DISTRIBUTION LAWS OF FRIABLE DIVISORS

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Abstract

A classical result due to Deshouillers, Dress and Tenenbaum asserts that, on average, the distribution of the divisors of the integers follows the arcsine law. In this paper, we investigate the distribution of friable divisors of the integers, that is, those divisors which are free of large prime factors. We show that, on average, these divisors are distributed according to a probability law that we will describe.

1. Introduction

Let $n \geqslant 1$ be an integer. We denote by $P(n)$ the largest prime divisor of $n \geqslant 2$ and we set $P(1) = 1$. Let $y \in]1, +\infty[$ be a real number. Consider the set of y-friable divisors of n, that is, those divisors of n which are free of prime factors exceeding y:

$$\mathcal{D}_{n,y} := \{d|n : P(d) \leqslant y\},$$

and denote by $\tau(n, y)$ its cardinality. For each integer $n \geqslant 1$ and for each real number $y > 1$, we define the random variable

$$X_{n,y} : \mathcal{D}_{n,y} \longrightarrow [0, 1],$$

which takes the values $\log d/ \log n$ with uniform probability $1/\tau(n, y)$, and for $v \in [0, 1]$, its distribution function

$$F_{n,y}(v) := \mathbb{P}(X_{n,y} \leqslant v) = \frac{1}{\tau(n, y)} \sum_{d|n, d\leqslant n^v, P(d)\leqslant y} 1.$$

It is easy to see that the sequence $(F_{n,y})_{n\geqslant 1}$ does not converge pointwise in $[0,1]$. We consider its mean in the interval $[1,x]$, $x\geqslant 2$,

$$\frac{1}{x}\sum_{n\leqslant x}F_{n,y}(v)=\frac{1}{x}\sum_{n\leqslant x}\mathbb{P}(X_{n,y}\leqslant v)\cdot \tag{1}$$

The aim of this paper is to show that this mean converges to a distribution function which will be described. Deshouillers, Dress and Tenenbaum [4] studied the analogue of this question by considering all divisors of n, that is, without constraint on the size of their prime factors. Denote by X_n the analogue of the random variable $X_{n,y}$ defined on the set of all divisors of n, they showed that

$$\frac{1}{x}\sum_{n\leqslant x}\mathbb{P}(X_n\leqslant v)=\frac{2}{\pi}\,\arcsin(\sqrt{v})+O\left(\frac{1}{\sqrt{\log x}}\right) \tag{2}$$

uniformly for $x\geqslant 2$ and $v\in[0,1]$. This arcsin law is a Dirichlet law in one dimension with parameter equal to $(1/2,1/2)$. Studying the distribution law of couples of divisors, the authors of the present paper showed that they are distributed according to a two-dimensional Dirichet Law [8] . The method works in higher dimensions but becomes very technical. De La Bretèche and Tenenbaum [2] also studied the distribution law of couples of divisors by using a probabilistic model that preserves the equiprobability of the first marginal law and allowed them to deduce the second marginal law. They also obtained a Dirichlet law.
Recently, Basquin [1] studied the question of the distribution law of divisors of friable integers n. This question is naturally connected to the de Bruijn function:

$$\Psi(x,y):=\sharp\{n\leqslant x: P(n)\leqslant y\}\cdot$$

The asymptotic behavior of de Bruijn's function is known in a large range of the xy-plane. It is connected to Dickman's function ρ, which is the continuous solution in $]0,+\infty[$ to the differential-difference equation with initial conditions:

$$\begin{cases} u\rho'(w)+\rho(w-1)=0, \quad (w>1)\\ \rho(w)=1, \qquad\qquad (0\leqslant w\leqslant 1)\\ \rho(w)=0, \quad (w<0),\end{cases}$$

for which the asymptotic behavior is well-known. For example, we have [7]

$$\log\rho(w)=-(1+0(1))w\log w, \quad (w\to+\infty).$$

Before quoting Basquin's result and formulate the behavior of $\Psi(x,y)$, let us introduce some notations that will be maintained throughout the rest of this paper. For $1<y\leqslant x$, we set

$$u:=\frac{\log x}{\log y}$$

and we denote by (H_ϵ) the subset of $\mathbb{R}^2$ defined by the conditions

$$x \geqslant x_0(\epsilon), \quad \exp\left((\log\log x)^{\frac{5}{3}+\epsilon}\right) \leqslant y \leqslant x,$$

where $x_0(\epsilon) > 0$ is a sufficiently large constant depending on $\epsilon > 0$. Here, it is sufficient to quote the following asymptotic formula for $\Psi(x,y)$ due to Hildebrand [6] and valid in the range (H_ϵ)

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right).$$

Let us introduce the functions ρ_k for $k \in]0,+\infty[$. Each function ρ_k is the continuous solution to the differential-difference equation with initial conditions:

$$\begin{cases} w\rho_k'(w) + (1-k)\rho_k(w) + k\rho_k(w-1) = 0, & (w > 1) \\ \rho_k(w) = \frac{1}{\Gamma(k)}\, w^{k-1}, & (0 < w \leqslant 1) \\ \rho_k(w) = 0, & (w \leqslant 0). \end{cases}$$

In particular, we have $\rho_1 = \rho$. The function ρ_k is the k-th fractional convolution power of ρ – see Hensley's work [5]. Its asymptotic behavior is well-known – see in particular Smida's papers [9] and [10], where the properties of this function, as well as its connection to the asymptotic behavior of Dickman's function ρ, are given. In particular, we have the formula [9]:

$$\rho_k(u) = k^{u(1+O(\frac{1}{\log u}))}\rho(u), \quad (u \to +\infty).$$

Basquin showed that

$$\frac{1}{\Psi(x,y)} \sum_{\substack{n \leqslant x \\ P(n) \leqslant y}} \mathbb{P}(X_n \leqslant v) = \frac{\int_0^{uv} \rho_{\frac{1}{2}}(s)\rho_{\frac{1}{2}}(u-s)ds}{\rho(u)} + O\left(\frac{\log(u+1)}{\log y} + \frac{1}{\sqrt{\log y}}\right),$$

uniformly for $v \in [0,1]$ and $(x,y) \in (H_\epsilon)$, and he deduced that as $u \longrightarrow +\infty$, the distribution function converges to the normal distribution. More precisely, he showed that

$$\frac{1}{\rho(u)} \int_0^{uv} \rho_{\frac{1}{2}}(s)\rho_{\frac{1}{2}}(u-s)\, ds = \Phi\left(u\sqrt{2\xi'(u)}(v - \frac{1}{2})\right) + O\left(\frac{1}{u}\right),$$

where $\xi'(u) \sim 1/u$ as $u \to +\infty$, and

$$\Phi(w) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{w} e^{-t^2}\, dt$$

is the normal distribution function.

2. Statements of the Results

To state our results, let us introduce the Buchstab function ω. This function discovered by Bushstab comes from the study of the uncanceled elements in the sieve of Eratosthenes – see de Bruijn's beautiful article [3]. It is the unique continuous solution for $v > 1$ to the differential-difference equation, with initial conditions:

$$\begin{cases} v\omega'(v) + \omega(v) - \omega(v-1) = 0, \quad (v > 2) \\ \omega(v) = \frac{1}{v}, \quad (1 \leqslant v \leqslant 2) \\ \omega(v) = 0, \quad v < 1. \end{cases}$$

Its asymptotic behavior is known – see de Bruijn [3] and Tenenbaum's book [11, chap. III.6]. In particular for $v \geqslant 1$, we have

$$\omega(v) = e^{-\gamma} + O\left(\rho(v) e^{\frac{-c\,v}{\log^2(v+2)}}\right), \tag{3}$$

where γ is the Euler constant and c is a positive constant. In the first theorem below, we show the convergence of the mean of distribution functions (1) to a distribution function. In the second one, we describe the limit law as $u \longrightarrow +\infty$ and in the third one we give, as an example, expressions of the limit law for $1 < u \leqslant 2$.

