A NOTE ON A GENERALIZATION OF FUNDAMENTAL GAPS IN NUMERICAL SEMIGROUPS

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Abstract

Let S be a numerical semigroup. A positive integer n is a fundamental gap of S if $n \notin S$ whereas $kn \in S$ for each integer $k > 1$. We call a positive integer n a fundamental gap of order λ, $\lambda \in \mathbb{N}$, if $kn \notin S$ for each $k \in \{1, \ldots, \lambda\}$ and $kn \in S$ for each integer $k > \lambda$. We investigate the set $\mathrm{FG}(S, \lambda)$ of fundamental gaps of S of order λ when S has embedding dimension two.

1. Introduction

By $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ we mean the set of non-negative integers and the set of positive integers, respectively. A numerical semigroup is a subset S of $\mathbb{Z}_{\geq 0}$ that is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is a finite set. If $a_1, \ldots, a_k \in \mathbb{N}$, we set

$$\langle a_1, \ldots, a_k \rangle = \Big\{ a_1x_1 + \cdots + a_kx_k : x_i \in \mathbb{Z}_{\geq 0} \Big\}.$$

Then $\langle a_1, \ldots, a_k \rangle$ is a numerical semigroup if and only if $\gcd(a_1, \ldots, a_k) = 1$.

The set $A = \{a_1, \ldots, a_k\}$ is called a system of generators of the semigroup $S = \langle a_1, \ldots, a_k \rangle$. For a semigroup S, A is a minimal system of generators if A generates S and no proper subset of A generates S. Every numerical semigroup has a unique minimal system of generators. This system of generators is finite, and the cardinality of this minimal system of generators is called the embedding dimension of S, and is denoted by $e(S)$.

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Given a numerical semigroup S, we define the set $\mathrm{G}(S)$ of gaps of S to be $\mathbb{N} \setminus S$. An element $n \in \mathrm{G}(S)$, following Rosales et al [4], is defined to be a fundamental gap if $kn \in S$ for each $k > 1$. It is easy to see that the set of fundamental gaps $\mathrm{FG}(S)$ determines the numerical semigroup S uniquely, as any element in $\mathrm{G}(S)$ is just a divisor of some fundamental gap. Fundamental gaps are a special case of special gaps. Special gaps are of interest, as they help in solving the problem of oversemigroups of a given numerical semigroup. The set $\mathrm{FG}(S)$ of all fundamental gaps of S has been determined by Rosales [2] in the case where S has embedding dimension 2. Rosales and García-Sánchez [3] provide a comprehensive introduction to the study of Numerical Semigroups.

In this note, we extend the notion of fundamental gaps to the notion of fundamental gaps of an arbitrary order λ, $\lambda \in \mathbb{N}$, and determine the set $\mathrm{FG}(S, \lambda)$ of all fundamental gaps of order λ where the embeding dimension of S is 2, and derive $\mathrm{FG}(S)$ as a special case. Whereas the set of fundamental gaps, $\mathrm{FG}(S)$, uniquely determine the semigroup, in general, the set of fundamental gaps of order λ, $\mathrm{FG}(S, \lambda)$, for $\lambda > 1$ do not usually determine the semigroup. For instance, taking two numerical semigroups S and T, neither of which is an ordinary semigroup, with $\mathrm{F}(S) = \mathrm{F}(T)$, it is easy to check that $\mathrm{FG}(S, \mathrm{F}(S)) = \mathrm{FG}(T, \mathrm{F}(T)) = \emptyset$. In fact, if $\lambda > \mathrm{F}(S)$, then $\mathrm{FG}(S, \lambda) = \emptyset$. In particular, when $\lambda > 1$, the set $\mathrm{FG}(S, \lambda)$ does not provide a way to test for membership of the numerical semigroup.

For a numerical semigroup S and a positive integer n, the set $\frac{S}{n}$ is given by $\{m \in \mathbb{Z}_{\geq 0} : mn \in S\}$. It is well known that every proportionally modular numerical semigroup is of the form $\frac{\langle a,b \rangle}{n}$ for some positive integers a, b, n where $\gcd(a, b) = 1$ (see [5]). Also, note that

$$\mathrm{FG}(S, \lambda) = \left(\bigcap_{k > \lambda} \tfrac{S}{k} \right) \setminus \left(S \cup \tfrac{S}{2} \cup \cdots \cup \tfrac{S}{\lambda} \right).$$

Hence it is not surprising to see that modular inequalities appear in Theorem 1.

