CONGRUENCE PROPERTIES FOR THE TRINOMIAL COEFFICIENTS

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Abstract

In this paper, we state and prove some congruence properties for the trinomial coefficients, one of which is similar to Wolstenholme's theorem.

1. Introduction

In 1819, Babbage [4] showed for any odd prime p,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}. \tag{1}$$

In 1862, Wolstenholme [21] proved that the above congruence holds modulo p^3 for any prime $p \geq 5$, i.e.,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}, \tag{2}$$

which is known as Wolstenholme's theorem. It is well-known that Wolstenholme's theorem is a fundamental congruence in combinatorial number theory. We refer to [14] for various extensions of Wolstenholme's theorem.

In the past few years, (q-)congruences for sums of binomial coefficients have attracted the attention of many researchers (see, for instance, [2, 3, 6, 7, 8, 9, 10, 11, 19, 20]). In 2011, Sun and Tauraso [20] proved that for any prime $p \geq 5$,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \tag{3}$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{1}{k+1}\equiv\frac{3}{2}\left(\frac{p}{3}\right)-\frac{1}{2}\pmod{p^2},\tag{4}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Note that $\binom{2k}{k}\frac{1}{k+1}$ is the n-th Catalan number C_n, which plays an important role in various counting problems. Extensions of (3) and (4) have been established in [3, 10].

In 2018, the first author [2] conjectured two congruences on sums of the super Catalan numbers (named by Gessel [5]):

$$\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}}\equiv\left(\frac{p}{3}\right)\pmod{p},$$

$$\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}(3i+3j+1)\frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}}\equiv-7\left(\frac{p}{3}\right)\pmod{p},$$

which were confirmed by the second author [11].

In this paper, we will study the congruence properties for the trinomial coefficients. Here we consider the coefficients of the trinomial

$$(1+x+x^{-1})^n=\sum_{j=-n}^{n}\left(\binom{n}{j}\right)x^j.$$

Two immediate consequences of this definition are

$$\left(\binom{n}{j}\right)=\left(\binom{n}{-j}\right),$$

and

$$\left(\binom{n}{j}\right)=\left(\binom{n-1}{j-1}\right)+\left(\binom{n-1}{j}\right)+\left(\binom{n-1}{j+1}\right).$$

We have the following multinomial theorem (see [18, page 17]):

$$(x+y+z)^n=\sum_{a+b+c=n}\frac{n!}{a!b!c!}x^ay^bz^c.\tag{5}$$

Letting $y=1$ and $z=1/x$ in (5), we get

$$\left(\binom{n}{j}\right)=\sum_{\substack{a+b+c=n\\ a-c=j}}\frac{n!}{a!b!c!}=\sum_{c=0}^{n}\binom{n}{c}\binom{n-c}{c+j}.\tag{6}$$

We first prove a congruence for the trinomial coefficients, which is similar to Wolstenholme's theorem.

Theorem 1. *For any prime $p \geq 5$, we have*

$$\left(\binom{2p}{p}\right) \equiv 2 + \frac{2p^2}{3}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}, \tag{7}$$

where $B_n(x)$ is the Bernoulli polynomial of order n.

The second result consists of the following two congruences on single sums of trinomial coefficients.

Theorem 2. *For any prime $p \geq 5$, we have*

$$\sum_{j=0}^{p}\left(\binom{p}{j}\right) \equiv \frac{1}{2}\left(1+3^p\right) \pmod{p^2}, \tag{8}$$

$$\sum_{j=0}^{p-1}\left(\binom{p-1}{j}\right) \equiv \frac{1}{2}\left(1+\left(\frac{p}{3}\right)\right) \pmod{p}. \tag{9}$$

The third aim of the paper is to establish a congruence on double sums of trinomial coefficients.

Theorem 3. *For any prime $p \geq 5$ and integer j with $0 < j < p$, we have*

$$\sum_{k=0}^{p-1}\left(\binom{k}{j}\right) \equiv \frac{(-1)^j+1}{2}\cdot(-1)^{\frac{p-j-1}{2}} \pmod{p}, \tag{10}$$

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left(\binom{m}{n}\right) \equiv \frac{1}{2}\left((-1)^{\frac{p-1}{2}}+1\right) \pmod{p}. \tag{11}$$

The rest of this paper is organized as follows. We shall prove Theorems 1, 2 and 3 in Sections 2, 3 and 4, respectively. An open problem on q-congruence is proposed in the last section for further research.