Theorem 2.1. *Uniformly for $v \in [0,1]$ and (x, y) in (H_ϵ), we have*

$$\begin{aligned} \tfrac{1}{x}\textstyle\sum_{n \leqslant x} \mathbb{P}(X_{n,y} \leqslant v) \quad &= \quad \textstyle\int_0^{uv} \left(\int_0^{u-s-1} \rho_{\frac{1}{2}}(z)\omega(u-s-z)\,dz\right)\rho_{\frac{1}{2}}(s)\,ds \\ &+ \quad \textstyle\int_0^{uv} \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds \\ &+ \quad O\left(\tfrac{\log(u+1)}{\log y} + \tfrac{1}{\sqrt{\log y}}\right)\cdot \end{aligned}$$

We notice that for $y = x$, that is to say $u = 1$, the formula of Theorem 2.1 is reduced to formula (2) obtained in [4]. Indeed, the first integral vanishes because $\rho_{\frac{1}{2}}(z) = 0$ for $z \leqslant 0$ and

$$\int_0^v \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(1-s)\,ds = \frac{2}{\pi}\int_0^{\sqrt{v}} \frac{dt}{\sqrt{1-t^2}} = \frac{2}{\pi}\arcsin(\sqrt{v}).$$

Let us denote

$$F(u,v) = \int_0^{uv}\left(\int_0^{u-s-1} \rho_{\frac{1}{2}}(z)\omega(u-s-z)\,dz\right)\rho_{\frac{1}{2}}(s)\,ds + \int_0^{uv} \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds.$$

We then have

Theorem 2.2. *For $v \in [0,1]$ and as $u \to +\infty$, we uniformly have*

$$F(u,v) = \frac{1}{\sqrt{e^\gamma}}\int_0^{uv} \rho_{\frac{1}{2}}(s)ds + O\left(\rho_2(u)\right),$$

where γ is the Euler constant.

Theorem 2.3. *1. For $v \in [0, \frac{u-1}{u}]$ and $1 < u \leqslant 2$, we have*

$$F(u,v) = \frac{2}{\pi}\arcsin(\sqrt{v}) + \frac{1}{\pi}\log(u(1-v))\arcsin\left(\sqrt{\frac{uv}{u-1}}\right) - \frac{1}{2}\log(1-v).$$

2. For $v \in [\frac{u-1}{u}, \frac{1}{u}]$ and $1 < u \leqslant 2$, we have

$$F(u,v) = \frac{2}{\pi}\arcsin(\sqrt{v}) + \frac{1}{2}\log u.$$

3. For $v \in [\frac{1}{u}, 1]$ and $1 < u \leqslant 2$, we have

$$F(u,v) = \frac{2}{\pi}\arcsin(\sqrt{v}) + \frac{1}{\pi}\log(uv)\arcsin\left(\sqrt{\frac{u(1-v)}{u-1}}\right) - \frac{1}{2}\log(v)\cdot$$

Let us set

$$\begin{aligned} S(x,y,v) &:= \textstyle\sum_{n\leqslant x}\mathbb{P}(X_{n,y}\leqslant v) = \sum_{n\leqslant x}\frac{1}{\tau(n,y)}\sum_{d\mid n, d\leqslant n^v, P(d)\leqslant y} 1 \\ &= S_1(x,y,v) - S_2(x,y,v), \end{aligned}$$

with

$$S_1(x,y,v) := \sum_{n\leqslant x}\frac{1}{\tau(n,y)}\sum_{\substack{d\mid n, d\leqslant x^v\\ P(d)\leqslant y}} 1 = \sum_{d\leqslant x^v, P(d)\leqslant y}\ \sum_{m\leqslant x/d}\frac{1}{\tau(dm,y)}$$

and

$$S_2(x,y,v) := \sum_{n\leqslant x}\frac{1}{\tau(n,y)}\sum_{\substack{d\mid n, n^v<d\leqslant x^v\\ P(d)\leqslant y}} 1.$$

We will show that the main contribution to Theorem 2.1 comes from the estimation of $S_1(x,y,v)$. The proof rests on the estimation of

$$\sum_{n\leqslant x}\frac{1}{\tau(dn,y)}$$

for (x,y) in (H_ϵ) and $d \geqslant 1$, y-friable.

3. Preparatory Lemmas and Proof of Theorem 2.1

3.1. Lemmas

Let us introduce some notations which will be used in the sequel. For each fixed integer $d \geqslant 1$, we define the multiplicative function

$$\gamma_d(n) = \frac{\tau(d)}{\tau(dn)},$$

where τ is the divisor function. For each prime number p, we denote by $v_p(d)$ the p-adic valuation of d. We have

$$\gamma_d(p^\alpha) = \frac{v_p(d)+1}{v_p(d)+\alpha+1}\cdot$$

We consider the Dirichlet series of $\gamma_d(n)$

$$F_d(s) := \sum_{n\geqslant 1} \frac{\gamma_d(n)}{n^s}, \quad (\Re(s) > 1).$$

We note that $F_d(s) = \zeta^{\frac{1}{2}}(s) G_d(s)$ in the half-plane $\Re(s) > 1$, where ζ is the Riemann zeta function and

$$\begin{aligned} G_d(s) &:= \textstyle\prod_p \left(1-\frac{1}{p^s}\right)^{\frac{1}{2}} \left(\sum_{\alpha\geqslant 0} \frac{v_p(d)+1}{(v_p(d)+\alpha+1)p^{\alpha s}}\right) \\ &= B(s)K_d(s), \end{aligned}$$

with

$$B(s) := \prod_p \left(1-\frac{1}{p^s}\right)^{\frac{1}{2}} \left(1+\sum_{\alpha\geqslant 1} \frac{1}{(\alpha+1)p^{\alpha s}}\right)$$

and

$$K_d(s) := \prod_{p^\beta \| d} \left(1+\sum_{\alpha\geqslant 1} \frac{\beta+1}{(\beta+\alpha+1)p^{\alpha s}}\right) \left(1+\sum_{\alpha\geqslant 1} \frac{1}{(\alpha+1)p^{\alpha s}}\right)^{-1}.$$

For each fixed integer $d \geqslant 1$, we define Dirichlet series

$$K_d(s) = \sum_{n\geqslant 1} \frac{\delta_d(n)}{n^s}; \quad B(s) = \sum_{n\geqslant 1} \frac{b(n)}{n^s}; \quad G_d(s) = \sum_{n\geqslant 1} \frac{h_d(n)}{n^s}\cdot$$

Then we obtain

$$F_d(s) = \sum_{n\geqslant 1} \frac{\gamma_d(n)}{n^s} = \zeta^{\frac{1}{2}}(s) \sum_{n\geqslant 1} \frac{h_d(n)}{n^s} \tag{4}$$

and

$$\sum_{n\geqslant 1} \frac{h_d(n)}{n^s} = \left(\sum_{n\geqslant 1} \frac{b(n)}{n^s}\right) \left(\sum_{n\geqslant 1} \frac{\delta_d(n)}{n^s}\right)\cdot \tag{5}$$

Lemma 3.1. *Let $d \geqslant 1$ be a fixed integer. There exists a real number $0 < \eta < 1/3$ such that the series*

$$\sum_{n \geqslant 1} \frac{h_d(n)}{n^s}$$

is absolutely convergent in the half-plane $\Re(s) = \sigma \geqslant 1 - \eta$ and for $d \geqslant 1$ we uniformly have

$$\sum_{n \geqslant 1} \frac{h_d(n)}{n^\sigma} \ll_\eta \prod_{p|d} (1 + \frac{2}{p^\sigma}).$$

The result of this lemma can be deduced from a general study developed in [4]. Indeed, by Lemma 1 of [4], the series

$$\sum_{n \geqslant 1} \frac{\delta_d(n)}{n^s} = K_d(s)$$

is absolutely convergent in the half-plane $\Re(s) = \sigma \geqslant 1 - \eta$, and satisfies

$$\sum_{n \geqslant 1} \frac{|\delta_d(n)|}{n^\sigma} \ll_\eta \prod_{p|d} (1 + \frac{2}{p^\sigma}). \tag{6}$$

Lemma 2 of [4] applies to the series

$$\sum_{n \geqslant 1} \frac{b(n)}{n^s} = B(s),$$

with the exponent $\alpha = 1/2$ and $\psi(n) = 1/\tau(n)$. We obtain

$$\sum_{n \geqslant 1} \frac{|b(n)|}{n^\sigma} \leqslant \prod_{p} (1 + O(\frac{1}{p^{2-2\eta}})) \ll_\eta 1 \tag{7}$$

in the half-plane $\Re(s) = \sigma \geqslant 1 - \eta$. Lemma 3.1 follows from (5), (6) and (7). $\square$