2. Main Result

Given a numerical semigroup S and a positive integer λ, in this section, we define a fundamental gap of S of order λ, and explicitly determine the set of fundamental gaps of S of order λ in the case where S has embedding dimension 2.

Definition 1. Let S be a numerical semigroup and let $\lambda \in \mathbb{N}$. We say that a positive integer n is a *Fundamental Gap of Order* λ if $kn \notin S$ for $1 \leq k \leq \lambda$ and $kn \in S$ for $k > \lambda$. We denote the set of all fundamental gaps of S of order λ by $\mathrm{FG}(S, \lambda)$.

Lemma 1. *Let S be a numerical semigroup and let $\lambda \in \mathbb{N}$. Then $n \in \mathrm{FG}(S,\lambda)$ if and only if $kn \notin S$ for $1 \leq k \leq \lambda$ and $kn \in S$ for $\lambda+1 \leq k \leq 2\lambda+1$.*

Proof. Suppose n is a positive integer such that $kn \in S$ for $\lambda+1 \leq k \leq 2\lambda+1$. Choose any $k > \lambda$, and write $k = q(\lambda+1)+r$, with $q \geq 1$ and $0 \leq r \leq \lambda$. Then $kn = \big((q-1)(\lambda+1)+(\lambda+r+1)\big)n \in S$, since $(\lambda+1)n \in S$ and $(\lambda+r+1)n \in S$. This proves the sufficiency. The necessity is obvious. $\square$

The Farey sequence $\mathcal{F}_n$ of order n is the ascending sequence of rational numbers $\frac{h}{k}$ between 0 and 1 with $\gcd(h,k)=1$ and $1 \leq k \leq n$. Farey sequences are fascinating in their own right, and useful in the study of Diophantine Approximations. The following results are useful in the explicit determination of $\mathrm{FG}(S,\lambda)$ when S has embedding dimension 2.

Proposition 1. ([1, Theorem 5.8, p. 44])
The term immediately succeeding $\frac{a}{b}$ in the Farey sequence $\mathcal{F}_n$ of order n is given by $\frac{c}{d}$, where

$$bc - ad = 1 \ \text{ and } \ 0 \leq n-b < d \leq n.$$

Lemma 2. *Let $\lambda \in \mathbb{N}$. Then*

$$\left[\tfrac{1}{\lambda}, \tfrac{2\lambda-1}{2\lambda}\right) = \bigcup_{1\leq c<k\leq\lambda} \left[\frac{c}{k}, \frac{\left\lceil\frac{\lambda+1}{k}\right\rceil c+1}{\left\lceil\frac{\lambda+1}{k}\right\rceil k}\right).$$

Proof. Throughout the proof, let $m(k) = \left\lceil\frac{\lambda+1}{k}\right\rceil$. Let

$$X = \left[\tfrac{1}{\lambda}, \tfrac{2\lambda-1}{2\lambda}\right) \text{ and } Y = \bigcup_{1\leq c<k\leq\lambda} \left[\tfrac{c}{k}, \tfrac{m(k)\cdot c+1}{m(k)\cdot k}\right).$$

Let $x \in X$. Then $x \in \left[\frac{u}{v}, \frac{u'}{v'}\right)$, where $\frac{u}{v}, \frac{u'}{v'}$ are consecutive terms in $\mathcal{F}_\lambda \setminus \{0,1\}$, or $x \in \left[\frac{\lambda-1}{\lambda}, \frac{2\lambda-1}{2\lambda}\right)$.

In the first case, choose $c=u$, $k=v$. By Proposition 1, $0 \leq \lambda-k < v' \leq \lambda$, and so $\lambda+1 \leq k+v' \leq kv'$. Hence $m(k)\cdot(uv'-u'v) = m(k) \leq v'$, and so $\frac{u'}{v'} \leq \frac{m(k)\cdot c+1}{m(k)\cdot k}$.

In the second case, choose $c = \lambda-1$, $k=\lambda$, and so $m(k)=2$.

