A SUPERCONGRUENCE INVOLVING CUBES OF CATALAN NUMBERS

Roberto Tauraso
Dipartimento di Matematica, Università di Roma "Tor Vergata," Roma, Italy
tauraso@mat.uniroma2.it

Received: 7/16/19, Accepted: 5/19/20, Published: 5/26/20

Abstract

We mainly show a supercongruence for a truncated series with cubes of Catalan numbers which extends a result due to Zhi-Wei Sun.

1. Introduction

This short paper grew out as a complement to the author's recent paper [9], where various supercongruences for truncated series of ${}_3F_2(1)$ were provided.

Let us consider the sum

$$\sum_{k=0}^{n}\left(\frac{C_k}{4^k}\right)^d$$

where d is a positive integer and $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.

For $d = 1$ and $d = 2$, it is easy to find a closed formula:

$$\sum_{k=0}^{n}\frac{C_k}{4^k} = 2 - \frac{\binom{2n+1}{n}}{4^n} \quad \text{and} \quad \sum_{k=0}^{n}\frac{C_k^2}{16^k} = -4 + (5+4n)\frac{\binom{2n+1}{n}^2}{16^n}.$$

Thus, as $n \to \infty$, we are able to evaluate the series

$$\sum_{k=0}^{\infty}\frac{C_k}{4^k} = 2 \quad \text{and} \quad \sum_{k=0}^{\infty}\frac{C_k^2}{16^k} = -4 + \frac{16}{\pi}.$$

Moreover, for any prime $p > 2$, the above explicit formulas allow getting the congruences

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{4^k} \equiv_{p^2} 2 - 2(-1)^{(p-1)/2}p \quad , \quad \sum_{k=0}^{(p-1)/2}\frac{C_k^2}{16^k} \equiv_{p^3} -4 + 12p^2$$

where we use the notation $a \equiv_m b$ to mean $a \equiv b \pmod{m}$.

On the other hand, when $d = 3$, it seems that there is no closed formula for the partial sum. However, by Dixon's theorem [1, p.13], we find

$$\sum_{k=0}^{\infty} \frac{C_k^3}{64^k} = 8\left(1 - {}_3F_2\left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, 1 \end{matrix}; 1\right]\right) = 8 - \frac{384\pi}{\Gamma^4(\frac{1}{4})}.$$

What about the related congruence? We are going to show that for any prime $p > 2$, a surprisingly *similar* formula modulo p^3 holds:

$$\sum_{k=0}^{(p-1)/2} \frac{C_k^3}{64^k} \equiv_{p^3} \begin{cases} 8 - \dfrac{24p^2}{\Gamma_p^4\left(\frac{1}{4}\right)} & \text{if } p \equiv_4 1, \\ 8 - \dfrac{384}{\Gamma_p^4\left(\frac{1}{4}\right)} & \text{if } p \equiv_4 3, \end{cases} \tag{1}$$

where Γ_p is Morita's p-adic Gamma function which is defined as the continuous extension to the set of all p-adic integers $\mathbb{Z}_p$ of the sequence

$$n \to (-1)^n \prod_{\substack{0 \le k < n \\ (k,p)=1}} k$$

(see [6, Chapter 7] for a detailed introduction to Γ_p).

The above congruence (1) modulo p has been established by Zhi-Wei Sun in [7, Theorem 1.2]. Modulo p^2, the case $p \equiv_4 1$ is implied by [8, Theorem 1.3] (see also [4, Theorem 1.1]).

After some preliminary work done in the next section, the proof of (1) will be given in the third section.