We set

$$M_{\eta,d} := \prod_{p|d} (1 + \frac{2}{p^{1-\eta}}).$$

From the Equation (4) we define a mutiplicative function h_d by the convolution identity $\gamma_d = \tau_{\frac{1}{2}} * h_d$. By Lemma 3.1, its Dirichlet series satisfies the conditions (1.18) of Theorem 3 of [10], since we have

$$\sum_{\substack{n>t \\ P(d) \leqslant y}} \frac{|h_d(n)|}{n} \leqslant \frac{1}{t^{\eta/2}} \sum_{\substack{n>t \\ P(d) \leqslant y}} \frac{|h_d(n)|}{n^{1-(\eta/2)}} \leqslant \frac{1}{t^{\eta/2}} \sum_{n \geqslant 1} \frac{|h_d(n)|}{n^{1-(\eta/2)}} \ll_\eta \frac{M_{\eta,d}}{t^{\eta/2}}.$$

The proof of Theorem 3 of [10] works and we obtain the first result of Lemma 3.2 below. The second result is a consequence of Theorem T of [4] applied to the function $1/\tau(n)$ and the fact that for $0 < u \leqslant 1$

$$\rho_{\frac{1}{2}}(u) = \frac{1}{\sqrt{\pi}\sqrt{u}} = \frac{\sqrt{\log y}}{\sqrt{\pi}\,\sqrt{\log x}}. \tag{8}$$

Lemma 3.2. *1. let $\eta \in]0, \frac{1}{2}[$ and let $\epsilon > 0$ be fixed . For $d \geqslant 1$ and (x, y) in (H_ϵ), we uniformly have*

$$\sum_{\substack{n \leqslant x \\ P(n) \leqslant y}} \gamma_d(n) = \frac{x}{\sqrt{\log y}}\, \rho_{\frac{1}{2}}(u) \left(G_d(1) + O\left(M_{\eta,d} \left(\frac{\log(u+1)}{\log y} + \frac{1}{\sqrt{\log y}} \right) \right) \right).$$

2. For $1 < x \leqslant y$ and $d \geqslant 1$, we uniformly have

$$\sum_{\substack{n \leqslant x \\ P(n) \leqslant y}} \gamma_d(n) = \frac{x}{\sqrt{\log y}}\, \rho_{\frac{1}{2}}(u) \left(G_d(1) + O\left(\frac{M_{\eta,d}}{\log x} \right) \right).$$

Basquin [2] obtained the first result of this lemma by using another convolution identity and by applying a general result of Tenenbaum and Wu [12].

We set the multiplicative function

$$g(d) := \frac{K_d(1)}{\tau(d)} = \prod_{p^\beta \| d} \left(\sum_{\alpha \geqslant 0} \frac{1}{(\beta + \alpha + 1)p^\alpha} \right) \left(\sum_{\alpha \geqslant 0} \frac{1}{(\alpha + 1)p^\alpha} \right)^{-1}.$$

For $\Re(s) > 1$, we have

$$\sum_{n \geqslant 1} \frac{g(n)}{n^s} = \zeta^{\frac{1}{2}}(s) \sum_{n \geqslant 1} \frac{\beta(n)}{n^s},$$

where β is a multiplicative function satisfying $g = \tau_{\frac{1}{2}} * \beta$. We have

$$H(s) := \sum_{n \geqslant 1} \frac{\beta(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s} \right)^{\frac{1}{2}} \left(\sum_{\alpha \geqslant 0} \frac{g(p^\alpha)}{p^{\alpha s}} \right).$$

For later use, note that

$$G_d(1) = K_d(1)B(1) = \tau(d)g(d)B(1). \tag{9}$$

Lemma 2 of [4] applies to the series $H(s)$ with exponent $\alpha = 1/2$ and the function $\psi(n) = g(n)$. It follows that the series $H(s)$ is absolutely convegent in the half-plane $\Re(s) = \sigma \geqslant 1 - \eta$ and we have

$$\sum_{n \geqslant 1} \frac{|\beta(n)|}{n^\sigma} \ll_\eta 1.$$

The conditions (1.18) of application of Theorem 3 of [10] are satisfied and we obtain the first result of the following Lemma 3.3. The second result is an immediate consequence of lemma 3 of [4] and the relation (8) above.

Lemma 3.3. *1. Uniformly in the range* (H_ϵ)*, we have*

$$\sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} g(n) = H(1)\,\frac{x}{\sqrt{\log y}}\,\rho_{\frac12}(u)\left(1+O\left(\frac{\log(u+1)}{\log y}+\frac{1}{\sqrt{\log y}}\right)\right)\cdot$$

2. For $1<x\leqslant y$*, we have*

$$\sum_{n\leqslant x} g(n) = H(1)\,\frac{x}{\sqrt{\log y}}\,\rho_{\frac12}(u)\left(1+O\left(\frac{1}{\log x}\right)\right)\cdot$$

Remark 3.1. We set

$$N(d) := \frac{M_{\eta,d}}{\tau(d)}\cdot$$

The function N is multiplicative, positive, satisfies

$$N(p)=1+O(\frac{1}{\sqrt{p}}),\ N(p^\alpha)\ll 1.$$

So, from [1, Lemma 3.1], a partial summation yields

$$\sum_{\substack{d\leqslant x\\ P(d)\leqslant y}} \frac{N(d)}{d} \ll_\eta \sqrt{\log y}$$

uniformly in (H_ϵ).

Lemma 3.4. *We have* $B(1)H(1)=1$.

Proof. We have

$$B(1)=\prod_p\left(1-\frac1p\right)^{\frac12}\left(\sum_{j\geqslant 0}\frac{1}{(j+1)p^j}\right),\quad H(1)=\prod_p\left(1-\frac1p\right)^{\frac12}\left(\sum_{\alpha\geqslant 0}\frac{g(p^\alpha)}{p^\alpha}\right)$$

and

$$\sum_{\alpha\geqslant 0}\frac{g(p^\alpha)}{p^\alpha}=\sum_{\alpha\geqslant 0}\left(\sum_{j\geqslant 0}\frac{1}{(\alpha+j+1)p^{\alpha+j}}\right)\left(\sum_{j\geqslant 0}\frac{1}{(j+1)p^j}\right)^{-1}.$$

Therfore

$$B(1)H(1)=\prod_p\left(1-\frac1p\right)\sum_{\alpha\geqslant 0}\left(\sum_{j\geqslant 0}\frac{1}{(\alpha+j+1)p^{j+\alpha}}\right)\cdot$$

By noticing that for $0<|x|<1$,

$$\frac{x^{\alpha+j}}{\alpha+j+1}=\frac{1}{x}\int_0^x t^{\alpha+j}dt,$$

we get

$$\sum_{\alpha\geqslant 0}\left(\sum_{j\geqslant 0}\frac{x^{\alpha+j}}{\alpha+j+1}\right)=\frac{1}{x}\int_0^x\frac{dt}{(1-t)^2}=\frac{1}{1-x},$$

and finally $B(1)H(1)=1$. □

Let us denote by $P_-(n)$ the smallest prime factor of $n>1$ and set $P_-(1)=+\infty$. The function

$$\Phi(x,y):=\sum_{\substack{n\leqslant x\\ P_-(n)>y}}1$$

has been studied by de Bruijn [3]. By [11, III.6. Cor 6.14] and Mertens' Formula we readily obtain the following lemma which is sufficient for our purpose.

Lemma 3.5. *In the range (H_ϵ) we uniformly have*

$$\Phi(x,y)=\frac{x\,\omega(u)}{\log y}-\frac{y}{\log y}+O\left(\frac{x}{(\log y)^2}\right).$$

Lemma 3.6. *Uniformly for each integer $d\geqslant 1$ such that $P(d)\leqslant y$ and (x,y) in (H_ϵ) we have*

$$\begin{aligned}\textstyle\sum_{n\leqslant x}\frac{1}{\tau(dn,y)} &= \tfrac{x}{\sqrt{\log y}}\tfrac{G_d(1)}{\tau(d)}\left(\textstyle\int_0^{u-1}\rho_{\frac12}(z)\omega(u-z)\,dz+\rho_{\frac12}(u)\right)\\ &+ O\left(\tfrac{M_{\eta,d}}{\tau(d)}\left(\tfrac{x\,\log(u+1)}{(\log y)^{\frac32}}+\tfrac{x}{\log y}\right)\right).\end{aligned}$$

Proof. We write $n=ab$ with $P(a)\leqslant y$ and $P_-(b)>y$. Then for d y-friable, we have

$$\tau(dn,y)=\tau(dab,y)=\tau(da),$$

so

$$T_d(x,y):=\sum_{n\leqslant x}\frac{1}{\tau(dn,y)}=\sum_{a\leqslant x,P(a)\leqslant y}\frac{1}{\tau(da)}\,\Phi(\frac{x}{a},y).$$