Hence $x \in \left[\frac{c}{k}, \frac{m(k)\cdot c+1}{m(k)\cdot k}\right)$ in each case, and so $X \subseteq Y$.

To show $X=Y$, it now suffices to show

$$\begin{aligned} \min\left\{\tfrac{c}{k} : 1 \leq c < k, 2 \leq k \leq \lambda\right\} &= \tfrac{1}{\lambda}, \\ \max\left\{\tfrac{m(k)\cdot c+1}{m(k)\cdot k} : 1 \leq c < k \leq \lambda\right\} &= \tfrac{2\lambda-1}{2\lambda}. \end{aligned}$$

The first claim is obvious. To prove the second claim, first note that $m(k) \geq 2 \geq 1 + \frac{k}{2\lambda-k} = \frac{2\lambda}{2\lambda-k}$. Hence $c \leq k-1 \leq \frac{(2\lambda-1)m(k)\cdot k-2\lambda}{2\lambda m(k)}$, so that $\frac{m(k)\cdot c+1}{m(k)\cdot k} \leq \frac{2\lambda-1}{2\lambda}$. To show the bound is attained, choose $c=\lambda-1$, $k=\lambda$, and so $m(k)=2$. This completes the claim that $X=Y$. $\square$

Our main result is the determination of $\mathrm{FG}(S,\lambda)$ when $S=\langle a,b\rangle$, $\gcd(a,b)=1$. Corresponding to each nonzero element n in a numerical semigroup S, we define the Apéry set $\mathrm{Ap}(S,n)$ of S. Apéry sets are essential tools in the study of numerical semigroups.

Definition 2. Let S be a numerical semigroup and let $n \in S\setminus\{0\}$. The *Apéry set* of S corresponding to n is defined as

$$\mathrm{Ap}(S,n)=\{s\in S : s-n\notin S\}.$$

Theorem 1. *Let* $S=\langle a,b\rangle$*, where* $\gcd(a,b)=1$*. The set of Fundamental Gaps of order* λ *is given by*

$$\mathrm{FG}(S,\lambda)=A\cup B\cup C\cup D\cup E,$$

where

$$\begin{aligned}
A&=\left\{bs-ar : \tfrac{(\lambda-1)b}{\lambda}<r\le \tfrac{\lambda b}{\lambda+1}, \tfrac{2\lambda a}{2\lambda+1}\le s\le a-1\right\},\\
B&=\left\{bs-ar : \tfrac{(\lambda-1)b}{\lambda}<r\le \tfrac{(2\lambda-1)b}{2\lambda+1}, \tfrac{(2\lambda-1)a}{2\lambda}\le s< \tfrac{2\lambda a}{2\lambda+1}\right\},\\
C&=\left\{bs-ar : 1\le r\le \tfrac{b}{2\lambda+1}, s=\tfrac{a}{\lambda+1}\right\},\\
D&=\left\{bs-ar : 1\le r\le \tfrac{b}{2\lambda}, \tfrac{a}{\lambda+1}<s<\tfrac{a}{\lambda}, 0<a \bmod s\le \tfrac{s}{2}\right\},\\
E&=\left\{bs-ar : 1\le r\le \tfrac{b}{2\lambda+1}, \tfrac{a}{\lambda+1}<s<\tfrac{a}{\lambda}, a \bmod s> \tfrac{s}{2}\right\}.
\end{aligned}$$

Proof. For $S=\langle a,b\rangle$, note that $\mathrm{Ap}(S,a)=\{bs : 0\le s\le a-1\}$. Thus, any $n\in\mathbb{Z}$ is of the form $bs-ar$ with $0\le s\le a-1$ and $r\in\mathbb{Z}$; see [3, Lemma 2.4, p. 8], for instance. Hence, $n\in S$ if and only if $r\le 0$. Therefore, $n\notin S$ if and only if $r>0$. Therefore $n\in\mathbb{N}\setminus S$ if and only if $1\le s\le a-1$ and $1\le r<\frac{bs}{a}$.

