Abstract
In this paper, we state and prove some congruence properties for the trinomial coefficients, one of which is similar to Wolstenholme’s theorem.

1. Introduction
In 1819, Babbage [4] showed for any odd prime $p$,
\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}.
\] (1)
In 1862, Wolstenholme [21] proved that the above congruence holds modulo $p^3$ for any prime $p \geq 5$, i.e.,
\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^3},
\] (2)
which is known as Wolstenholme’s theorem. It is well-known that Wolstenholme’s theorem is a fundamental congruence in combinatorial number theory. We refer to [14] for various extensions of Wolstenholme’s theorem.

In the past few years, ($q$-)congruences for sums of binomial coefficients have attracted the attention of many researchers (see, for instance, [2, 3, 6, 7, 8, 9, 10, 11, 19, 20]). In 2011, Sun and Tauraso [20] proved that for any prime $p \geq 5$,
\[
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \binom{p}{3} \pmod{p^2},
\] (3)
\[
\sum_{k=0}^{p-1} \binom{2k}{k} \frac{1}{k+1} \equiv \frac{3}{2} \binom{p}{3} - \frac{1}{2} \pmod{p^2},
\]

where \( \binom{p}{3} \) denotes the Legendre symbol. Note that \( \binom{2k}{k+1} \) is the \( n \)-th Catalan number \( C_n \), which plays an important role in various counting problems. Extensions of (3) and (4) have been established in [3, 10].

In 2018, the first author [2] conjectured two congruences on sums of the super Catalan numbers (named by Gessel [5]):

\[
\sum_{i+j=0}^{p-1} \binom{2i}{i} \binom{2j}{j} \equiv \binom{p}{3} \pmod{p},
\]

\[
\sum_{i+j=0}^{p-1} (3i+3j+1) \binom{2i}{i} \binom{2j}{j} \equiv -7 \binom{p}{3} \pmod{p},
\]

which were confirmed by the second author [11].

In this paper, we will study the congruence properties for the trinomial coefficients. Here we consider the coefficients of the trinomial

\[(1 + x + x^{-1})^n = \sum_{j=-n}^{n} \binom{n}{j} x^j.\]

Two immediate consequences of this definition are

\[\binom{n}{j} = \binom{n}{n-j},\]

and

\[\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j} + \binom{n-1}{j+1}.\]

We have the following multinomial theorem (see [18, page 17]):

\[(x + y + z)^n = \sum_{a+b+c=n} \frac{n!}{a!b!c!} x^a y^b z^c.\]

(5)

Letting \( y = 1 \) and \( z = 1/x \) in (5), we get

\[\binom{n}{j} = \sum_{a+b+c=n} \frac{n!}{a!b!c!} \binom{n}{c} \binom{n-c}{c+j}.\]

(6)

We first prove a congruence for the trinomial coefficients, which is similar to Wolstenholme’s theorem.
**Theorem 1.** For any prime $p \geq 5$, we have
\[
\binom{2p}{p} \equiv 2 + \frac{2p^2}{3} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3},
\]
where $B_n(x)$ is the Bernoulli polynomial of order $n$.

The second result consists of the following two congruences on single sums of trinomial coefficients.

**Theorem 2.** For any prime $p \geq 5$, we have
\[
\sum_{j=0}^{p} \binom{p}{j} \equiv \frac{1}{2} (1 + 3^p) \pmod{p^2},
\]
and
\[
\sum_{j=0}^{p-1} \binom{p-1}{j} \equiv \frac{1}{2} \left( 1 + \left( \frac{p}{3} \right) \right) \pmod{p}.
\]

The third aim of the paper is to establish a congruence on double sums of trinomial coefficients.

**Theorem 3.** For any prime $p \geq 5$ and integer $j$ with $0 < j < p$, we have
\[
\sum_{k=0}^{p-1} \binom{k}{j} \equiv \frac{(-1)^j}{2} + 1 \cdot (-1)^{\frac{p^2-j-1}{2}} \pmod{p},
\]
and
\[
\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \binom{m}{n} \equiv \frac{1}{2} \left( (-1)^{\frac{p^2}{2}} + 1 \right) \pmod{p}.
\]

The rest of this paper is organized as follows. We shall prove Theorems 1, 2 and 3 in Sections 2, 3 and 4, respectively. An open problem on $q$-congruence is proposed in the last section for further research.