First, we consider the range

$$\exp\left((\log\log x)^{\frac53+\epsilon}\right)\leqslant y\leqslant\frac{x}{a}\,,\quad x\geqslant x_0(\epsilon).$$

Write

$$\sum_{n\leqslant x}\frac{1}{\tau(dn,y)}=\sum_{\substack{a\leqslant x/y\\ P(a)\leqslant y}}\frac{1}{\tau(da)}\,\Phi(\frac{x}{a},y)+\sum_{\substack{x/y<a\leqslant x\\ P(a)\leqslant y}}\frac{1}{\tau(da)}:=T_d(x,y)+\overline{T}_d(x,y).$$

Lemma 3.2 gives

$$\overline{T}_d(x,y) = \frac{G_1(d)}{\tau(d)}\, \frac{x}{\sqrt{\log y}}\, \rho_{\frac{1}{2}}(u) + O\left(\frac{M_{\eta,d}}{\tau(d)}\left(\frac{x\,\log(u+1)}{(\log y)^{\frac{3}{2}}} + \frac{x}{\log y}\right)\right)\cdot$$

To study $T_d(x,y)$ we apply Lemma 3.5:

$$\begin{aligned} T_d(x,y) &= \tfrac{x}{\log y}\sum_{\substack{a\leqslant x/y\\ P(a)\leqslant y}} \tfrac{1}{a\tau(da)}\omega(u-\tfrac{\log a}{\log y}) + O\left(\tfrac{x}{\log^2 y}\sum_{\substack{a\leqslant x/y\\ P(a)\leqslant y}} \tfrac{1}{a\tau(da)}\right) \\ &=: \tfrac{x}{\log y}\; L_1 + O\left(\tfrac{x}{\log^2 y}\; L_2\right)\cdot \end{aligned}$$

We first study L_1. Partial summation and Lemma 3.2 give

$$\begin{aligned} L_1 &= \tfrac{1}{\tau(d)}\int_{1^-}^{x/y} \tfrac{1}{t}\omega(u-\tfrac{\log t}{\log y})\, d\left(\textstyle\sum_{a\leqslant t, P(a)\leqslant y}\gamma_d(a)\right) \\ &= \tfrac{1}{\tau(d)}\int_1^{x/y}\left(\sum_{\substack{a\leqslant t\\ P(a)\leqslant y}}\gamma_d(a)\right)\left(\tfrac{1}{t^2}\omega(u-\tfrac{\log t}{\log y}) + \omega'(u-\tfrac{\log t}{\log y})\tfrac{1}{t^2\log y}\right)dt \\ &+ O\left(\tfrac{M_{\eta,d}}{\tau(d)}\tfrac{1}{\sqrt{\log y}}\right) \\ &= \tfrac{1}{\sqrt{\log y}}\tfrac{G_d(1)}{\tau(d)}\int_1^{x/y}\left(\tfrac{1}{t}\omega(u-\tfrac{\log t}{\log y})\; + \omega'(u-\tfrac{\log t}{\log y})\tfrac{1}{t\log y}\right)\;\rho_{\frac{1}{2}}(\tfrac{\log t}{\log y})\,dt \\ &+ O\left(\tfrac{M_{\eta,d}}{\tau(d)}\int_1^{x/y}\left(\tfrac{\log(\frac{\log t}{\log y}+1)}{(\log y)^{\frac{3}{2}}} + \tfrac{1}{\log y}\right)\omega(u-\tfrac{\log t}{\log y})\rho_{\frac{1}{2}}(\tfrac{\log t}{\log y})\tfrac{dt}{t}\right) \\ &+ O\left(\tfrac{M_{\eta,d}}{\tau(d)}\int_1^{x/y}\left(\tfrac{\log(\frac{\log t}{\log y}+1)}{(\log y)^{\frac{3}{2}}} + \tfrac{1}{\log y}\right)|\omega'(u-\tfrac{\log t}{\log y})|\rho_{\frac{1}{2}}(\tfrac{\log t}{\log y})\tfrac{dt}{t\log y}\right) \\ &+ O\left(\tfrac{M_{\eta,d}}{\tau(d)}\tfrac{1}{\sqrt{\log y}}\right)\cdot \end{aligned}$$

By the change of variable $z = \frac{\log t}{\log y}$, we obtain

$$L_1 = \sqrt{\log y}\,\tfrac{G_d(1)}{\tau(d)}\int_0^{u-1}\rho_{\frac{1}{2}}(z)\omega(u-z)\,dz + O\left(\tfrac{M_{\eta,d}}{\tau(d)}\left(1+\tfrac{\log u}{(\log y)^{\frac{1}{2}}}\right)\right)\cdot$$

We now estimate L_2. In the same way, Lemma 3.2 and partial summation give

$$L_2 \ll \sqrt{\log y}\,\frac{M_{\eta,d}}{\tau(d)}\int_0^u \rho_{\frac{1}{2}}(z)dz \ll \sqrt{\log y}\,\frac{M_{\eta,d}}{\tau(d)}\cdot$$

We get our result in the considered range from these different estimates. In the range $\frac{x}{a} < y \leqslant x$ we have $\Phi(x/a,y) = 1$. The result follows immediately from Lemma 3.2. □

Lemma 3.7. *Set* $\epsilon_x = \frac{\log 2}{\log x}$. *Uniformly for* (x, y) *in* (H_ϵ) *and* $0 \leqslant v < \epsilon_x$, *we have*

$$S(x, y, v) = O\left(\frac{x}{\sqrt{\log y}}\right)\cdot$$

Proof. The condition on v implies $d = 1$. By Lemma 3.6 with $d = 1$ we get

$$S(x, y, v) = \sum_{n \leqslant x} \frac{1}{\tau(n, y)} \sum_{d|n, d \leqslant n^v, P(d) \leqslant y} 1 \leqslant \sum_{n \leqslant x} \frac{1}{\tau(n, y)} \ll \frac{x}{\sqrt{\log y}}\cdot$$

□

Lemma 3.8. *Let* $\epsilon_x = \frac{\log 2}{\log x}$. *Uniformly for* (x, y) *in* (H_ϵ) *and* $\epsilon_x \leqslant v \leqslant 1$, *we have*

$$S_2(x, y, v) = O\left(\frac{x}{\log y}\right)\cdot$$

Proof. We have

$$S_2(x, y, v) = \sum_{n \leqslant x} \frac{1}{\tau(n, y)} \sum_{d|n, n^v < d \leqslant x^v, P(d) \leqslant y} 1 \leqslant \sum_{d \leqslant x^v, P(d) \leqslant y} \sum_{m \leqslant x^{1-v}} \frac{1}{\tau(dm, y)}\cdot$$

If $1 - \epsilon_x \leqslant v \leqslant 1$, then $x^{1-v} < 2$. In this case, we have

$$S_2(x, y, v) \leqslant \sum_{d \leqslant x, P(d) \leqslant y} \frac{1}{\tau(d)} \ll \frac{x}{\sqrt{\log y}}\, \rho_{\frac{1}{2}}(u) \ll \frac{x}{\sqrt{\log y}}\cdot$$

Now suppose that $\epsilon_x \leqslant v \leqslant 1 - \epsilon_x$. For $y \leqslant \min\{x^v, x^{1-v}\}$, Lemmas 3.6 and 3.3, give

$$\begin{aligned} S_2(x, y, v) &\ll \tfrac{x^{1-v}}{\sqrt{\log y}} \left(\textstyle\int_0^{(1-v)u-1} \rho_{\frac{1}{2}}(z)\omega((1-v)u - z)dz + \rho_{\frac{1}{2}}((1-v)u)\right) \times \\ & \quad \textstyle\sum_{d \leqslant x^v, P(d) \leqslant y} \frac{G_d(1)}{\tau(d)} \\ &\ll \tfrac{x^{1-v}}{\sqrt{\log y}} \textstyle\sum_{d \leqslant x^v, P(d) \leqslant y} g(d) \\ &\ll \tfrac{x}{\log y}\, \rho_{\frac{1}{2}}(uv) \ll \tfrac{x}{\log y}, \end{aligned}$$

since $uv \geqslant 1$ and $(1 - v)u \geqslant 1$. For $x^{1-v} < y \leqslant x^v$, the inner sum is

$$\sum_{m \leqslant x^{1-v}} \frac{1}{\tau(dm, y)} = \sum_{m \leqslant x^{1-v}} \frac{1}{\tau(dm)} \ll \frac{x^{1-v}}{\sqrt{(1-v)\log x}} \ll x^{1-v}$$

from Lemma 3.2. We apply Lemma 3.3 to the outer sum and obtain

$$S_2(x,y,v) \ll \frac{x}{\sqrt{\log y}}\rho_{\frac{1}{2}}(uv) \ll \frac{x}{\sqrt{\log y}}\cdot$$

