FINITE SUMS OF THE FLOOR FUNCTIONS SERIES

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Abstract

This paper deals with new formulas for finite sums of the floor function series. The analogy between the sum of the successive natural numbers and the floor functions is applied to define the sums of a finite series of floor functions of different powers. Formulas for different powers of the floor functions are developed and proved by mathematical induction. The main goal of this paper is the development and proof of the formula that enables the calculation of the sums of finite series of different powers of the floor functions as a closed-form expression.

1. Introduction

The floor function, denoted floor(x) = $[x]$, has been a subject of interest of mathematicians since the late eighteenth century and continues to attract interest in modern number theory.

From the definition of the floor function, the following can be easily obtained:

$$x \leq y \text{ implies } [x] \leq [y]\,, \tag{1}$$

$$[x+k] = [x] + k, \text{ for } k \in Z, \tag{2}$$

$$[x+y] \geq [x] + [y]\,, \tag{3}$$

$$\left[\frac{x+1}{2}\right] + \left[\frac{x}{2}\right] = [x]\,. \tag{4}$$

More advanced formulas can be found in the literature [1, 3, 5]:

$$\left[\frac{n(n+1)}{4n-2}\right] = \left[\frac{n+1}{4}\right], \tag{5}$$

$$\left[\frac{a^{n+1}+1}{a^{n-1}+1}\right]=a^2-1 \quad \text{for} \quad a\geq 2, \tag{6}$$

$$\sum_{i=1}^{n}\left[\frac{i}{4}\right]^2=\left[\frac{n\left(n+2\right)\left(2n-1\right)}{24}\right], \tag{7}$$

$$\sum_{i=n}^{n+m-1}\left[\frac{i}{m}\right]=n. \tag{8}$$

2. Sums of Finite Series of the Floor Function

In this paper, finite sums of the floor function series will be denoted as:

$$\mathrm{S}_{\left[\frac{n}{k}\right]}\left(r\right)=\sum_{i=k}^{n}\left[\frac{i}{k}\right]^r, \tag{9}$$

where $r, n \geq k \geq 1 \in N$.

Theorem 2.1. *The sums of finite series of different powers of floor functions can be expressed as:*

$$\mathrm{S}_{\left[\frac{n}{k}\right]}\left(r\right)=\left[\frac{n}{k}\right]^r\left(n+1-k\left[\frac{n}{k}\right]\right)+\frac{k}{r+1}\sum_{j=0}^{r}\left(-1\right)^j\binom{r+1}{j}B_j\left[\frac{n}{k}\right]^{r+1-j}, \tag{10}$$

where B_j are the Bernoulli numbers of the first kind.

Proof. Each integer $n \geq k$ can be written as $n = kp + m$, where $p = \left[\frac{n}{k}\right]$, $0 \leq m \leq k-1$. And thus it is possible to find an integer $1 \leq l \leq p-1$, for $k \leq i \leq kp-1$, which means that $\left[\frac{i}{k}\right]^r = l^r$. Additionally, for $kp \leq i \leq n$ we have $\left[\frac{i}{k}\right]^r = p^r$. This allows us to express $\mathrm{S}_{\left[\frac{n}{k}\right]}\left(r\right)$ as a finite sum as follows:

$$\mathrm{S}_{\left[\frac{n}{k}\right]}\left(r\right)=k\left(1^r+2^r+3^r+...+\left(p-1\right)^r\right)+\left(m+1\right)p^r=k\sum_{l=1}^{p-1}l^r+\left(m+1\right)p^r. \tag{11}$$

The first term is a product of the number k and the Bernoulli sum of next natural numbers in r power $l_1^r + l_2^r + ... + l_{p-1}^r$, thus it calculates the following dependence:

$$\mathrm{S}_l\left(r\right)=k\left(1^r+2^r+3^r+...+\left(p-1\right)^r\right) \tag{12}$$

in the closed-form expression. Finally, because $m = n - kp$, we have:

$$\mathrm{S}_{\left[\frac{n}{k}\right]}(r)=\left[\frac{n}{k}\right]^r\left(n+1-k\left[\frac{n}{k}\right]\right)+\frac{k}{r+1}\sum_{j=0}^{r}\left(-1\right)^j\binom{r+1}{j}B_j\left[\frac{n}{k}\right]^{r+1-j}. \tag{13}$$

