ON THE FINITENESS OF CARMICHAEL NUMBERS WITH FERMAT FACTORS AND $L = 2^\alpha P^2$.

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Abstract
Let m be a Carmichael number and let L be the least common multiple of $p-1$, where p runs over the prime factors of m. We determine all the Carmichael numbers m with a known Fermat prime factor such that $L = 2^\alpha P^2$, where $\alpha \in \mathbb{N}$ and P is an odd prime number. There are eleven such Carmichael numbers.

1. Introduction

Fermat's little theorem states that a prime number p divides $a^p - a$ for any $a \in \mathbb{N}$. It would be interesting if prime numbers are the only integers (except for 1) having this property. Some composites m, however, satisfies $a^m \equiv a \pmod{m}$ for all integers a. Such a composite is called a *Carmichael number*. The smallest Carmichael number is 561 and it was found by Carmichael in 1910 [2]. Prior to Carmichael's discovery, Korselt [4] had provided the following simple test for Carmichael numbers in 1899:

(Korselt's criterion). *A composite m is a Carmichael number if and only if m is squarefree and $p-1$ divides $m-1$ for all prime divisors p of m.*

Let $L = L_m$ be the least common multiple of $p_i - 1$, where $m = p_1 \cdots p_n$ is the prime decomposition of a squarefree composite. Then Korselt's criterion can be rephrased as follows: m is a Carmichael number if and only if $L \mid m-1$.

It is known that there are infinitely many Carmichael numbers. The result was proved by Alfred, Granville, and Pomerance in 1994 [1] based on Erdős' heuristic argument [3]. The idea is to construct an integer L' so that it is divisible by $p-1$ for a large number of primes p. Then if the product $m = p_1 \cdots p_k$ of some of these primes is congruent to 1 modulo L', then m is a Carmichael number by Korselt's criterion since we have $L_m \mid L' \mid m-1$. [1] showed that in fact there are infinitely many such L' and such products, hence the infinitude of Carmichael numbers follows. Wright noted that as L' contains a sizable number of prime factors, it is likely that L_m corresponding to the Carmichael numbers m obtained in this manner contains many prime factors as well. This led him the study of Carmichael numbers with restricted L. Wright [5] proved that there are no Carmichael numbers with $L = 2^\alpha$ and determined all the Carmichael numbers with $L = 2^\alpha P$ for some odd prime P under the assumption that the Fermat primes conjecture is true. Assuming that 3, 5, 17, 257, and 65537 are the only Fermat primes, there are only eight Carmichael numbers with $L = 2^\alpha P$, and P is one of 3, 5, 7 or 127. The prime factorizations of these numbers are given in [5].

In this article, we extend Wright's result to the next simplest case $L = 2^\alpha P^2$. Let m be a Carmichael number with $L = 2^\alpha P^2$. Then each prime factor of m is one of the primes of the form $p = 2^k + 1$, $q = 2^l P + 1$, or $r = 2^s P^2 + 1$. (We call these prime factors *Type 1*, *Type 2*, and *Type 3* primes, respectively.) We assume that m is divisible by at least one of the known Fermat prime numbers. Under this assumption, we prove the following theorem.

Theorem 1. *Let m be a Carmichael number with $L = 2^\alpha P^2$ for some odd prime P. Suppose that m is divisible by one of the known Fermat primes and not divisible by any Fermat prime not yet known. Then m must be one of the following 11 Carmichael numbers. In particular, P is either* 5 *or* 3*.*