The case $x^v < y \leqslant x^{1-v}$ is similar. Lastly for $y > \max\{x^{1-v}, x^v\}$, we set $\epsilon'_x = \frac{\log 2}{\sqrt{\log x}}$ and consider three cases $\epsilon_x \leqslant v \leqslant \epsilon'_x$, $\epsilon'_x < v \leqslant 1-\epsilon'_x$ et $1-\epsilon'_x < v \leqslant 1-\epsilon_x$. In each situation, we have $\frac{1}{\sqrt{v(1-v)}} \ll \sqrt{\log x}$. By applying Lemmas 3.2 and 3.3 we get

$$S_2(x,y,v) \ll \frac{x}{\log x}\frac{1}{\sqrt{v(1-v)}} \ll \frac{x}{\sqrt{\log x}}\cdot$$

□

3.2. Proof of Theorem 2.1

Taking into account Lemmas 3.7 and 3.8, it remains to estimate $S_1(x,y,v)$ for $\epsilon_x \leqslant v \leqslant 1$. We consider the two following situations: $\epsilon_x \leqslant v \leqslant 1-\epsilon_x$ and $1-\epsilon_x < v \leqslant 1$. First, suppose that $1-\epsilon_x < v \leqslant 1$. In this case $x/2 < x^v$. We write

$$\begin{aligned} S_1(x,y,v) &= \sum_{\substack{d\leqslant x/2\\ P(d)\leqslant y}} \sum_{m\leqslant x/d} \frac{1}{\tau(dm,y)} + \sum_{\substack{x/2<d\leqslant x^v\\ P(d)\leqslant y}} \sum_{m\leqslant x/d} \frac{1}{\tau(dm,y)} \\ &=: \widehat{S}_1 + \widehat{S}_2. \end{aligned}$$

Let us study $\widehat{S}_2$. Since $x/2 < d$ then $m = 1$. Hence

$$\widehat{S}_2 = \sum_{x/2<d\leqslant x^v, P(d)\leqslant y} \frac{1}{\tau(d)} \ll \sum_{d\leqslant x, P(d)\leqslant y} \frac{1}{\tau(d)} \ll \frac{x}{\sqrt{\log y}}\rho_{\frac{1}{2}}(u) \ll \frac{x}{\sqrt{\log y}}\cdot$$

The evaluation of $\widehat{S}_1$ is similar to the evaluation of $S_1(x,y,v)$ under the complementary condition $\epsilon_x \leqslant v \leqslant 1-\epsilon_x$, since $x/2 = x^{1-\epsilon_x}$. Let us study $S_1(x,y,v)$ under the condition $\epsilon_x \leqslant v \leqslant 1-\epsilon_x$. First, consider (x,y) in the range

$$\exp\left((\log\log x)^{\frac{5}{3}+\epsilon}\right) \leqslant y \leqslant x/d\,, \quad x \geqslant x_0(\epsilon).$$

Write

$$\begin{aligned} S_1(x,y,v) &:= \sum_{n\leqslant x} \frac{1}{\tau(n,y)} \sum_{\substack{d|n, d\leqslant x^v\\ P(d)\leqslant y}} 1 = \sum_{d\leqslant x^v, P(d)\leqslant y} \sum_{m\leqslant x/d} \frac{1}{\tau(dm,y)} \\ &= \sum_{d\leqslant x^v, P(d)\leqslant y} T_d(\tfrac{x}{d}, y) \end{aligned}$$

and set

$$u_d := u - \frac{\log d}{\log y}\cdot$$

By Lemma 3.6 we get $S_1(x,y,v) = S_{1,1} + O(S_{1,2})$, where

$$S_{1,1} := \frac{x}{\sqrt{\log y}} \sum_{d\leqslant x^v, P(d)\leqslant y} \frac{G_d(1)}{d\tau(d)} \left(\int_0^{u_d-1} \rho_{\frac{1}{2}}(z)\omega(u_d - z)\, dz + \rho_{\frac{1}{2}}(u_d) \right)$$

and

$$S_{1,2} := x \left(\frac{1}{\log y} + \frac{\log(u+1)}{(\log y)^{\frac{3}{2}}} \right) \sum_{d\leqslant x^v, P(d)\leqslant y} \frac{M_{\eta,d}}{d\tau(d)} \ll x \left(\frac{1}{\sqrt{\log y}} + \frac{\log(u+1)}{\log y} \right),$$

by Remark 3.1. It remains to estimate $S_{1,1}$. From (9) and partial summation we get

$$\begin{aligned}
S_{1,1} &= \tfrac{x}{\sqrt{\log y}}\, B(1) \textstyle\sum_{d\leqslant x^v, P(d)\leqslant y} \frac{g(d)}{d} \left(\int_0^{u_d-1} \rho_{\frac{1}{2}}(z)\omega(u_d - z)\, dz + \rho_{\frac{1}{2}}(u_d) \right) \\
&= \tfrac{B(1)x^{1-v}}{\sqrt{\log y}} \left(\int_0^{(1-v)u-1} \rho_{\frac{1}{2}}(z)\omega((1-v)u - z)\, dz + \rho_{\frac{1}{2}}((1-v)u) \right) \times \\
&\quad \left(\textstyle\sum_{n\leqslant x^v, P(n)\leqslant y} g(n) \right) - \tfrac{x}{\sqrt{\log y}} B(1) \times \\
&\quad \int_1^{x^v} \left(\textstyle\sum_{n\leqslant t, P(n)\leqslant y} g(n) \right)\, d\left(\frac{1}{t} \left(\int_0^{u_t-1} \rho_{\frac{1}{2}}(z)\omega(u_t - z)\, dz + \rho_{\frac{1}{2}}(u_t) \right) \right) \\
&:= R_1 - R_2.
\end{aligned}$$

From Lemmas 3.3 and 3.4,

$$R_1 \ll \frac{x}{\log y}\, \rho_{\frac{1}{2}}(uv) \left(1 + \rho_{\frac{1}{2}}((1-v)u) \right) \ll \frac{x}{\sqrt{\log y}},$$

where the last upper bound is proved in the same way as in the proof of Lemma 3.8. Now consider R_2. We have

$$d\left(\frac{1}{t} \left(\int_0^{u_t-1} \rho_{\frac{1}{2}}(z)\, \omega(u_t - z)\, dz + \rho_{\frac{1}{2}}(u_t) \right) \right) = D_1 + D_2,$$

with

$$D_1 := -\left(\int_0^{u_t-1} \rho_{\frac{1}{2}}(z)\omega(u_t - z)\, dz + \rho_{\frac{1}{2}}(u_t) \right) \frac{dt}{t^2}$$

and

$$D_2 := \frac{1}{t} d\left(\int_0^{u_t-1} \rho_{\frac{1}{2}}(z)\, \omega(u_t - z)\, dz + \rho_{\frac{1}{2}}(u_t) \right).$$

Write $R_2 = R_{2,1} + R_{2,2}$, with

$$R_{2,1} := \frac{x}{\sqrt{\log y}}\, B(1) \int_1^{x^v} \left(\sum_{n\leqslant t, P(n)\leqslant y} g(n) \right) D_1$$

and

$$R_{2,2} := \frac{x}{\sqrt{\log y}} B(1) \int_1^{x^v} \left(\sum_{n \leqslant t, P(n) \leqslant y} g(n) \right) D_2.$$

To evaluate $R_{2,1}$ we apply Lemmas 3.3 and 3.4. We get

$$\begin{aligned} R_{2,1} &= -\tfrac{x}{\log y} \int_1^{x^v} \rho_{\frac{1}{2}}(\tfrac{\log t}{\log y}) \left(\int_0^{u_t - 1} \rho_{\frac{1}{2}}(z) \omega(u_t - z)\, dz + \rho_{\frac{1}{2}}(u_t) \right) \times \\ &\quad \left(1 + O\left(\tfrac{\log(\frac{\log t}{\log y} + 1)}{\log y} + \tfrac{1}{\sqrt{\log y}} \right) \right) \tfrac{dt}{t}. \end{aligned}$$