2. A Bunch of Identities

The following one-parameter formula is the identity 6.34 in Gould's collection [3] (for $j = 0$ see [2]):

$$\sum_{k=0}^{n} A(n,k,j) = \frac{1 + (-1)^{n-j}}{2} \binom{n+j}{(n+j)/2} \binom{n-j}{(n-j)/2} \tag{2}$$

where

$$A(n,k,j) = \binom{n}{k}\binom{n+k}{k}\binom{2k}{k+j}(-4)^{n-k}.$$

We have that

$$\frac{A(n,k,1)}{A(n,k,0)} = 1 - \frac{1}{k+1},$$

and, by (2), we get

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}}{(-4)^k(k+1)}=\frac{\binom{2\lfloor n/2\rfloor}{\lfloor n/2\rfloor}^2}{16^{\lfloor n/2\rfloor}}\cdot\begin{cases}1 & \text{if } n\equiv_2 0,\\ \dfrac{n}{n+1} & \text{if } n\equiv_2 1.\end{cases}\tag{3}$$

In like manner, by using the partial fraction expansion,

$$\frac{A(n+1,k+1,1)-(n^2-n)A(n,k,2)}{A(n,k,0)}=4+n-n^2+\frac{4n^2+4n-2}{k+1}-\frac{2n(n+1)}{(k+1)^2},$$

and

$$\frac{A(n+1,k+1,0)}{A(n,k,0)}=4+\frac{8n+2}{k+1}+\frac{4n^2-2}{(k+1)^2}-\frac{2n(n+1)}{(k+1)^3},$$

yield respectively

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}}{(-4)^k(k+1)^2}=\frac{\binom{2\lfloor n/2\rfloor}{\lfloor n/2\rfloor}^2}{16^{\lfloor n/2\rfloor}}\cdot\begin{cases}2 & \text{if } n\equiv_2 0,\\ \dfrac{2n^2+2n-1}{(n+1)^2} & \text{if } n\equiv_2 1,\end{cases}\tag{4}$$

and

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}}{(-4)^k(k+1)^3}=-\frac{2}{n(n+1)}+\frac{\binom{2\lfloor n/2\rfloor}{\lfloor n/2\rfloor}^2}{16^{\lfloor n/2\rfloor}}\cdot\begin{cases}\dfrac{(2n+1)^2}{n(n+1)} & \text{if } n\equiv_2 0,\\ \dfrac{4n^4+8n^3+3n^2-n+1}{n(n+1)^3} & \text{if } n\equiv_2 1.\end{cases}\tag{5}$$

We would like to point out that, along the same lines, we are able to provide a closed formula for

$$\sum_{k=0}^{n}\frac{Q(k)\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}}{(-4)^k(k+1)^3}$$

where Q is any polynomial in $\mathbb{Z}[x]$. For example

$$\sum_{k=0}^{n}\frac{k^3\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}}{(-4)^k}=\frac{\binom{2\lfloor n/2\rfloor}{\lfloor n/2\rfloor}^2}{16^{\lfloor n/2\rfloor}}\cdot\begin{cases}\dfrac{n^2(n+1)^2(2n+1)^2}{15} & \text{if } n\equiv_2 0,\\ -\dfrac{n^2(4n^4+8n^3+3n^2-n+1)}{15} & \text{if } n\equiv_2 1.\end{cases}\tag{6}$$

Moreover, by using the same approach outlined in [9, Lemma 4.2], we get two key identities which play a major role in the proof of our main result. For any positive even number n,

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}H_{k+1}^{(2)}}{(-4)^k(k+1)^3}=\frac{\binom{n}{n/2}^2}{4^n}\left(16+\frac{(2n+1)^2\sum_{k=1}^{n}\frac{(-1)^k}{k^2}}{n(n+1)}\right)$$
$$-\frac{4^n}{\binom{n}{n/2}^2}\cdot\frac{4n^4+8n^3+3n^2-n+1}{n^3(n+1)^3}, \tag{7}$$