$$
\begin{array}{rl}
3 \cdot 11 \cdot 17 \cdot 401 \cdot 641 \cdot 1601 & L = 2^2 \cdot 5^2 \\
5 \cdot 7 \cdot 17 \cdot 19 \cdot 73 & L = 2^4 \cdot 3^2 \\
5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 257 & L = 2^8 \cdot 3^2 \\
5 \cdot 13 \cdot 257 \cdot 577 \cdot 1153 & L = 2^8 \cdot 3^2 \\
5 \cdot 37 \cdot 73 \cdot 257 \cdot 577 \cdot 769 & L = 2^8 \cdot 3^2 \\
5 \cdot 37 \cdot 73 \cdot 193 \cdot 257 \cdot 1153 & L = 2^8 \cdot 3^2 \\
5 \cdot 13 \cdot 257 \cdot 577 \cdot 1153 \cdot 18433 & L = 2^{11} \cdot 3^2 \\
5 \cdot 257 \cdot 37 \cdot 73 \cdot 577 \cdot 12289 \cdot 18433 & L = 2^{12} \cdot 3^2 \\
5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 193 \cdot 257 \cdot 577 \cdot 1153 & L = 2^8 \cdot 3^2 \\
5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 257 \cdot 577 \cdot 769 \cdot 1153 & L = 2^8 \cdot 3^2 \\
5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 769 \cdot 12289 \cdot 147457 \cdot 65537 \cdot 1179649 \cdot 786433 & L = 2^{18} \cdot 3^2
\end{array}
$$

In Section 2, we review a theorem (Theorem 2) that played an important role in [5] and will be used extensively in this article in a slightly extended form (Theorem 3). Suppose that $n = 2^\beta x + 1$ is an integer, where x is odd. Then we call the exponent β of 2 the 2-*power* of n. Theorem 2 proves that if a Carmichael number is expressed as the product of several odd integers, then there cannot be a unique smallest 2-power among these integers.

Several lemmas will be given in Section 3 that narrow down the number of possible primes P. In particular, we will see that a Carmichael number under our assumption must have at least two Fermat primes, and P is a divisor of $R - 1$, where R is the product of Fermat prime factors of m.

Section 4 provides a general procedure to obtain all Carmichael numbers for a given P. Theorem 3, which will be referred to as the minimality argument, implies the existence of some prime factor or a pair of prime factors of a Carmichael number with relatively small 2-power. As there are not so many possible such prime numbers or pairs, the procedure terminates and produces all Carmichael number for P, or proves that there are no Carmichael numbers for P.

By Theorem 2, at least two of the smallest 2-powers of Type 1, 2, and 3 primes must be the same. Hence there are three cases to consider according to which two 2-powers are the same.

The rest of the paper will be devoted to a careful scrutiny of Carmichael numbers in these three cases.

2. General Results on Carmichael Numbers

Let m be a Carmichael number. Let L be the least common multiple of $p-1$, where p runs over the prime factors of m. Korselt's criterion yields that L divides $m-1$.

The following result is crucial, which is proved in [5, Theorem 2.1]. For completeness, we give its proof.

Theorem 2. *Let m be a Carmichael number and write*

$$m = \prod_{i=1}^{n}(2^{\alpha_i} D_i + 1),$$

where D_i are odd integers, $n \geq 2$, and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Then if $2^{\alpha_1+1} \mid L$, then $\alpha_1 = \alpha_2$.

Proof. Seeking a contradiction, assume that $\alpha_1 < \alpha_2$. Then we have

$$m = \prod_{i=1}^{n}(2^{\alpha_i} D_i + 1) \equiv 2^{\alpha_1} D_1 + 1 \pmod{2^{\alpha_1+1}}.$$

Since $2^{\alpha_1+1} \mid L$ by assumption and $L \mid m-1$ by Korselt's criterion, we obtain

$$m \equiv 1 \pmod{2^{\alpha_1+1}}.$$

It follows that

$$2^{\alpha_1} D_1 + 1 \equiv 1 \pmod{2^{\alpha_1+1}}.$$

However, this implies that D_1 is even, which contradicts that D_1 is odd. Thus, we have $\alpha_1 = \alpha_2$. □

Observe that Theorem 2 does not assume that each factor $2^{\alpha_i} D_i + 1$ to be prime.

3. Lemmas

In this section, we prove several lemmas that reduce the possible factors of Carmichael numbers.

Let m be a Carmichael number. Let L be the least common multiple of $p-1$, where p runs over the prime factors of m. In this paper, we assume that

$$L = 2^{\alpha}P^2,$$

where $\alpha \in \mathbb{N}$ and P is an odd prime number. This implies that each of the prime factors of m is one of the followings:

$p = 2^k + 1$	Type 1 (Fermat's prime)
$q = 2^l P + 1$	Type 2
$r = 2^s P^2 + 1$	Type 3.

We further assume that m has at least one Type 1 prime factor.