The change of variable $s = \frac{\log t}{\log y}$ gives

$$\begin{aligned} R_{2,1} &= -x \int_0^{uv} \rho_{\frac{1}{2}}(s) \left(\int_0^{u-s-1} \rho_{\frac{1}{2}}(z) \omega(u - s - z) dz + \rho_{\frac{1}{2}}(u - s) \right) \times \\ &\quad \left(1 + O\left(\tfrac{\log(s+1)}{\log y} + \tfrac{1}{\sqrt{\log y}} \right) \right) ds \\ &= -x \int_0^{uv} \rho_{\frac{1}{2}}(s) \left(\int_0^{u-s-1} \rho_{\frac{1}{2}}(z) \omega(u - s - z)\, dz \right) ds - \\ &\quad x \int_0^{uv} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u - s) ds + O\left(\tfrac{x \log(u+1)}{\log y} + \tfrac{x}{\sqrt{\log y}} \right). \end{aligned}$$

Now consider $R_{2,2}$. An easy calculation gives

$$\begin{aligned} D_2 &= -\tfrac{1}{t^2 (u_t - 1) \log y} \left(\int_0^{u_t - 1} \rho_{\frac{1}{2}}(z)\, \omega(u_t - z)\, dz \right) dt \\ &- \tfrac{1}{t^2 (u_t - 1) \log y} \left(\int_0^{u_t - 1} z \rho'_{\frac{1}{2}}(z)\, \omega(u_t - z)\, dz \right) dt \\ &- \tfrac{1}{t^2 (u_t - 1) \log y} \left(\int_0^{u_t - 1} (u_t - z) \rho_{\frac{1}{2}}(z)\, \omega'(u_t - z)\, dz \right) dt \\ &- \tfrac{1}{t^2 \log y} \rho'_{\frac{1}{2}}(u_t) \\ &:= D_{2,1} + D_{2,2} + D_{2,3} + D_{2,4}. \end{aligned}$$

Thus

$$\begin{aligned} R_{2,2} &= \tfrac{x}{\sqrt{\log y}} B(1) \times \\ &\quad \int_1^{x^v} \left(\textstyle\sum_{n \leqslant t, P(n) \leqslant y} g(n) \right) (D_{2,1} + D_{2,2} + D_{2,3} + D_{2,4})\, dt \\ &:= \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3 + \widetilde{D}_4, \end{aligned}$$

where

$$\widetilde{D}_i = \frac{x}{\sqrt{\log y}} B(1) \int_1^{x^v} \left(\sum_{n \leqslant t, P(n) \leqslant y} g(n) \right) D_{2,i}$$

for $i \in \{1, 2, 3, 4\}$. By the same method as previously we get

$$\widetilde{D}_1 \ll \frac{x}{\log y} \int_0^{uv} \frac{\rho_{\frac{1}{2}}(s)}{u-s-1} \left(\int_0^{u-s-1} \rho_{\frac{1}{2}}(z)\omega(u-s-z)\, dz \right) ds.$$

For $s \geqslant u-2$, $\omega(u-s-z) = 0$ since $0 \leqslant z \leqslant u-s-1 \leqslant 1$ and for $s \leqslant u-2$, $u-s-1 \geqslant 1$. It follows that

$$\widetilde{D}_1 \ll \frac{x}{\log y} \int_0^{u-2} \rho_{\frac{1}{2}}(s) ds \int_0^{u-1} \rho_{\frac{1}{2}}(z) dz \ll \frac{x}{\log y}.$$

The study of $\widetilde{D}_2$, $\widetilde{D}_3$ and $\widetilde{D}_4$ is similar by using propreties of $\rho'_{\frac{1}{2}}$ and ω' and gives the same result. We omit details. Assembling these estimates we get our result in the considered range. In the complementary range $x/d < y \leqslant x$, the proof is similar, we omit it. □

4. Preparatory Lemmas and Proof of Theorem 2.2

4.1. Lemmas

We need a weak form of the following lemma.

Lemma 4.1. *For each fixed integer $N \geqslant 1$and for $w > 1$ we uniformly have*

$$\int_w^{\infty} \rho_{\frac{1}{2}}(t) dt = 2 \sum_{k=1}^{N} (w+k) \rho_{\frac{1}{2}}(w+N) + O\left(\frac{\rho_{\frac{1}{2}}(w)}{w^N} \right).$$

Proof. Set

$$I(w) := \int_w^{\infty} \rho_{\frac{1}{2}}(t) dt.$$

The change of variable $z = t+1$ and the differerential equation satisfied by $\rho_{\frac{1}{2}}$ give

$$\begin{aligned} I(w) &= \textstyle\int_{w+1}^{\infty} \rho_{\frac{1}{2}}(z-1) dz = -2 \int_{w+1}^{\infty} z \rho'_{\frac{1}{2}}(z) dz - \int_{w+1}^{\infty} \rho_{\frac{1}{2}}(z) dz \\ &= -2J(w) - I(w+1), \end{aligned} \tag{10}$$

with

$$J(w) := \int_{w+1}^{\infty} z \rho'_{\frac{1}{2}}(z) dz.$$

An integration by parts gives

$$J(w) = -(w+1)\rho_{\frac{1}{2}}(w+1) - \int_{w+1}^{\infty} \rho_{\frac{1}{2}}(z)dz = -(w+1)\rho_{\frac{1}{2}}(w+1) - I(w+1).$$

Inserting in (10) we obtain

$$I(w) - I(w+1) = 2(w+1)\rho_{\frac{1}{2}}(w+1). \tag{11}$$

After replacing w by $w+k-1$ we obtain for every integer $k \geqslant 1$,

$$I(w+k-1) - I(w+k) = 2(w+k)\rho_{\frac{1}{2}}(w+k).$$

Summing these inequalities, we obtain

$$I(w) = 2(w+1)\rho_{\frac{1}{2}}(w+1) + 2\sum_{k=2}^{\infty}(w+k)\rho_{\frac{1}{2}}(w+k).$$

The lemma follows from the preceding formula and the bound [9]

$$\rho_{\frac{1}{2}}(w+k) = O\left(\frac{\rho_{\frac{1}{2}}(w)}{w^k}\right),$$

uniformly for $w > 1$ and $k > 0$. □

Lemma 4.2. *For $v \in [0,1]$ and $u \to \infty$ we uniformly have*

$$H(u,v) := \frac{1}{e^\gamma}\int_0^{uv}\left(\int_0^{u-s-1}\rho_{\frac{1}{2}}(w)dw\right)\rho_{\frac{1}{2}}(s)ds = \frac{1}{\sqrt{e^\gamma}}\int_0^{uv}\rho_{\frac{1}{2}}(w)dw + O\left(u\rho(u)\right),$$

where γ is Euler's constant.

Proof. We have

$$\begin{aligned} H(u,v) &= \tfrac{1}{e^\gamma}\int_0^{uv}\left(\int_0^{+\infty}\rho_{\frac{1}{2}}(w)dw - \int_{u-s-1}^{\infty}\rho_{\frac{1}{2}}(w)dw\right)\rho_{\frac{1}{2}}(s)ds \\ &= \tfrac{1}{\sqrt{e^\gamma}}\int_0^{uv}\rho_{\frac{1}{2}}(s)ds - \tfrac{1}{e^\gamma}\int_0^{uv}\left(\int_{u-s-1}^{\infty}\rho_{\frac{1}{2}}(w)dw\right)\rho_{\frac{1}{2}}(s)ds, \end{aligned}$$

since by using Laplace transform, we have [11, III.5]

$$\int_0^{+\infty}\rho_{\frac{1}{2}}(w)dw = \widehat{\rho_{\frac{1}{2}}}(0) = (\widehat{\rho}(0))^{1/2} = \sqrt{e^\gamma}.$$

Lemma 4.1 in a weak version yields

$$\begin{aligned} H(u,v) &= \tfrac{1}{\sqrt{e^\gamma}}\int_0^{uv}\rho_{\frac{1}{2}}(s)ds + O\left(u\int_0^{uv}\rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds\right) \\ &= \tfrac{1}{\sqrt{e^\gamma}}\int_0^{uv}\rho_{\frac{1}{2}}(s)ds + O(u\rho(u)), \end{aligned}$$

since

$$\int_0^{uv}\rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds \leqslant \int_0^{u}\rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds = (\rho_{\frac{1}{2}} * \rho_{\frac{1}{2}})(u) \ll \rho(u).$$