whereas for any positive odd number n,

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}H_{k+1}^{(2)}}{(-4)^k(k+1)^3}=\frac{\binom{n-1}{(n-1)/2}^2}{4^{n-1}}\left(\frac{(4n^4+8n^3+3n^2-n+1)\sum_{k=1}^{n}\frac{(-1)^k}{k^2}}{n(n+1)^3}\right.$$
$$\left.+\frac{16n^8+64n^7+92n^6+52n^5+5n^4-2n^3+2n^2+n+1}{n^3(n+1)^5}\right)$$
$$-\frac{4^{n-1}}{\binom{n-1}{(n-1)/2}^2}\cdot\frac{(2n+1)^2}{n^3(n+1)} \tag{8}$$

where $H_k^{(r)}=\sum_{j=1}^{k}1/j^r$ is the k-th harmonic number of order $r\geq 1$.

3. Proof of Supercongruence (1)

First of all, we need a stronger version of Lemma 3 of [9]: for any prime $p>3$,

$$\frac{\binom{2\lfloor n/2\rfloor}{\lfloor n/2\rfloor}^2}{16^{\lfloor n/2\rfloor}}\equiv_{p^2}\begin{cases}-\Gamma_p^4\left(\frac{1}{4}\right)\left(1+\frac{p^2E_{p-3}}{2}\right) & \text{if } p\equiv_4 1,\\ \frac{16+32p+p^2(48-8E_{p-3})}{\Gamma_p^4\left(\frac{1}{4}\right)} & \text{if } p\equiv_4 3,\end{cases} \tag{9}$$

where $n=(p-1)/2$ and E_k is the k-th Euler number. Indeed, for $p\equiv_4 3$,

$$\frac{\binom{2\lfloor n/2\rfloor}{\lfloor n/2\rfloor}^2}{16^{\lfloor n/2\rfloor}}=\frac{\Gamma_p^2\left(\frac{p+3}{4}\right)}{\left(\frac{p-1}{4}\right)^2\Gamma_p^2\left(\frac{p+1}{4}\right)}\equiv_{p^3}\frac{\Gamma_p^2\left(\frac{3}{4}\right)\left(1-\frac{(-1)^n}{16}p^2H_{\lfloor p/4\rfloor}^{(2)}\right)^2}{\left(\frac{p-1}{4}\right)^2\Gamma_p^2\left(\frac{1}{4}\right)}$$
$$\equiv_{p^3}\frac{16+32p+p^2(48-8E_{p-3})}{\Gamma_p^4\left(\frac{1}{4}\right)}$$

because, by [5, (20)], $H_{\lfloor p/4\rfloor}^{(2)}\equiv_p(-1)^n 4E_{p-3}$.

The proof for the case $p\equiv_4 1$ is similar:

$$\frac{\binom{2\lfloor n/2\rfloor}{\lfloor n/2\rfloor}^2}{16^{\lfloor n/2\rfloor}}=-\frac{\Gamma_p^2\left(\frac{p+1}{4}\right)}{\Gamma_p^2\left(\frac{p+3}{4}\right)}\equiv_{p^3}-\frac{\Gamma_p^2\left(\frac{1}{4}\right)\left(1+\frac{(-1)^n}{16}p^2H_{\lfloor p/4\rfloor}^{(2)}\right)^2}{\Gamma_p^2\left(\frac{3}{4}\right)}$$
$$\equiv_{p^3}-\Gamma_p^4\left(\frac{1}{4}\right)\left(1+\frac{p^2E_{p-3}}{2}\right).$$

Furthermore, for $0 \leq k \leq n$,

$$\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{2k}{k}\frac{\prod_{j=1}^{k}((2j-1)^2-p^2)}{4^k(2k)!} \equiv_{p^3} \frac{\binom{2k}{k}^2}{16^k}\left(1-p^2\sum_{j=1}^{k}\frac{1}{(2j-1)^2}\right).$$