From now on, we reserve the letters p, q, r (and the versions with subscripts p_i, q_i, r_i) for Type 1, 2, 3 primes, respectively. Similarly we reserve letters k, l, s for the exponent of 2 in these prime numbers.

Lemma 1. *Let P be an odd prime number. Suppose that 2^k+1, 2^lP+1, and 2^sP^2+1 are prime numbers. Then:*

1. *k is a power of 2.*
2. *If $P \equiv 1 \pmod 3$, then l is even.*
3. *If $P \equiv 2 \pmod 3$, then l is odd.*
4. *If $P \neq 3$, then s is even.*

Let $t = 2^a x + 1$ be a prime number, where x is an odd integer. Then we say that t is an *a-prime*. A pair (t_1, t_2) of a-prime numbers is called *a-pair*.

Corollary 1. *With the notation of Lemma 1, when $P \equiv 2 \pmod 3$, there is no a-pair between Type 2 and Type 3 primes for any integer a.*

In the next lemma, a Carmichael number m might or might not have Type 1 prime factors.

Lemma 2. *Let m be a Carmichael number with $L = 2^{\alpha}P^2$. Suppose that m is not of the form $p_1q_1r_1$ with $k_1 = l_1 = s_1$. If $P \equiv 1 \pmod 3$, then $P \not\equiv 1 \pmod 4$.*

Proof. Let

$$m = \prod_{i \geq 1} v_i,$$

where v_i is a prime of Type 1, 2, or 3. Note that any Carmichael number has three or more prime factors. Let β_i be the 2-power of v_i and suppose that $0 < \beta_1 \le \beta_2 \le \beta_3 \le \cdots$. Observe that $\beta_1 = \beta_2$ by Theorem 2 because of the assumption of the shape of m. Suppose that $P \equiv 1 \pmod 3$. Then both l_i and s_i are even by Lemma 1. Hence $k_1 \neq 1$, otherwise k_1 is the unique smallest power of 2 and this contradicts Theorem 2. So each k_i is even, and hence every β_i is even. Let

$$v_i = 2^{\beta_i} P^{\delta_i} + 1,$$

where $\delta_i = 0, 1, 2$ depending on the type of v_i. Then we have

$$\begin{aligned} v_1 v_2 &= (2^{\beta_1} P^{\delta_1} + 1)(2^{\beta_1} P^{\delta_2} + 1) \\ &= 2^{2\beta_1} P^{\delta_1+\delta_2} + 2^{\beta_1} P^{\delta_1} + 2^{\beta_1} P^{\delta_2} + 1 \\ &= 2^{\beta_1}(2^{\beta_1} P^{\delta_1+\delta_2} + P^{\delta_1} + P^{\delta_2}) + 1. \end{aligned}$$

Now seeking a contradiction, assume that $P \equiv 1 \pmod 4$. Then we have

$$2^{\beta_1} P^{\delta_1+\delta_2} + P^{\delta_1} + P^{\delta_2} \equiv 2 \pmod 4$$

since β_1 is even. It follows that the 2-power of $v_1 v_2$ is $\beta_1 + 1$.

Then consider the expression

$$m = (v_1 v_2) \prod_{i \ge 3} v_i.$$

If $\beta_1 < \beta_3$, then $\beta_3 \ge \beta_1 + 2$ as each β_i is even. Thus $2^{\beta_1+2} | L$ and 2^{β_1+1} is the unique smallest power of 2 in the above expression of m. This contradicts Theorem 2. Hence $\beta_3 = \beta_1$. Then we have $v_1 v_2 v_3 = p_1 q_1 r_1$ with $k_1 = l_1 = s_1$, and by assumption that $m \neq p_1 q_1 r_1$, there exists v_4 with $\beta_4 \ge \beta_1 + 2$. Then in the expression

$$m = v_3 (v_1 v_2) \prod_{i \ge 4} v_i,$$

v_3 has the unique smallest power β_1 of 2, and $2^{\beta_1+1} \mid 2^{\beta_4} \mid L$. This contradicts Theorem 2. Hence we conclude that $P \not\equiv 1 \pmod 4$. □