Recall that n is a fundamental gap of order λ if and only if $kn\notin S$ for $1\le k\le\lambda$ and $kn\in S$ for $\lambda+1\le k\le 2\lambda+1$. The latter is equivalent to

$$k(bs-ar)<b\big(ks \bmod a\big) \qquad \text{if } 1\le k\le\lambda; \tag{1}$$

$$k(bs-ar)\ge b\big(ks \bmod a\big) \qquad \text{if } \lambda+1\le k\le 2\lambda+1. \tag{2}$$

Fix $s\in\{1,\dots,a-1\}$, and write $a=qs+\rho$, $0\le\rho<s$.

`Case I.` $\big(q<\lambda\big)$

With $s=a-t$, $t\ge 1$, $q<\lambda$ is equivalent to $t<\frac{(\lambda-1)a}{\lambda}$. We consider three subcases: (i) $1\le t\le\frac{a}{2\lambda+1}$; (ii) $\frac{a}{2\lambda+1}<t\le\frac{a}{2\lambda}$; (iii) $\frac{a}{2\lambda}<t<\frac{(\lambda-1)a}{\lambda}$.

In subcases (i) and (ii), if $1\le k\le 2\lambda$, then $ks\equiv -kt \pmod a$ and $1\le kt\le a$. Therefore $ks \bmod a = a-kt$.

In subcase (i), if $k = 2\lambda+1$, then $ks \equiv -kt \pmod{a}$ and $2\lambda+1 \le (2\lambda+1)t \le a$. Therefore $ks \bmod a = a - kt$.

In subcase (ii), if $k = 2\lambda+1$, then $ks \equiv -kt \pmod{a}$ and $a < (2\lambda+1)t \le \frac{(2\lambda+1)a}{2\lambda} < 2a$. Therefore $ks \bmod a = 2a - kt$.

In subcase (i), inequality (1) reduces to $k\big(b(a-t)-ar\big) < b\big(a-kt\big)$, $1 \le k \le \lambda$, and inequality (2) reduces to $k\big(b(a-t)-ar\big) \ge b\big(a-kt\big)$, $\lambda+1 \le k \le 2\lambda+1$. Hence $n \in \mathrm{FG}(S,\lambda)$ if and only if $r > \frac{(k-1)b}{k}$ for $1 \le k \le \lambda$ and $r \le \frac{(k-1)b}{k}$ for $\lambda+1 \le k \le 2\lambda+1$. Therefore $n \in \mathrm{FG}(S,\lambda)$ if and only if

$$\max\left\{\tfrac{(k-1)b}{k} : 1 \le k \le \lambda\right\} = \tfrac{(\lambda-1)b}{\lambda} < r \le \min\left\{\tfrac{(k-1)b}{k} : \lambda+1 \le k \le 2\lambda+1\right\} = \tfrac{\lambda b}{\lambda+1}.$$

This leads to $n \in A$.

In subcase (ii), inequality (1) reduces to $k\big(b(a-t)-ar\big) < b\big(a-kt\big)$, $1 \le k \le \lambda$, and inequality (2) reduces to $k\big(b(a-t)-ar\big) \ge b\big(a-kt\big)$, $\lambda+1 \le k \le 2\lambda$ and to $k\big(b(a-t)-ar\big) \ge b\big(2a-kt\big)$ for $k = 2\lambda+1$. Hence $n \in \mathrm{FG}(S,\lambda)$ if and only if $r > \frac{(k-1)b}{k}$ for $1 \le k \le \lambda$, $r \le \frac{(k-1)b}{k}$ for $\lambda+1 \le k \le 2\lambda$ and $r \le \frac{(2\lambda-1)b}{2\lambda+1}$. Therefore $n \in \mathrm{FG}(S,\lambda)$ if and only if

$$\begin{aligned}
\max\left\{\tfrac{(k-1)b}{k} : 1 \le k \le \lambda\right\} &= \tfrac{(\lambda-1)b}{\lambda} \\
&< r \\
&\le \min\left\{\min\left\{\tfrac{(k-1)b}{k} : \lambda+1 \le k \le 2\lambda\right\}, \tfrac{(2\lambda-1)b}{2\lambda+1}\right\} \\
&= \tfrac{(2\lambda-1)b}{2\lambda+1}.
\end{aligned}$$

This leads to $n \in B$.