Hence, by (5) and (9), congruence (1) is implied by

$$\sum_{k=0}^{(p-1)/2}\frac{C_k^3}{64^k}\sum_{j=1}^{k}\frac{1}{(2j-1)^2} \equiv_p \begin{cases} -8-\dfrac{24}{\Gamma_p^4\left(\frac{1}{4}\right)}-4\Gamma_p^4\left(\dfrac{1}{4}\right) & \text{if } p \equiv_4 1, \\ -8+\dfrac{192(5-E_{p-3})}{\Gamma_p^4\left(\frac{1}{4}\right)} & \text{if } p \equiv_4 3, \end{cases} \tag{10}$$

which is interesting in its own right. In order to show (10), notice that

$$\frac{C_k}{4^k} \equiv_p \frac{(-1)^k}{k+1}\binom{n}{k} = \frac{(-1)^k}{n+1}\binom{n+1}{k+1}$$

and, since $H_n^{(2)} \equiv_p 0$ (see [5, (19)]), it follows that

$$\sum_{j=1}^{k}\frac{1}{(2j-1)^2} \equiv_p \frac{1}{4}\sum_{j=1}^{k}\frac{1}{(n+1-j)^2} = \frac{H_n^{(2)}-H_{n-k}^{(2)}}{4} \equiv_p -\frac{H_{n-k}^{(2)}}{4}.$$

Then

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{C_k^3}{64^k}\sum_{j=1}^{k}\frac{1}{(2j-1)^2} &\equiv_p \frac{1}{4(n+1)^3}\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k+1}^3 H_{n+1-(k+1)}^{(2)} \\ &= \frac{(-1)^n}{4}\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\frac{H_{k+1}^{(2)}}{(k+1)^3}-\frac{H_{n+1}^{(2)}}{4(n+1)^3} \\ &\equiv_p \frac{(-1)^n}{4}\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}H_{k+1}^{(2)}}{(-4)^k(k+1)^3}-2H_{n+1}^{(2)}\end{aligned}$$

and we find (10) from (7), (8), (9),

$$H_{n+1}^{(2)} \equiv_p H_n^{(2)}+4 \equiv_p 0+4 \quad \text{and} \quad \sum_{k=1}^{n}\frac{(-1)^k}{k^2} = -H_n^{(2)}+\frac{H_{\lfloor p/4\rfloor}^{(2)}}{2} \equiv_p 0+(-1)^n 2E_{p-3}.$$

□

As a final remark, we notice that, following the same procedure, from identity (6), we find that for any prime $p > 5$,

$$\sum_{k=0}^{(p-1)/2}\frac{k^3\binom{2k}{k}^3}{64^k} \equiv_{p^3} \begin{cases} -\dfrac{p^2}{40\Gamma_p^4\left(\frac{1}{4}\right)} & \text{if } p \equiv_4 1, \\ -\dfrac{2}{5\Gamma_p^4\left(\frac{1}{4}\right)} & \text{if } p \equiv_4 3. \end{cases} \tag{11}$$

This congruence modulo p appeared in [7] whereas, the case $p \equiv_4 1$ modulo p^2 is implied by Theorem 1.3 of [8].

References

[1] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935.

[2] E. T. Bell, Problem 3457, *Amer. Math. Monthly* **37** (1930), 507–508.

[3] H. W. Gould, *Combinatorial Identities*, Morgantown, WV, 1972.

[4] G.-S. Mao and H. Pan, On the divisibility of some truncated hypergeometric series, arXiv:1801.02213v2 [math.NT].

[5] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Ann. of Math. (2)* **39** (1938), 350–360.

[6] A. Robert, *A Course in p-adic Analysis*, Springer-Verlag, New York, 2000.

[7] Z.-W. Sun, On congruences related to central binomial coefficients, *J. Number Theory* **131** (2011), 2219–2238.

[8] Z.-W. Sun, On sums of Apéry polynomials and related congruences, *J. Number Theory* **132** (2012), 2673–2690.

[9] R. Tauraso, Supercongruences related to ${}_3F_2(1)$ involving harmonic numbers, *Int. J. Number Theory* **14** (2018), 1093–1109.