3.1. Fermat Primes and the Prime P

Lemma 3. *Let m be a Carmichael number with $L = 2^\alpha P^2$. Let $p_1, \ldots, p_n$ be the Type 1 prime divisors of m. Then*

$$p_1 \cdots p_n \equiv 1 \pmod P.$$

Proof. By Korselt's criterion, we have $P \mid L \mid m - 1$. It follows that

$$1 \equiv m \equiv p_1 \cdots p_n \pmod P$$

since Type 2 and Type 3 primes are congruent to 1 modulo P. □

Corollary 2. *Let m be a Carmichael number with $L = 2^\alpha P^2$. Assume that m has a Type 1 (Fermat) prime factor. Then m has at least two distinct Fermat prime factors.*

Proof. If there is a unique Type 1 divisor $p_1 = 2^{k_1} + 1$ of m, then Lemma 3 yields that

$$p_1 = 2^{k_1} + 1 \equiv 1 \pmod{P}.$$

Thus $P \mid 2^{k_1}$ and this is impossible because P is an odd prime. □

For the rest of the paper, we assume that m is a Carmichael number with $L = 2^\alpha P^2$ and m has a Type 1 factor. Also we restrict Type 1 factors to be known Fermat's primes: $p_i = 3, 5, 17, 257, 65537$.

By Lemma 3 and Corollary 2, the prime P appears in the prime factorization of $R - 1$, where R is the product of two or more known Fermat's primes.

Table 1 can be found in [5, Table 1], though we rearranged the table by k_1.

3.2. Cases

Let m be a Carmichael number with $L = 2^\alpha P^2$ and write

$$m = \left(\prod_{i=1}^{n_1} p_i\right)\left(\prod_{i=1}^{n_2} q_i\right)\left(\prod_{i=1}^{n_3} r_i\right) = \left(\prod_{i=1}^{n_1} 2^{k_i} + 1\right)\left(\prod_{i=1}^{n_2} 2^{l_i} P + 1\right)\left(\prod_{i=1}^{n_3} 2^{s_i} P^2 + 1\right)$$

with $k_1 < k_2 < \cdots < k_{n_1}$, $l_1 < l_2 < \cdots < l_{n_2}$, and $s_1 < s_2 < \cdots < s_{n_3}$. We are assuming $n_1 \geq 1$. By Theorem 2, there cannot be a unique smallest 2-power. Thus, there are three cases to consider.

`Case A` $k_1 = l_1 \leq s_1$.

`Case B` $k_1 = s_1 < l_1$.

`Case C` $l_1 = s_1 < k_1$.

We also consider the case when m has no Type 2 factor in Case B.

4. Procedure

The next theorem will play a key role in our procedure to find Carmichael numbers. This theorem holds for any Carmichael number.

Theorem 3 (The minimality argument). *Let m be a Carmichael number. Suppose that $2^a \mid L$ for some $a \in \mathbb{N}$. Write*

$$m = (2^b x + 1)(2^{b_1} x_1 + 1) \cdots (2^{b_f} x_f + 1) \prod_{i=1}^{g} v_i,$$

Table 1: Products of Fermat primes

Combination of Primes (R)	Factorization of $R-1$	k_1
$3*5$	$2*7$	1
$3*17$	$2*5^2$	1
$3*257$	$2*5*7*11$	1
$3*65537$	$2*5*19661$	1
$3*5*17$	$2*127$	1
$3*5*257$	$2*41*47$	1
$3*17*257$	$2*6553$	1
$3*5*65537$	$2*491527$	1
$3*17*65537$	$2*127*13159$	1
$3*257*65537$	$2*25264513$	1
$3*5*17*257$	$2*7*31*151$	1
$3*5*257*65537$	$2*7*18046081$	1
$3*5*17*65537$	$2*8355967$	1
$3*17*257*65537$	$2*19*22605091$	1
$3*5*17*257*65537$	$2*2147483647$	1
$5*17$	2^2*3*7	2
$5*257$	$2^2*3*107$	2
$5*65537$	$2^2*3*7*47*83$	2
$5*17*257$	$2^2*43*127$	2
$5*17*65537$	$2^2*131*10631$	2
$5*257*65537$	$2^2*467*45083$	2
$5*17*257*65537$	$2^2*3*7*11*31*151*331$	2
$17*257$	2^4*3^3*7*13	4
$17*65537$	2^4*3^3*2579	4
$17*257*65537$	$2^4*29*43*113*127$	4
$257*65537$	$2^8*3*7*13*241$	8

where $x, x_1, \ldots, x_f$ are odd integers, v_i is a prime factor of m, and $g \geq 1$. Let β_i be the 2-power of v_i. That is, $2^{\beta_i}D + 1 = v_i$ for some odd D. Assume that

$$b < b_1 \leq b_2 \leq \cdots \leq b_f \text{ and } \beta_1 \leq \beta_2 \leq \cdots \leq \beta_g.$$