Expressions (10) and (13) are identical, and thus the statement of the theorem is true. □

Using formula (13), it is possible to get the expression for the sum of extended floor functions for $a \in \mathrm{Z}$:

$$S_{\left[\frac{n+a}{k}\right]}(r) = \sum_{i=1}^{n+a} \left[\frac{i+a}{k}\right]^r - \sum_{i=1}^{a} \left[\frac{i}{k}\right]^r. \tag{14}$$

Using the above dependencies, it is possible to define formulas for finite series of the floor function of any power. As an example, we show these formulas for $r = 1$, 2, 3 and 4:

$$S_{[\frac{n}{k}]}(1) = \left[\frac{n}{k}\right]\Big(n+1-k\left[\frac{n}{k}\right]\Big) + k\Big(\frac{1}{2}\left[\frac{n}{k}\right]^2 - \frac{1}{2}\left[\frac{n}{k}\right]\Big) = \left[\frac{n}{k}\right]\Big(n+1-\frac{k}{2}\Big(\left[\frac{n}{k}\right]+1\Big)\Big), \tag{15}$$

$$\begin{aligned} S_{[\frac{n}{k}]}(2) &= \left[\frac{n}{k}\right]^2\Big(n+1-k\left[\frac{n}{k}\right]\Big) + k\Big(\frac{1}{3}\left[\frac{n}{k}\right]^3 - \frac{1}{2}\left[\frac{n}{k}\right]^2 + \frac{1}{6}\left[\frac{n}{k}\right]\Big) \\ &= \left[\frac{n}{k}\right]^2\Big(n+1-\frac{k}{2}\Big(\left[\frac{n}{k}\right]+1\Big)\Big) - \frac{1}{6}k\left[\frac{n}{k}\right]\Big(\left[\frac{n}{k}\right]^2 - 1\Big), \end{aligned} \tag{16}$$

$$\begin{aligned} S_{[\frac{n}{k}]}(3) &= \left[\frac{n}{k}\right]^3\Big(n+1-k\left[\frac{n}{k}\right]\Big) + k\Big(\frac{1}{4}\left[\frac{n}{k}\right]^4 - \frac{1}{2}\left[\frac{n}{k}\right]^3 + \frac{1}{4}\left[\frac{n}{k}\right]^2\Big) \\ &= \left[\frac{n}{k}\right]^3\Big(n+1-\frac{k}{2}\Big(\left[\frac{n}{k}\right]+1\Big)\Big) - \frac{1}{4}k\left[\frac{n}{k}\right]^2\Big(\left[\frac{n}{k}\right]^2 - 1\Big), \end{aligned} \tag{17}$$

$$\begin{aligned} S_{[\frac{n}{k}]}(4) &= \left[\frac{n}{k}\right]^4\Big(n+1-k\left[\frac{n}{k}\right]\Big) + k\Big(\frac{1}{3}\left[\frac{n}{k}\right]^5 - \frac{1}{2}\left[\frac{n}{k}\right]^4 + \frac{1}{3}\left[\frac{n}{k}\right]^3 - \frac{1}{30}\left[\frac{n}{k}\right]\Big) \\ &= \left[\frac{n}{k}\right]^4\Big(n+1-\frac{k}{2}\Big) - \frac{1}{30}k\left[\frac{n}{k}\right]\Big(24\left[\frac{n}{k}\right]^4 - 10\left[\frac{n}{k}\right]^2 + 1\Big). \end{aligned} \tag{18}$$

The above expressions can be easily obtained by using formula (10). Additionally, they can be proved by mathematical induction.