□

4.2. Proof of Theorem 2.2

We have

$$F(u,v)=\int_0^{uv}\left(\int_0^{u-s-1}\rho_{\frac12}(z)\omega(u-s-z)\,dz\right)\rho_{\frac12}(s)\,ds+O(\rho(u))$$

and

$$\int_0^{uv}\left(\int_0^{u-s-1}\rho_{\frac12}(z)\omega(u-s-z)\,dz\right)\rho_{\frac12}(s)\,ds=\tfrac{1}{\sqrt{e^\gamma}}\int_0^{uv}\rho_{\frac12}(s)ds-$$

$$\tfrac{1}{\sqrt{e^\gamma}}\int_0^{uv}\rho_{\frac12}(s)\left(\int_{u-s-1}^{\infty}\rho_{\frac12}(z)dz\right)ds+$$

$$O\left(\int_0^{uv}\rho_{\frac12}(s)\left(\int_0^{u-s-1}\rho_{\frac12}(z)\rho(u-s-z)dz\right)\,ds\,\right)\cdot$$

By formula (3) and Lemma 4.2, we get

$$\int_0^{uv}\left(\int_0^{u-s-1}\rho_{\frac12}(z)\omega(u-s-z)\,dz\right)\rho_{\frac12}(s)\,ds=$$

$$\tfrac{1}{e^\gamma}\int_0^{uv}\rho_{\frac12}(s)\left(\int_0^{u-s-1}\rho_{\frac12}(z)dz\right)\,ds+$$

$$O\left(\int_0^{uv}\rho_{\frac12}(s)\left(\int_0^{u-s-1}\rho_{\frac12}(z)\rho(u-s-z)dz\right)\,ds\,\right)$$

$$=\tfrac{1}{\sqrt{e^\gamma}}\int_0^{uv}\rho_{\frac12}(s)ds+O(u\rho(u))+O\left(\rho_2(u)\right)$$

$$=\tfrac{1}{\sqrt{e^\gamma}}\int_0^{uv}\rho_{\frac12}(s)ds+O\left(\rho_2(u)\right)$$

since

$$\begin{aligned}&\int_0^{uv}\rho_{\frac12}(s)\left(\int_0^{u-s-1}\rho_{\frac12}(z)\rho(u-s-z)dz\right)\,ds\leqslant\\&\int_0^{uv}\rho_{\frac12}(s)(\rho_{\frac12}*\rho)(u-s)\,ds\leqslant(\rho_{\frac12}*\rho_{\frac12}*\rho)(u)\\&\ll(\rho*\rho)(u)\ll\rho_2(u)\end{aligned}$$

and $u\rho(u)\ll\rho_2(u)$. □

5. Preparatory Lemmas and Proof of Theorem 2.3

5.1. Lemmas

Lemma 5.1. *For $0\leqslant\xi<1$ we have*

$$R(\xi):=\frac{1}{\pi}\int_0^{\xi}\int_0^{-s'+\xi}\frac{ds'\,dz'}{\sqrt{s'}\sqrt{z'}(1-s'-z')}=-\log(1-\xi)\cdot$$

Proof. We use the change of variables $(s', z') \mapsto (w, r) = (\frac{s'}{s'+z'}, s' + z')$. Then $ds'dz' = rdwdr$ and

$$R(\xi) = \left(\frac{1}{\pi}\int_0^1 w^{-\frac{1}{2}}(1-w)^{-\frac{1}{2}}\ dw\right)\left(\int_0^{\xi}(1-r)^{-1}\ dr\right) = -\log(1-\xi)$$

since

$$\frac{1}{\pi}\int_0^1 w^{-\frac{1}{2}}(1-w)^{-\frac{1}{2}}\ dw = \frac{B(\frac{1}{2},\frac{1}{2})}{\pi} = \frac{\Gamma^2(\frac{1}{2})}{\pi} = 1.$$

□

Lemma 5.2. *For $0 \leqslant \beta \leqslant 1 - \frac{1}{u}$ and $u > 1$ we have*

$$\begin{aligned} S(\beta) \quad &:= \quad \tfrac{1}{\pi}\textstyle\int_0^{\beta}\left(\int_0^{-s+1-\frac{1}{u}} \frac{ds\,dz}{\sqrt{s}\,\sqrt{z}\,(1-s-z)}\right) \\ &= \quad \tfrac{2}{\pi}\,(\ \log(u) + \log(1-\beta)\ )\arcsin\left(\sqrt{\tfrac{u\beta}{u-1}}\right) - \log(1-\beta)\cdot \end{aligned}$$

Proof. We write $S(\beta) = I_1 - I_2$ with

$$I_1 = \frac{1}{\pi}\int_0^{1-\frac{1}{u}}\left(\int_0^{-s+1-\frac{1}{u}} \frac{ds\,dz}{\sqrt{s}\,\sqrt{z}\,(1-s-z)}\right)$$

and

$$I_2 = \frac{1}{\pi}\int_{\beta}^{1-\frac{1}{u}}\left(\int_0^{-s+1-\frac{1}{u}} \frac{ds\,dz}{\sqrt{s}\,\sqrt{z}\,(1-s-z)}\right).$$

From Lemma 5.1, $I_1 = R(1-\frac{1}{u}) = \log(u)$. Let us study I_2. By using the change of variables $(s', z') \mapsto (s, z) = (s' + \beta, z')$ we obtain

$$I_2 = \frac{1}{\pi}\int_0^{1-\frac{1}{u}-\beta}\left(\int_0^{-s'+1-\frac{1}{u}-\beta} \frac{ds'\,dz'}{\sqrt{s'+\beta}\,\sqrt{z'}\,(1-s'-\beta-z')}\right).$$

Now we put the change of variables $(r, w) \mapsto (s', z') = (rw - \beta, r(1-w) - \beta)$. Then $ds'dz' = rdwdr$ and

$$I_2 = \frac{1}{\pi}\int_{\frac{u\beta}{u-1}}^1 \frac{dw}{\sqrt{w}\sqrt{1-w}}\int_{\beta}^{1-\frac{1}{u}} \frac{dr}{1-r}\cdot$$

Since

$$\int_{\beta}^{1-\frac{1}{u}} \frac{dr}{1-r} = \log(u) + \log(1-\beta),$$

the change of variable $t = \sqrt{w}$, yields

$$\frac{1}{\pi}\int_{\frac{u\beta}{u-1}}^1 \frac{dw}{\sqrt{w}\sqrt{1-w}} = \frac{2}{\pi}\int_{\sqrt{\frac{u\beta}{u-1}}}^1 \frac{dt}{\sqrt{1-t^2}} = 1 - \frac{2}{\pi}\arcsin\left(\sqrt{\frac{u\beta}{u-1}}\right)\cdot$$

Finally,

$$I_2 = (\ \log(u) + \log(1-\beta)\ )\left(1 - \frac{2}{\pi}\arcsin\left(\sqrt{\frac{u\beta}{u-1}}\right)\right)\cdot$$

Hence

$$S(\beta) = \frac{2}{\pi}(\log(u) + \log(1-\beta))\arcsin\left(\sqrt{\frac{u\beta}{u-1}}\right) - \log(1-\beta).$$

□

Lemma 5.3. *For* $0 < w \leqslant 1$*,*

$$\rho_{\frac{1}{2}}(w) = \frac{1}{\sqrt{\pi}\sqrt{w}}$$

and for $1 \leqslant w \leqslant 2$*,*

$$\rho_{\frac{1}{2}}(w) = \frac{1}{\sqrt{\pi}\sqrt{w}} - \frac{\log(\sqrt{w} + \sqrt{w-1})}{\sqrt{\pi}\sqrt{w}}.$$

Proof. The first formula is the definition of $\rho_{\frac{1}{2}}$ for $0 < w \leqslant 1$ and the second one follows from the differential equation satisfied by $\rho_{\frac{1}{2}}$ pour $1 \leqslant w \leqslant 2$. □