1. *If $b < a$, then v_1 is a b-prime or (v_1, v_2) is a β_1-pair with $\beta_1 < b$.*
2. *If $b \geq a$, then either v_1 is a β_1-prime with $a \leq \beta_1 \leq b$ or we have a β_1-pair (v_1, v_2) with $\beta_1 < a$.*

Proof. 1. Since $b < a$, we have $2^{b+1} \mid 2^a \mid L$. It follows from Theorem 2 that b cannot be the unique smallest 2-power. Thus, we have the following possibilities: $b = \beta_1$ or $\beta_1 = \beta_2 < b$ by Theorem 2. The first case yield that v_1 is a b-prime. The second case implies that (v_1, v_2) is a β_1-pair and $\beta_1 < b$.

2. Assume that $\beta_1 > b$. Then we have $2^{b+1} \mid 2^{\beta_1} \mid L$. This implies that b is the unique smallest 2-power, which contradicts Theorem 2. Thus we have $\beta_1 \leq b$.

   If $\beta_1 < a$, then as $\beta_1 < a \leq b$ we must have $\beta_1 = \beta_2$ to avoid a unique smallest 2-power. Thus, in this case we have a β_1-pair (v_1, v_2).

$\square$

We explain the procedure to find all Carmichael numbers for a given P. In the sequel, we use the letter x to denote an odd number but its actual value could be different in each occurrence. For example, we write

$$5 \cdot 13 = (2^2x + 1)(2^2x + 1) = 2^6x + 1.$$

In this case, the actual values are $x = 1, 3, 1$ in this order.

Let us fix P. Then each of Case A, B, C, we start with two prime numbers with minimal 2-power together with Type 1 primes that give P. Let a be the largest integer among the a-prime factors of these known factors of m. Then $2^a \mid L$.

Step 1 We multiply some or all of these known factors and find the decomposition of the form $(2^{a_1}x + 1) \cdots (2^{a_n}x + 1)$, with $a_1 < a_2 \leq a_3 \leq \cdots \leq a_n$. It is possible that we have only one term. Then a Carmichael number is of the form

$$m = (2^{a_1}x + 1) \cdots (2^{a_n}x + 1) \prod v_i,$$

where v_i are primes of Type 1, 2, 3, or the product could be empty.

Step 2 Check whether $(2^{a_1}x + 1) \cdots (2^{a_n}x + 1)$ is a Carmichael number by Korselt's criterion. If the product $\prod_{i \geq 1} v_i$ from Step 1 is not empty, then it must contain a prime number or a pair of prime numbers satisfying some condition by the minimality argument (Theorem 3). There are only finitely many possible cases.

Step 3 We multiply some or all of the factors we obtained in Step 1 and Step 2. Then we have

$$m = (2^{b_1}x + 1) \cdots (2^{b_f}x + 1) \prod v_i,$$

where v_i can be a prime of Type 1, 2, 3 that did not appear in Step 1 and Step 2 (recall that every Carmichael number is squarefree), or the product $\prod v_i$ could be empty.

We repeat Step 2 and 3 until there is no possible prime or a pair of primes in Step 2.

We remark that there is no theoretical guarantee that this process terminates. However, our computation below terminates after a finite number of steps.

5. Case A: $k_1 = l_1 \leq s_1$.

We classify Carmichael numbers with $k_1 = l_1 \leq s_1$ in this section. We reduce the number of the possible P from Table 1 by removing those (P, k_1) such that $2^{k_1}P+1$ are not prime. The possible primes are

Table 2: After removing composite $2^{k_1}P+1$

Combination of Primes (R)	Possible factors P of $R-1$	k_1
$3*17$	5	1
$3*257$	$5, 11$	1
$3*65537$	$5, 19661$	1
$3*5*257$	41	1
$5*17$	$3, 7$	2
$5*257$	3	2
$5*65537$	$3, 7$	2
$5*17*257$	$43, 127$	2
$5*17*257*65537$	$3, 7$	2

$$P = 3, 5, 7, 11, 41, 43, 127, 19661.$$

In the sequel, we actually prove that none of these produces a Carmichael number except for $P = 3, 5$.