2. Proof of Theorem 1

By (2) and (6), we have

$$\begin{aligned}\left(\binom{2p}{p}\right) &= \sum_{k=0}^{\frac{p-1}{2}}\binom{2p}{k}\binom{2p-k}{k+p} \\ &\equiv 2 + \sum_{k=1}^{\frac{p-1}{2}}\binom{2p}{k}\binom{2p-k}{k+p} \pmod{p^3}. \end{aligned} \tag{12}$$

Let H_k denote the k-th harmonic number:

$$H_k = \sum_{j=1}^{k} \frac{1}{j}.$$

For any integer s and $0 \le k \le p-1$, we have

$$\binom{sp-1}{k} \equiv (-1)^k \left(1 - spH_k\right) \pmod{p^2}, \tag{13}$$

and

$$\binom{sp+k}{k} \equiv 1 + spH_k \pmod{p^2}. \tag{14}$$

It follows from (1), (13) and (14) that for $1 \le k \le \frac{p-1}{2}$,

$$\binom{2p}{k} = \frac{2p}{k}\binom{2p-1}{k-1} \equiv \frac{2p(-1)^{k-1}}{k}\left(1 - 2pH_{k-1}\right) \pmod{p^3}, \tag{15}$$

and

$$\begin{aligned}\binom{2p-k}{k+p} &= \frac{\binom{2p}{p}\binom{2k}{k}}{4} \cdot \frac{\binom{p-1}{2k-1}}{\binom{2p-1}{k-1}\binom{p+k}{k}} \\ &\equiv (-1)^{k-1}\binom{2k}{k} \cdot \frac{pH_{2k-1}-1}{2(1-2pH_{k-1})(1+pH_k)} \pmod{p^2}.\end{aligned} \tag{16}$$

Substituting (15) and (16) into (12) gives

$$\left(\!\!\binom{2p}{p}\!\!\right) \equiv 2 + p\sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}(pH_{2k-1}-1)}{k(1+pH_k)} \pmod{p^3}. \tag{17}$$

Note that

$$\frac{pH_{2k-1}-1}{1+pH_k} \equiv -1 + (H_k + H_{2k-1})p \pmod{p^2}. \tag{18}$$

Combining (17) and (18), we arrive at

$$\begin{aligned}\left(\!\!\binom{2p}{p}\!\!\right) &\equiv 2 - p\sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{k} + p^2\sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{k}(H_k + H_{2k-1}) \pmod{p^3} \\ &= 2 + p\left(p\sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{k}H_{2k} - \sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{k}\right) + p^2\sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{k}H_k - \frac{p^2}{2}\sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{k^2}.\end{aligned}$$

Using the following congruence [12, (2.8)]:

$$p\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k}H_{2k}-\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k}\equiv\frac{5p}{2}\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k^2}-2p\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k}H_k\pmod{p^2},$$

we have

$$\left(\!\!\binom{2p}{p}\!\!\right)\equiv 2+2p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k^2}-p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k}H_k\pmod{p^3}.\tag{19}$$

We have the following two congruences (see [12, (1.1)] and [13, page 156]):

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k}H_k\equiv\frac{1}{3}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\pmod{p},\tag{20}$$

and

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k^2}\equiv\frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\pmod{p}.\tag{21}$$

Finally, combining (19)–(21), we complete the proof of (7).

3. Proof of Theorem 2

Proof of (8). We begin with the following identity, which is A027914 of the Online Encyclopedia of Integer Sequences [17]:

$$\sum_{j=0}^{n}\left(\!\!\binom{n}{j}\!\!\right)=\frac{1}{2}\left(3^n+\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}\right).\tag{22}$$

Letting $n=p$ in the above gives

$$\sum_{j=0}^{p}\left(\!\!\binom{p}{j}\!\!\right)=\frac{1}{2}\left(3^p+\sum_{k=0}^{\frac{p-1}{2}}\binom{p}{k}\binom{p-k}{k}\right).$$

Note that for $1\le k\le\frac{p-1}{2}$,

$$\begin{aligned}\binom{p}{k}\binom{p-k}{k}&=\frac{p(p-1)\cdots(p-k+1)(p-k)(p-k-1)\cdots(p-2k+1)}{k!^2}\\&\equiv-\frac{p}{2k}\binom{2k}{k}\pmod{p^2}.\end{aligned}$$

Thus,

$$\sum_{j=0}^{p}\left(\!\!\binom{p}{j}\!\!\right) \equiv \frac{1}{2}\left(3^p + 1 - \frac{p}{2}\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k}\right) \pmod{p^2}.$$

We have the following congruence [15, (1.6)]:

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{k} \equiv 0 \pmod{p}.$$

It follows that

$$\sum_{j=0}^{p}\left(\!\!\binom{p}{j}\!\!\right) \equiv \frac{1}{2}\left(3^p + 1\right) \pmod{p^2},$$

as desired. □

Proof of (9). Letting $n = p-1$ in (22), we obtain

$$\sum_{j=0}^{p-1}\left(\!\!\binom{p-1}{j}\!\!\right) = \frac{1}{2}\left(3^{p-1} + \sum_{k=0}^{\frac{p-1}{2}}\binom{p-1}{k}\binom{p-1-k}{k}\right).$$