### 2. Proof of Theorem 1

By (2) and (6), we have
\[
\binom{2p}{p} = \sum_{k=0}^{p-1} \binom{2p}{k} \binom{2p-k}{k+p} \equiv 2 + \sum_{k=1}^{p-1} \binom{2p}{k} \binom{2p-k}{k+p} \pmod{p^3}.
\]
Let $H_k$ denote the $k$-th harmonic number:

$$H_k = \sum_{j=1}^{k} \frac{1}{j}.$$ 

For any integer $s$ and $0 \leq k \leq p - 1$, we have

$$\binom{sp - 1}{k} \equiv (-1)^k (1 - spH_k) \pmod{p^2}, \quad (13)$$

and

$$\binom{sp + k}{k} \equiv 1 + spH_k \pmod{p^2}. \quad (14)$$

It follows from (1), (13) and (14) that for $1 \leq k \leq \frac{p-1}{2}$,

$$\binom{2p}{k} = \frac{2p}{k} \binom{2p - 1}{k - 1} \equiv \frac{2p(-1)^{k-1}}{k} (1 - 2pH_{k-1}) \pmod{p^3}, \quad (15)$$

and

$$\binom{2p - k}{k + p} = \frac{(2p)}{4} \binom{2k}{k} \frac{(2p-1)}{(2k-1)} \binom{p+k}{k}$$

$$\equiv (-1)^{k-1} \binom{2k}{k} \frac{pH_{2k-1} - 1}{2(1 - 2pH_{k-1})(1 + pH_k)} \pmod{p^2}. \quad (16)$$

Substituting (15) and (16) into (12) gives

$$\left( \binom{2p}{p} \right) \equiv 2 + p \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} \frac{pH_{2k-1} - 1}{k(1 + pH_k)} \pmod{p^3}. \quad (17)$$

Note that

$$\frac{pH_{2k-1} - 1}{1 + pH_k} \equiv -1 + (H_k + H_{2k-1}) \pmod{p^2}. \quad (18)$$

Combining (17) and (18), we arrive at

$$\left( \binom{2p}{p} \right) \equiv 2 - p \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} H_{2k} - \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} \left( H_k + H_{2k-1} \right) \pmod{p^3}$$

$$= 2 + p \left( \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} H_{2k} - \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} H_k \right) + p^2 \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} - \frac{p^2}{2} \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k^2}. \quad (19)$$
Using the following congruence [12, (2.8)]:
\[
p \sum_{k=1}^{p^{-1}} \frac{(2k)}{k} H_{2k} - \sum_{k=1}^{p^{-1}} \frac{(2k)}{k} = \frac{5p}{2} \sum_{k=1}^{p^{-1}} \frac{(2k)}{k^2} - 2p \sum_{k=1}^{p^{-1}} \frac{(2k)}{k} H_k \quad \text{(mod } p^2),
\]
we have
\[
\left( \binom{2p}{p} \right) \equiv 2 + 2p^2 \sum_{k=1}^{p^{-1}} \frac{(2k)}{k^2} - p^2 \sum_{k=1}^{p^{-1}} \frac{(2k)}{k} H_k \quad \text{(mod } p^3). \tag{19}
\]

We have the following two congruences (see [12, (1.1)] and [13, page 156]):
\[
\sum_{k=1}^{p^{-1}} \frac{(2k)}{k} H_k \equiv \frac{1}{3} \left( \frac{p}{3} \right)^2 B_{p-2} \left( \frac{1}{3} \right) \quad \text{(mod } p), \tag{20}
\]
and
\[
\sum_{k=1}^{p^{-1}} \frac{(2k)}{k^2} \equiv \frac{1}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \quad \text{(mod } p). \tag{21}
\]

Finally, combining (19)–(21), we complete the proof of (7).

3. Proof of Theorem 2

Proof of (8). We begin with the following identity, which is A027914 of the Online Encyclopedia of Integer Sequences [17]:
\[
\sum_{j=0}^{n} \binom{n}{j} = \frac{1}{2} \left( 3^n + \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k} \right). \tag{22}
\]

Letting \( n = p \) in the above gives
\[
\sum_{j=0}^{p} \binom{p}{j} = \frac{1}{2} \left( 3^p + \sum_{k=0}^{p^{-1}} \binom{p}{k} \binom{p-k}{k} \right).
\]

Note that for \( 1 \leq k \leq \frac{p-1}{2} \),
\[
\binom{p}{k} \left( \begin{array}{c} p-k \\ k \end{array} \right) = \frac{p(p-1) \cdots (p-k)(p-k+1)(p-k-1) \cdots (p-2k+1)}{k!^2} \\
= -\frac{p}{2k} \binom{2k}{k} \quad \text{(mod } p^2).
\]
Thus,
\[ \sum_{j=0}^{p} \left( \binom{p}{j} \right) \equiv \frac{1}{2} \left( 3^p + 1 - \frac{p-1}{2} \sum_{k=1}^{p-1} \binom{2k}{k} \right) \pmod{p^2}. \]

We have the following congruence [15, (1.6)]:
\[ \sum_{k=1}^{p-1} \frac{(2k)!}{k!} \equiv 0 \pmod{p}. \]

It follows that
\[ \sum_{j=0}^{p} \left( \binom{p}{j} \right) \equiv \frac{1}{2} (3^p + 1) \pmod{p^2}, \]
as desired. \hfill \Box