5.1. The Impossible Case: $P = 43$

As in [5], let us start with 43. We prove that there is no Carmichael number with $P = 43$. This case illustrates the procedure explained in Section 4.

Let m be a Carmichael number with $L = 2^\alpha \cdot 43^2$. Since $P = 43$, the product of Type 1 primes must be $5 \cdot 17 \cdot 257$ from Table 2.

Since $k_1 = l_1 = 2$ (recall we are dealing with Case A), we have

$$q_1 = 2^2 \cdot 43 + 1 = 173.$$

Let

$$m = 5 \cdot 17 \cdot 257 \cdot 173 \prod_{i \geq 1} u_i,$$

where u_i is either a Type 2 or Type 3 prime (and there must be at least one Type 3 prime.) Note that $2^8 \mid L$. Then we find the decomposition

$$m = (2^4+1)(2^5 \cdot 27+1)(2^8+1) \prod_{i \geq 1} u_i.$$

By the minimality argument (Theorem 3), there exists a pair (u_1, u_2) of Type 2 and Type 3 primes with 2-power less than 4 (we call such a pair *n-pair* with

$n < 4$), or there exists u_1 of Type 2 or Type 3 with 2-power 4 (we call such a prime *4-prime*). Since $k_1 = l_1 = 2$ is already used, the 2-power of a pair must be greater than 2 and less than 4. Thus the 2-power of a pair must be 3. However, since $43 \equiv 1 \pmod 3$, we know l_i is even by Lemma 1. So there is no such pair.

Note that the numbers

$$2^4 \cdot 43 + 1 = 13 \cdot 53 \text{ and } 2^4 \cdot 43^2 + 1 = 5 \cdot 61 \cdot 97$$

are composite. So there is no 4-prime of Type 2 or 3. We have proved the following result.

Theorem 4. *If $P = 43$, then there is no Carmichael number in Case A.*

5.2. The Impossible Case: $P = 19661$

Let us next consider the case $P = 19661$. From Table 2, the product of Type 1 primes is $3 \cdot 65537$ and $k_1 = 1$. Since $k_1 = l_1 = 1$, we have $q_1 = 2 \cdot 19661 + 1 = 39323$. Let

$$m = 3 \cdot 65537 \cdot 39323 \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2 or Type 3. Note that $2^{16} \mid L$ and $P \equiv 2 \pmod 3$, hence l_i is odd by Lemma 1. We have

$$m = (2^4 \cdot 7373 + 1)(2^{16} + 1) \prod_{i \geq 1} u_i.$$

By the minimality argument and Corollary 1, there is no pair (u_1, u_2) of Type 2 and 3, and there must be a 4-prime of Type 2 or 3 (and this should be Type 3 as l_i is odd). As the number $2^4 \cdot 19661^2 + 1 = 3217 \cdot 1922561$ is composite, there is no 4-prime as well. This proves:

Theorem 5. *If $P = 19661$, then there is no Carmichael number for Case A.*

5.3. The Impossible Case: $P = 41$

If $P = 41$, then the product of Type 1 primes is $3 \cdot 5 \cdot 257$ from Table 2 and $k_1 = l_1 = 1$. So $q_1 = 2 \cdot 41 + 1 = 83$. Note $2^8 \mid L$ and $P \equiv 2 \pmod 3$, hence l_i is odd. Let

$$m = 3 \cdot 5 \cdot 257 \cdot 83 \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2 or Type 3. Then we have

$$m = (2^2 + 1)(2^3 \cdot 31 + 1)(2^8 + 1) \prod_{i \geq 1} u_i.$$

Consider the minimality argument. By Corollary 1, there is no available pair (u_1, u_2). Since l_i is odd, the only candidate for 2-prime is the Type 3 2-prime. However, the number $2^2 \cdot 41^2 + 1 = 5^2 \cdot 269$ is not prime. Hence we have:

Theorem 6. *If $P = 41$, then there is no Carmichael number for Case A.*

5.4. The Impossible Case: $P = 11$

When $P = 11$, the product of Type 1 primes is $3 \cdot 257$ from Table 2. Then we have $k_1 = l_1 = 1$, and thus $q_1 = 2 \cdot 11 + 1 = 23$. Using $\prod_{i \geq} u_i$ as in the previous cases, we have

$$m = 3 \cdot 257 \cdot 23 \prod_{i \geq 1} u_i = (2 \cdot 17 + 1)(2^8 + 1) \prod_{i \geq 1} u_i.$$

Since s_1 is even by Lemma 1, there is no 1-prime among u_i. Thus, 2^1 is the unique smallest power and $2^8 \mid L$. We conclude that m cannot be a Carmichael number in this case. So we have:

Theorem 7. *If $P = 11$, then there is no Carmichael number for Case A.*

6. The Impossible Case: $P = 7$

We still assume $k_1 = l_1$ (Case A) and prove that there is no Carmichael number with $P = 7$. From Table 2, there are three possibilities for the Type 1 primes:

$$5 \cdot 17, \qquad 5 \cdot 65537, \qquad 5 \cdot 17 \cdot 257 \cdot 65537.$$

In any case, $k_1 = l_1 = 2$, and thus $q_1 = 2^2 \cdot 7 + 1 = 29$.

Let us first deal with the case when 17 divides m.

6.1. Case 1: 17 Divides m.

Suppose that 17 divides m. Let

$$m = 5 \cdot 17 \cdot 29 \prod_{i \geq 1} v_i = (2^5 \cdot 77 + 1) \prod_{i \geq 1} v_i,$$

where v_i is a prime of Type 1, 2, or 3. The only possible Type 1 primes for v_i are 257 and 65537. Note that $2^4 \mid L$. By the minimality argument (Theorem 3), either we must have an n-pair (v_1, v_2) with $n < 4$, or an n-prime with $n = 4$ or $n = 5$. Table 3 lists data of the numbers of the form $2^l \cdot 7 + 1$, $2^s \cdot 7^2 + 1$, and $2^k + 1$. Note that since $7 \equiv 1 \pmod 3$, both l_i, s_i are even by Lemma 1. From the table, we see that there is no such pair. As 17 is already used, we have a unique 4-prime: 113.

Hence we have

$$m = 5 \cdot 17 \cdot 29 \cdot 113 \prod_{i \geq 2} v_i = (2^4 \cdot 3 \cdot 7 \cdot 829 + 1) \prod_{i \geq 2} v_i.$$

Since the product still must contain a Type 3 prime, it is not empty. By the minimality argument, there must be an n-pair with $n < 4$ or 4-prime, but this is impossible from Table 3 as there is no more 4-primes.

Table 3: Primality for $2^l \cdot 7+1$, $2^s \cdot 7^2+1$, 2^k+1

n	$2^n \cdot 7+1$	$2^n \cdot 7^2+1$	2^n+1
2	29 prime	197 prime	5 prime
4	113 prime	×	17 prime
6	449 prime	3137 prime	×
8	×	×	257 prime
10	×	50177 prime	×
12	×	×	×
14	114689 prime	×	×
16	×	×	65537

×=composite, $7 \equiv 1 \pmod 3$

6.2. Case 2: 17 Does Not Divide m.

Suppose that m is not divisible by 17. Namely, the case when the type 1 combination is $5 \cdot 65537$. Then we have $k_1 = l_1 = 2$, and $q_1 = 2^2 \cdot 7 + 1 = 29$. Let

$$m = 5 \cdot 65537 \cdot 29 \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2 or 3. We have $2^{16} \mid L$. Then

$$m = (2^4 \cdot 9 + 1)(2^{16} + 1) \prod_{i \geq 1} u_i.$$

From Table 3, we see that there is no pair of u_i of 2-power smaller than 4. The only 4-prime u_1 is $q_2 = 113$ with $l_2 = 4$ (as we assume $17 \nmid m$). Since $5 \cdot 29 \cdot 113 = 2^{14} + 1$, we have

$$m = (2^{14} + 1)(2^{16} + 1) \prod_{i \geq 2} u_i$$

By the minimality argument, we must have either a pair (u_2, u_3) of Type 2 and 3 with 2-power between 5 and 13, or a 14-prime. From Table 3, we see that $(449, 3137)$ is a 6-pair and 114689 is a 14-prime.