We claim that $n \notin \mathrm{FG}(S,\lambda)$ in subcase (iii). Using $ks - \big(ks \bmod a\big) = \lfloor \frac{ks}{a} \rfloor a$ in inequality (1) yields

$$b\left\lfloor \tfrac{ks}{a} \right\rfloor < kr, \quad 1 \le k \le \lambda, \tag{3}$$

and inequality (2) yields

$$b\left\lfloor \tfrac{ks}{a} \right\rfloor \ge kr, \quad \lambda+1 \le k \le 2\lambda+1. \tag{4}$$

In subcase (iii), we have $n = bs - ar$, with $\frac{a}{\lambda} < s < \frac{(2\lambda-1)a}{2\lambda}$. Hence $\frac{s}{a} \in X$, so $\frac{s}{a} \in \left[\frac{c}{k}, \frac{m(k)\cdot c+1}{m(k)\cdot k}\right)$ with $1 \le c < k \le \lambda$ and $\lambda+1 \le m(k)\cdot k \le 2\lambda+1$, by Lemma 2. Since $1 \le k \le \lambda$, using $\frac{c}{k} \le \frac{s}{a}$, or the equivalent $c \le \frac{ks}{a}$ in inequality (3) we get $bc < kr$. Since $\lambda+1 \le m(k)\cdot k \le 2\lambda+1$, using $\frac{s}{a} < \frac{m(k)\cdot c+1}{m(k)\cdot k}$ in inequality (4) we get $m(k)\cdot kr \le m(k)\cdot bc$, or that $kr \le bc$. Thus, there is no r corresponding to s covered by the subcase (iii). Hence $n \notin \mathrm{FG}(S,\lambda)$ in subcase (iii).

Case II. $\left(q \geq \lambda\right)$

If $q > \lambda$, then inequality (2) fails to hold for $k = \lambda+1$ since $(\lambda+1)s \bmod a = (\lambda+1)s$ and $(\lambda+1)(bs-ar) < b(\lambda+1)s$ unless $\rho = 0$ and $q = \lambda+1$. For $\rho = 0$ and $q = \lambda+1$, inequality (2) holds for $k = \lambda+1$ because $(\lambda+1)s \bmod a = 0$.

For the rest of this proof, we assume $q = \lambda$, and when $\rho = 0$ we need to consider the additional $q = \lambda+1$.

Subcase (i). $\left(\rho = 0\right)$

If $q = \lambda$, then inequality (1) fails to hold for $k = q$ since $qs \bmod a = 0$.

Suppose $q = \lambda+1$. Then inequality (1) reduces to $k(bs-ar) < b(ks)$, and inequality (2) reduces to $k(bs-ar) \geq b(ks-a)$, since $a = (\lambda+1)s < (2\lambda+1)s < 2a$. It is evident that inequality (1) holds, and that inequality (2) holds if and only if $r \leq \frac{b}{k}$ holds for $k \in \{\lambda+1, \ldots, 2\lambda+1\}$. Hence $bs-ar \in \mathrm{FG}(S,\lambda)$ if and only if $s = \frac{a}{\lambda+1}$ and $1 \leq r \leq \frac{b}{2\lambda+1}$. This leads to $n \in C$.

Subcase (ii). $\left(0 < \rho \leq \frac{s}{2}\right)$

Suppose $q = \lambda$. Then inequality (1) reduces to $k(bs-ar) < b(ks)$, and this evidently holds. Since $a < (\lambda+1)s \leq 2\lambda s < 2a \leq (2\lambda+1)s < 3a$, inequality (2) reduces to $k(bs-ar) \geq b(ks-a)$ for $k \in \{\lambda+1, \ldots, 2\lambda\}$ and to $k(bs-ar) \geq b(ks-2a)$ for $k = 2\lambda+1$. Hence inequality (2) holds if and only if $r \leq \frac{b}{k}$ holds for $k \in \{\lambda+1, \ldots, 2\lambda\}$ and $r \leq \frac{2b}{k}$ holds for $k = 2\lambda+1$. Hence $bs-ar \in \mathrm{FG}(S,\lambda)$ if and only if $\lambda = q = \lfloor \frac{a}{s} \rfloor$ and $1 \leq r \leq \frac{b}{2\lambda}$. This leads to $n \in D$.