Let us consider the integral

$$I := \int_0^{uv}\left(\int_0^{u-s-1}\rho_{\frac{1}{2}}(z)\omega(u-s-z)\,dz\right)\rho_{\frac{1}{2}}(s)\,ds$$

for $1 < u \leqslant 2$. We notice that on one hand for $s > u-1$

$$\int_0^{u-s-1}\rho_{\frac{1}{2}}(z)\omega(u-s-z)\,dz = 0,$$

since $\rho_{\frac{1}{2}}(z) = 0$ pour $z \leqslant 0$ and then we can restrict the study of the integral on s to the interval $[0, M]$ where $M = \min\{u-1, uv\} \leqslant 1$. On the other hand, if $0 \leqslant z \leqslant u-s-1$ then $1 \leqslant u-s-z \leqslant u-s \leqslant u \leqslant 2$ and $\omega(u-s-z) = 1/(u-s-z)$. Hence

$$I = \frac{1}{\pi}\int_0^M\left(\int_0^{-s+u-1}\frac{dz}{\sqrt{z}\,(u-s-z)}\right)\frac{ds}{\sqrt{s}}\cdot \tag{12}$$

We will give two expessions of I. By the change of variable $t = \sqrt{z}$ in the inner integral (12) we get

$$I = \frac{2}{\pi}\int_0^M \frac{\log(\sqrt{u-s} + \sqrt{u-s-1})}{\sqrt{s}\sqrt{u-s}}\,ds, \tag{13}$$

and by the change of variables $(s', z') \mapsto (s, z) = (us', uz')$ and by puting $M' = \frac{M}{u} = \min\{1 - \frac{1}{u}, v\}$ in (13), we obtain

$$I = \frac{1}{\pi} \int_0^{M'} \int_0^{-s'+1-\frac{1}{u}} \frac{ds'\, dz'}{\sqrt{s'}\sqrt{z'}(1 - s' - z')}. \tag{14}$$

By using the notation of Lemma 5.2, we rewrite (13) and (14) in the form:

$$I = \begin{cases} S(1 - \frac{1}{u}) & \text{if} \quad M = u - 1, \\ S(v) & \text{if} \quad M = uv. \end{cases} \tag{15}$$

Now, we study the integral

$$\int_0^{uv} \rho_{\frac{1}{2}}(s)\, \rho_{\frac{1}{2}}(u - s)\, ds, \quad (1 < u \leqslant 2).$$

Lemma 5.4. *For $v \in [0, \frac{u-1}{u}]$ with $1 < u \leqslant 2$ we have*

$$\int_0^{uv} \rho_{\frac{1}{2}}(s)\, \rho_{\frac{1}{2}}(u - s)\, ds = \frac{2}{\pi} \arcsin(\sqrt{v}) - \frac{1}{2} S(v).$$

Proof. For $v \in [0, \frac{u-1}{u}]$ we have $uv \leqslant u - 1$ and therefore $0 \leqslant s \leqslant uv \leqslant u - 1 \leqslant 1$ and $1 \leqslant u - s \leqslant 2$. Applying Lemma 5.3 we get

$$\begin{aligned} \int_0^{uv} \rho_{\frac{1}{2}}(s)\, \rho_{\frac{1}{2}}(u - s)\, ds &= \frac{1}{\pi} \int_0^{uv} \frac{ds}{\sqrt{s}\sqrt{u-s}} - \frac{1}{\pi} \int_0^{uv} \frac{\log(\sqrt{u-s} + \sqrt{u-s-1})}{\sqrt{s}\sqrt{u-s}}\, ds \\ &:= J_1 - J_2. \end{aligned}$$

By the change of variable $t = \sqrt{s}$, we get

$$J_1 = \frac{2}{\pi} \int_0^{\sqrt{v}} \frac{dt}{\sqrt{1 - t^2}} = \frac{2}{\pi} \arcsin(\sqrt{v}).$$

From (15) we have $J_2 = \frac{1}{2} S(v)$. This completes the proof. ∎

Lemma 5.5. *For $v \in [\frac{u-1}{u}, \frac{1}{u}]$ with $1 < u \leqslant 2$ we have*

$$\int_0^{uv} \rho_{\frac{1}{2}}(s)\, \rho_{\frac{1}{2}}(u - s)\, ds = \frac{2}{\pi} \arcsin(\sqrt{v}) - \frac{1}{2} S(1 - \frac{1}{u}).$$

Proof. For $v \in [\frac{u-1}{u}, \frac{1}{u}]$ we have $u - 1 \leqslant uv \leqslant 1$. We write

$$\int_0^{uv} \rho_{\frac{1}{2}}(s)\, \rho_{\frac{1}{2}}(u-s)\, ds = \int_0^{u-1} \rho_{\frac{1}{2}}(s)\, \rho_{\frac{1}{2}}(u-s)\, ds + \int_{u-1}^{uv} \rho_{\frac{1}{2}}(s)\, \rho_{\frac{1}{2}}(u-s)\, ds = J_1 + J_2.$$

Consider J_1. As in the previous lemma, we have

$$\begin{aligned} J_1 &= \frac{1}{\pi} \int_0^{u-1} \frac{ds}{\sqrt{s}\sqrt{u-s}} - \frac{1}{\pi} \int_0^{u-1} \frac{\log(\sqrt{u-s} + \sqrt{u-s-1})}{\sqrt{s}\sqrt{u-s}}\, ds \\ &= \frac{2}{\pi} \arcsin(\sqrt{\tfrac{u-1}{u}}) - \frac{1}{2} S(1 - \tfrac{1}{u}). \end{aligned}$$

Now consider J_2. We have $u-1 \leqslant s \leqslant uv \leqslant 1$ and $u(1-v) \leqslant u-s \leqslant 1$. Hence

$$J_2 = \frac{1}{\pi}\int_{u-1}^{uv} \frac{ds}{\sqrt{s}\sqrt{u-s}} = \frac{2}{\pi}\arcsin(\sqrt{v}) - \frac{2}{\pi}\arcsin(\sqrt{\frac{u-1}{u}}).$$

This completes the proof. □

Lemma 5.6. *For $v \in [\frac{1}{u}, 1]$ with $1 < u \leqslant 2$, we have*

$$\int_0^{uv} \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds = -\frac{3}{2}S(1-\frac{1}{u}) + \frac{2}{\pi}\arcsin(\sqrt{v}) + S(1-v).$$

Proof. As $u \leqslant 2$, we have $\frac{u-1}{u} \leqslant \frac{1}{u} \leqslant v \leqslant 1$ and therefore $u-1 \leqslant uv \leqslant u$. We write

$$\int_0^{uv} \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds = \int_0^{u-1} \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds + \int_{u-1}^{uv} \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds = J_1 + J_2.$$

The integral J_1 has been studied in lemma 5.5. It remains to calculate J_2. As $\frac{1}{u} \leqslant v$, that is to say, $uv \geqslant 1$, we write

$$J_2 = \int_{u-1}^{1} \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds + \int_1^{uv} \rho_{\frac{1}{2}}(s)\,\rho_{\frac{1}{2}}(u-s)\,ds := J_{2,1} + J_{2,2}.$$

We have

$$J_{2,1} = \frac{1}{\pi}\int_{u-1}^{1} \frac{ds}{\sqrt{s}\sqrt{u-s}} = \frac{2}{\pi}\arcsin(\frac{1}{\sqrt{u}}) - \frac{2}{\pi}\arcsin(\sqrt{\frac{u-1}{u}}).$$

We write

$$J_{2,2} = \frac{1}{\pi}\int_1^{uv} \frac{ds}{\sqrt{s}\sqrt{u-s}} - \frac{1}{\pi}\int_1^{uv} \frac{\log(\sqrt{s}+\sqrt{s-1})}{\sqrt{s}\sqrt{u-s}}\,ds := \widehat{J_1} - \widehat{J_2}.$$

We have

$$\widehat{J_1} = \frac{2}{\pi}\arcsin(\sqrt{v}) - \frac{2}{\pi}\arcsin(\frac{1}{\sqrt{u}}).$$

Now it remains to study $\widehat{J_2}$. We put the change of variable $s' = u-s$. We get

$$\widehat{J_2} = \frac{1}{\pi}\int_{u(1-v)}^{u-1} \frac{\log(\sqrt{u-s'}+\sqrt{u-s'-1})}{\sqrt{s'}\sqrt{u-s'}}\,ds'.$$

As $v \geqslant \frac{1}{u}$ we have $u(1-v) \leqslant u-1 \leqslant 1$. Therefore, by using notations in (13), (14) and (15) as well as Lemmas 5.1 and 5.2 we obtain

$$\widehat{J_2} = \frac{1}{2}S(1-\frac{1}{u}) - \frac{1}{2}S(1-v).$$

We complete the proof by grouping different estimates above. □

5.2. Proof of Theorem 2.3

Theorem 2.3 follows from (15) and different lemmas of Section 5. □

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