**Proof of** (9). Letting \( n = p - 1 \) in (22), we obtain
\[ \sum_{j=0}^{p-1} \binom{p-1}{j} = \frac{1}{2} \left( 3^{p-1} + \sum_{k=0}^{p-1} \binom{p-1}{k} \left( p - 1 - k \right) \right). \]

Note that for \( 0 \leq k \leq \frac{p-1}{2} \),
\[ \binom{p-1}{k} \equiv (-1)^k \pmod{p}, \]
and
\[ \binom{p-1 - k}{k} = \frac{(p-1-k)(p-2-k)\cdots(p-2k)}{k!} \equiv (-1)^k \binom{2k}{k} \pmod{p}. \]

By the above two congruences and Fermat’s little theorem, we have
\[ \sum_{j=0}^{p-1} \left( \binom{p-1}{j} \right) \equiv \frac{1}{2} \left( 1 + \sum_{k=0}^{p-1} \binom{2k}{k} \right) \pmod{p}. \] (23)

Then the proof of (9) follows from (3) and (23). \hfill \Box
4. Proof of Theorem 3

Proof of (10). Exchanging the summation order, we get

\[
\sum_{k=0}^{p-1} \binom{k}{j} = \sum_{k=0}^{p-1} \sum_{i=0}^{k} \binom{k}{i} \binom{k-i}{i+j} = \sum_{i=0}^{p-1} \sum_{k=i}^{p-1} \binom{k}{i} \binom{k-i}{i+j}.
\]

Since

\[
\binom{k}{i} \binom{k-i}{i+j} = \binom{2i+j}{i} \binom{k}{2i+j},
\]

we have

\[
\sum_{k=0}^{p-1} \binom{k}{j} = \sum_{i=0}^{p-1} \binom{2i+j}{i} \sum_{k=i}^{p-1} \binom{k}{2i+j} = \sum_{i=0}^{p-1} \binom{2i+j}{i} p \binom{p}{2i+j+1},
\]

where we have utilized the identity (proved by induction):

\[
\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}.
\]

Note that for \(1 \leq k \leq p-1\),

\[
\binom{p}{k} \equiv 0 \pmod{p}.
\]

If \(j\) is odd, then

\[
\sum_{k=0}^{p-1} \binom{k}{j} \equiv 0 \pmod{p}.
\]

If \(j\) is even, then

\[
\sum_{k=0}^{p-1} \binom{k}{j} \equiv \binom{p-1}{\frac{p-j-1}{2}} \equiv (-1)^{\frac{p-j-1}{2}} \pmod{p}.
\]

This completes the proof of (10).
Proof of (11). By (10), we have
\[
\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \binom{m}{n} \equiv \sum_{n=0}^{p-1} \frac{(-1)^n + 1}{2} \cdot \left( -1 \right)^{\frac{p-n-1}{2}} \pmod{p}
\]
\[
= \sum_{n=0}^{p-1} (-1)^{\frac{p-2n-1}{2}}
\]
\[
= \frac{1}{2} \left( (-1)^{\frac{p-1}{2}} + 1 \right),
\]
as claimed. \hfill \Box

Remark. Theorems 2 and 3 can also be established by using the method of the first author and Zeilberger [3].

5. Concluding Remarks

We have three q-analogs corresponding to the trinomial coefficients as given in [16], namely,
\[
T_1(n, j, q) := \sum_{k=0}^{n} q^{k(j+k)} \binom{n}{k} q^{n-k},
\]
\[
T_2(n, j, q) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{2n-2k},
\]
\[
T_3(n, j, q) := \sum_{k=0}^{n} (-q)^k \binom{n}{k} q^{2n-2k},
\]
where the q-binomial coefficients \(\binom{n}{k}_q\) are defined as
\[
\binom{n}{k}_q = \begin{cases} 
(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})/(1-q)(1-q^2)\cdots(1-q^k), & \text{if } 0 \leq k \leq n, \\
0, & \text{otherwise.}
\end{cases}
\]

It is not hard to prove the following q-congruences.

Proposition 1. For any odd prime \(p\) and integer \(1 \leq s \leq 3\), we have
\[
\sum_{j=0}^{p-1} T_s(p, j, q) \equiv 1 \pmod{\lfloor p \rfloor_q},
\]
where the q-integers are given by \([n]_q = (1 - q^n)/(1 - q)\).
The proof of Proposition 1 is trivial and left to the interested reader.


$$\binom{2p - 1}{p - 1}_q \equiv q^{\frac{p(p-1)}{2}} (p)_q^2 \pmod{p},$$

for any odd prime $p$. It is natural to ask whether the congruence (7) possesses a $q$-analog. For convenience sake, let

$$\binom{n}{j}_q = T_1(n, j, q).$$

Numerical calculation suggests the following $q$-congruence, and we propose this conjecture for further research.

**Conjecture 1.** For any prime $p \geq 5$, we have

$$\binom{2p}{p}_q \equiv 2 \left( p + \frac{3}{6} \right) + 2 \pmod{p},$$

where $\lfloor x \rfloor$ denotes the integral part of real $x$.

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**References**


