

$$L_n^2 = L_{2n} + 2(-1)^n$$
$$L_n^3 = L_{3n} + 3(-1)^n L_n$$
$$L_n^4 = L_{4n} + 4(-1)^n L_{2n} + 6$$
$$L_n^5 = L_{5n} + 5(-1)^n L_{3n} + 10L_n$$
$$L_n^6 = L_{6n} + 6(-1)^n L_{4n} + 15L_{2n} + 20(-1)^n$$
$$L_n^7 = L_{7n} + 7(-1)^n L_{5n} + 21L_{3n} + 35(-1)^n L_n$$
$$L_n^8 = L_{8n} + 8(-1)^n L_{6n} + 28L_{4n} + 56(-1)^n L_{2n} + 70.$$

It is not too hard to guess the general formula from that:

$$L_n^j = \sum_{0\le s<\frac{j}{2}} L_{n(j-2s)} \binom{j}{s} (-1)^{sn} + (-1)^n [\![j \text{ even}]\!] \binom{j}{j/2}.$$

Once these formulae have been successfully guessed (the hard part), they can be proved using the Binet formulae and routine manipulations with binomial identities. We leave this for the interested reader.

Summations like

$$\sum_{0\le k\le n} 1 = n+1 \qquad \text{and} \qquad \sum_{0\le k\le n} (-1)^k = \frac{1}{2}\big(1+(-1)^n\big)$$

are also needed but of a trivial nature.

5. Frontczak's Results Revisited

Let us do an example computation:

$$\sum_{k=0}^{n} F_{mk}^3 = \sum_{k=0}^{n} \Big[\frac{1}{5}F_{3mk} - \frac{3}{5}(-1)^k F_{mk}\Big]$$
$$= \frac{1}{5}\,\frac{F_{2m} - F_{(n+1)3m} + (-1)^m F_{3nm}}{1 - F_{3m-1} - F_{3m+1} + (-1)^m} - \frac{3}{5}\,\frac{-F_m + (-1)^n F_{(n+1)m} + (-1)^{n+m} F_{nm}}{1 - F_{m-1} - F_{m+1} + (-1)^m}.$$

The other sums from [2] can be obtained in a similar way.

6. Why Can We Expand Powers of Fibonacci and Lucas Numbers?

The key to the success is the formula

$$x^n + y^n = \sum_{0 \le k \le n/2} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (x+y)^{n-2k} (xy)^k,$$

which is a consequence of classical formulae due to Girard and Waring; see, e.g., [3].

Set

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then, with $x = \alpha^m$ and $y = \beta^m$, the formula becomes

$$L_{mn} = \sum_{0 \le k \le n/2} \frac{n}{n-k} \binom{n-k}{k} L_{(n-2k)m};$$

if one sets $x = \alpha^m$ and $y = (-\beta)^m$, then

$$x + y = \begin{cases} L_m & \text{if } m \text{ is even,} \\ \sqrt{5} F_m & \text{if } m \text{ is odd} \end{cases}$$

and

$$x^n + y^n = \begin{cases} L_{mn} & \text{if } m \text{ or } n \text{ is even,} \\ \sqrt{5} F_{mn} & \text{if } m \text{ and } n \text{ are odd.} \end{cases}$$

Furthermore, $xy = 1$. Since $L_n = F_{n+1} + F_{n-1}$, everything could be expressed in Fibonacci number (alternatively, everything could be expressed in terms of Lucas numbers).

In [4], these formulae were derived from scratch.

So, F_{mn} (resp. L_{mn}) are expressed in terms (linear combinations) of powers of F_m resp. L_m. The formulae of the previous sections are just inverted versions of this, namely, powers of Fibonacci numbers are expressed as linear combinations of shifted Fibonacci numbers.

References

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