Subcase (iii). $\left(\rho > \frac{s}{2}\right)$

Suppose $q = \lambda$. Then inequality (1) reduces to $k(bs-ar) < b(ks)$, and inequality(2) reduces to $k(bs-ar) \geq b(ks-a)$, since $a < (\lambda+1)s < (2\lambda+1)s < 2a$. It is evident that inequality (1) holds, and that inequality (2) holds if and only if $r \leq \frac{b}{k}$ holds for $k \in \{\lambda+1, \ldots, 2\lambda+1\}$. Hence $bs-ar \in \mathrm{FG}(S,\lambda)$ if and only if $\lambda = q = \lfloor \frac{a}{s} \rfloor$ and $1 \leq r \leq \frac{b}{2\lambda+1}$. This leads to $n \in E$. □

Corollary 1. *Let $S = \langle a, b\rangle$, where $\gcd(a,b) = 1$. The set of Fundamental Gaps is given by*

$$\mathrm{FG}(S) = \left\{bs-ar : 1 \leq r \leq \tfrac{b}{3}, \tfrac{a}{2} \leq s < \tfrac{2a}{3}\right\} \bigcup \left\{bs-ar : 1 \leq r \leq \tfrac{b}{2}, \tfrac{2a}{3} \leq s \leq a-1\right\}.$$

Proof. For $\lambda = 1$, Theorem 1 gives

$$\begin{aligned}
A &= \left\{bs-ar : 0 < r \leq \tfrac{b}{2}, \tfrac{2a}{3} \leq s \leq a-1\right\},\\
B &= \left\{bs-ar : 0 < r \leq \tfrac{b}{3}, \tfrac{a}{2} \leq s < \tfrac{2a}{3}\right\},\\
C &= \left\{bs-ar : 1 \leq r \leq \tfrac{b}{3}, s = \tfrac{a}{2}\right\},\\
D &= \left\{bs-ar : 1 \leq r \leq \tfrac{b}{2}, \tfrac{a}{2} < s < a, 0 < a \bmod s \leq \tfrac{s}{2}\right\},\\
E &= \left\{bs-ar : 1 \leq r \leq \tfrac{b}{3}, \tfrac{a}{2} < s < a, a \bmod s > \tfrac{s}{2}\right\}.
\end{aligned}$$

When $\frac{a}{2} < s < a$, $\left\lfloor \frac{a}{s} \right\rfloor = 1$, and so $a \bmod s = a - \left\lfloor \frac{a}{s} \right\rfloor s = a - s$. Hence

$$C \cup E = \left\{ bs - ar : 1 \leq r \leq \tfrac{b}{3}, \tfrac{a}{2} \leq s < a, a \bmod s > \tfrac{s}{2} \right\} = B,$$
$$D = \left\{ bs - ar : 1 \leq r \leq \tfrac{b}{2}, \tfrac{2a}{3} \leq s < a \right\} = A.$$

Therefore $\mathrm{FG}(S) = A \cup B$. □

We illustrate Theorem 1 with the following example.

Example 1. *For* $S = \langle 13, 31 \rangle$*, we get*

λ	$\mathrm{FG}(S, \lambda)$
1	84, 87, 97, 100, 110, 113, 115, 118, 123, 126, 128, 131, 136, 139, 141, 144, 146, 149, 152, 154, 157, 159, 162, 165, 167, 170, 172, 175, 177, 178, 180, 183, 185, 188, 190, 191, 193, 196, 198, 201, 203, 204, 206, 209, 211, 214, 216, 219, 222, 224, 227, 229, 232, 235, 237, 240, 242, 245, 250, 253, 255, 258, 263, 266, 268, 271, 276, 281, 284, 289, 294, 297, 302, 307, 315, 320, 328, 333, 346, 359
2	76, 77, 81, 89, 90, 94, 95, 102, 103, 107, 108, 112, 116, 120, 121, 125, 129, 133, 134, 138, 142, 147, 151, 160, 164, 173
3	55, 59, 68, 72, 73, 85, 86, 98, 99, 111
4	54, 60, 67, 80
5	47
6	34, 36, 49
$[7, 11]$	$\emptyset$
12	18
≥ 13	$\emptyset$

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