Note that for $0 \le k \le \frac{p-1}{2}$,

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p},$$

and

$$\binom{p-1-k}{k} = \frac{(p-1-k)(p-2-k)\cdots(p-2k)}{k!} \equiv (-1)^k\binom{2k}{k} \pmod{p}.$$

By the above two congruences and Fermat's little theorem, we have

$$\sum_{j=0}^{p-1}\left(\!\!\binom{p-1}{j}\!\!\right) \equiv \frac{1}{2}\left(1 + \sum_{k=0}^{p-1}\binom{2k}{k}\right) \pmod{p}. \tag{23}$$

Then the proof of (9) follows from (3) and (23). □

4. Proof of Theorem 3

Proof of (10). Exchanging the summation order, we get

$$\begin{aligned}\sum_{k=0}^{p-1}\left(\!\binom{k}{j}\!\right) &= \sum_{k=0}^{p-1}\sum_{i=0}^{k}\binom{k}{i}\binom{k-i}{i+j}\\ &= \sum_{i=0}^{p-1}\sum_{k=i}^{p-1}\binom{k}{i}\binom{k-i}{i+j}.\end{aligned}$$

Since

$$\binom{k}{i}\binom{k-i}{i+j} = \binom{2i+j}{i}\binom{k}{2i+j},$$

we have

$$\begin{aligned}\sum_{k=0}^{p-1}\left(\!\binom{k}{j}\!\right) &= \sum_{i=0}^{p-1}\binom{2i+j}{i}\sum_{k=i}^{p-1}\binom{k}{2i+j}\\ &= \sum_{i=0}^{p-1}\binom{2i+j}{i}\binom{p}{2i+j+1},\end{aligned}$$

where we have utilized the identity (proved by induction):

$$\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}.$$

Note that for $1 \le k \le p-1$,

$$\binom{p}{k} \equiv 0 \pmod{p}.$$

If j is odd, then

$$\sum_{k=0}^{p-1}\left(\!\binom{k}{j}\!\right) \equiv 0 \pmod{p}.$$

If is j is even, then

$$\sum_{k=0}^{p-1}\left(\!\binom{k}{j}\!\right) \equiv \binom{p-1}{\frac{p-j-1}{2}} \equiv (-1)^{\frac{p-j-1}{2}} \pmod{p}.$$

This completes the proof of (10). □

Proof of (11). By (10), we have

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left(\binom{m}{n}\right) \equiv \sum_{n=0}^{p-1}\frac{(-1)^n+1}{2}\cdot(-1)^{\frac{p-n-1}{2}} \pmod p$$
$$= \sum_{n=0}^{\frac{p-1}{2}}(-1)^{\frac{p-2n-1}{2}}$$
$$= \frac{1}{2}\left((-1)^{\frac{p-1}{2}}+1\right),$$

as claimed. □

Remark. Theorems 2 and 3 can also be established by using the method of the first author and Zeilberger [3].

5. Concluding Remarks

We have three q-analogs corresponding to the trinomial coefficients as given in [16], namely,

$$T_1(n,j,q) := \sum_{k=0}^{n} q^{k(k+j)}\binom{n}{k}_q\binom{n-k}{k+j}_q,$$
$$T_2(n,j,q) := \sum_{k=0}^{n}(-1)^k\binom{n}{k}_{q^2}\binom{2n-2k}{n-k-j}_q,$$
$$T_3(n,j,q) := \sum_{k=0}^{n}(-q)^k\binom{n}{k}_{q^2}\binom{2n-2k}{n-k-j}_q,$$

where the q-binomial coefficients $\binom{n}{k}_q$ are defined as

$$\binom{n}{k}_q = \begin{cases} \dfrac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

It is not hard to prove the following q-congruences.

Proposition 1. *For any odd prime p and integer $1 \le s \le 3$, we have*

$$\sum_{j=0}^{p-1} T_s(p,j,q) \equiv 1 \pmod{[p]_q},$$

where the q-integers are given by $[n]_q = (1-q^n)/(1-q)$.

The proof of Proposition 1 is trivial and left to the interested reader.

In 1999, Andrews [1] established an interesting q-analog of Babbage's congruence (1):

$$\binom{2p-1}{p-1}_q \equiv q^{\frac{p(p-1)}{2}} \pmod{[p]_q^2},$$

for any odd prime p. It is natural to ask whether the congruence (7) possesses a q-analog. For convenience sake, let

$$\left(\binom{n}{j}\right)_q = T_1(n,j,q).$$

Numerical calculation suggests the following q-congruence, and we propose this conjecture for further research.

Conjecture 1. For any prime $p \geq 5$, we have

$$\left(\binom{2p}{p}\right)_q \equiv \left(2\left\lfloor\frac{p+3}{6}\right\rfloor + p\right)(q^p-1)+2 \pmod{[p]_q^2}, \tag{24}$$

where $\lfloor x \rfloor$ denotes the integral part of real x.

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