Theorem 2.2. *Formula* (10) *for $r = 1$ can be written as follows:*

$$S_n = \sum_{k}^{n} \left[\frac{n}{k}\right] = \left[\frac{n}{k}\right]\Big\{n+1-\frac{k}{2}\Big(\left[\frac{n}{k}\right]+1\Big)\Big\}. \tag{19}$$

Proof. We use induction. Each integer $n \geq k$ can be written as $n = kp + m$, where $p = \left[\frac{n}{k}\right]$, $0 \leq m \leq k-1$. Substituting the above expression into (9) we obtain

$$S_n = \left[\frac{kp+m}{k}\right]\Big\{kp+m+1-\frac{k}{2}\Big(\left[\frac{kp+m}{k}\right]+1\Big)\Big\}. \tag{20}$$

From the definition of the floor function we have $\left[\frac{kp+m}{k}\right] = p$, and thus

$$S_n = p\Big(\frac{kp}{2} + m + 1 - \frac{k}{2}\Big). \tag{21}$$

We can check this formula for $n = k$ $(p = 1,\ m = 0)$,

$$S_k = 1 \cdot \Big(\frac{k}{2} + 0 + 1 - \frac{k}{2}\Big) = 1. \tag{22}$$

For the case $m \le k - 2$, we can write

$$S_{n+1} = S_{kp+m+1} = \Big[\frac{kp+m+1}{k}\Big]\Big\{kp+m+1+1-\frac{k}{2}\Big(\Big[\frac{kp+m+1}{k}\Big]+1\Big)\Big\}, \tag{23}$$

and, after some manipulations,

$$S_{n+1} = p(\frac{kp}{2} + m + 2 - \frac{k}{2}). \tag{24}$$

From the other side, according to the series definition $S_{n+1} = S_n + \big[\frac{n+1}{k}\big]$, we obtain

$$\begin{aligned} S_{kp+m+1} = {} & \Big[\frac{kp+m}{k}\Big]\Big\{kp+m+1-\frac{k}{2}\Big(\Big[\frac{kp+m}{k}\Big]+1\Big)\Big\} \\ & + \Big[\frac{kp+m+1}{k}\Big] = p\Big(\frac{kp}{2} + m + 2 - \frac{k}{2}\Big). \end{aligned} \tag{25}$$

Both expressions are identical, which means that formula (19) is true.

For the case $m = k - 1$ we should check the expression $S_{n+1} = S_{kp+k}$. Substituting into (19) we obtain

$$S_{kp+k} = \Big[\frac{kp+k}{k}\Big]\Big\{kp+k+1-\frac{k}{2}(\Big[\frac{kp+k}{2}\Big]+1)\Big\}, \tag{26}$$

and, after some manipulations, this formula can be written as

$$S_{kp+k} = \Big(p+1\Big)\Big(\frac{kp}{2}+1\Big). \tag{27}$$

Now we get

$$S_{n+1} = S_n + \left[\frac{n+1}{k}\right] = S_{kp+k-1} + \left[\frac{kp+k}{2}\right], \tag{28}$$

and

$$\begin{aligned} S_{kp+k} = {} & \Big[\frac{kp+k-1}{k}\Big]\Big\{kp+k-1+1-\frac{k}{2}\Big(\Big[\frac{kp+k-1}{k}\Big]+1\Big)\Big\} \\ & + \Big[\frac{kp+k}{k}\Big] = \Big(p+1\Big)\Big(\frac{kp}{2}+1\Big). \end{aligned} \tag{29}$$

This proves that formula (19) is true. $\square$

Using the same method of the above proof, it is possible to prove formulas for the next sums of the floor functions of higher powers.

3. Summary

The formulas shown in this paper create the first step in the theory of the sums of the floor functions. The next step will include the expression for the sums of different forms of the floor functions.

Floor functions are widely applied in informatics (see, e.g. [2], [4]), but their application in number theory may lead to solving many significant problems. For example, the problem of "Partitio Numerorum", which has been analyzed since Euler without satisfactory results, might be solved using new results in the theory of floor functions.

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