6.2.1. Subcase 1: $(u_2, u_3) = (449, 3137)$

If $(u_2, u_3) = (449, 3137)$, then we have

$$\begin{aligned} m &= 5 \cdot 65537 \cdot 29 \cdot 113 \cdot 449 \cdot 3137 \prod_{i \geq 4} u_i \\ &= (2^9 \cdot 45075167 + 1)(2^{16} + 1) \prod_{i \geq 4} u_i. \end{aligned}$$

Since there is no 9-prime, and there is no n-pair for $7 \leq n \leq 8$ from Table 3, this case does not happen.

6.2.2. Subcase 2: $u_2 = q_3 = 114689$.

If $u_2 = q_3 = 114689$, then we have

$$\begin{aligned} m &= 5 \cdot 65537 \cdot 29 \cdot 113 \cdot 114689 \prod_{i \geq 3} u_i \\ &= (2^{16}+1)(2^{17} \cdot 3^5 \cdot 59 + 1) \prod_{i \geq 3} u_i. \end{aligned}$$

There is no 16-prime in Table 3. The only n-pair with $n < 16$ is $(449, 3137)$ but we dealt with this case in Subcase 1. Thus, there is no Carmichael numbers for Case 2 as well. This completes the proof of:

Theorem 8. *If $P = 7$, then there is no Carmichael number for Case A.*

7. The Impossible Case: $P = 127$

Next, we prove that there is no Carmichael number when $P = 127$ for $k_1 = l_1 \leq s_1$ (Case A).

From Table 2, there is only one Type 1 combination: $5 \cdot 17 \cdot 257$ with $k_1 = l_1 = 2$. So we have $q_1 = 2^2 \cdot 127 + 1 = 509$. Let

$$m = 5 \cdot 17 \cdot 257 \cdot 509 \prod_{i \geq 1} u_i = (2^9 \cdot 3^2 \cdot 19 \cdot 127 + 1) \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2 or Type 3. We have $2^8 \mid L$. Since $P = 127 \equiv 1 \pmod 3$, we know that l_i, s_i are even. By the minimality argument (Theorem 3), there is an n-pair with $n < 8$ or an n-prime with $n = 8$ or $n = 9$. From Table 4, we see that there is no such n-pair or n-prime. Thus, m cannot be a Carmichael number.

Table 4: Primality for $2^l \cdot 127 + 1$, $2^s \cdot 127^2 + 1$

n	$2^n \cdot 127 + 1$	$2^n \cdot 127^2 + 1$
2	509 prime	$\times$
4	$\times$	$\times$
6	$\times$	$\times$
8	$\times$	$\times$
10	$\times$	$\times$
12	520193 prime	$\times$
14	$\times$	$\times$
16	$\times$	$\times$

$\times$=composite, $127 \equiv 1 \pmod 3$

Thus we obtain:

Theorem 9. *If $P = 127$, then there is no Carmichael number for Case A.*

8. Case: $P = 5$

The combinations of Type 1 primes for $P = 5$ are the following three from Table 2:

$$3 \cdot 17, \qquad 3 \cdot 257, \qquad 3 \cdot 65537.$$

So in any case, we have $k_1 = l_1 = 1$, and hence $q_1 = 2 \cdot 5 + 1 = 11$. Since $P = 5 \equiv 2 \pmod 3$, we know that l_i is odd and s_i is even. In all cases, we have $2^4 \mid L$.

Let

$$m = 3 \cdot 11 \prod_{i \geq 1} v_i = (2^5 + 1) \prod_{i \geq 1} v_i,$$

where each v_i is a prime of Type 1, 2, or 3. If v_i is a Type 1 prime, then it must be one of $17, 257, 65537$.

Table 5: Primality for $2^l \cdot 5 + 1$, $2^s \cdot 5^2 + 1$

n	$2^n \cdot 5 + 1$	$2^n \cdot 5^2 + 1$	$2^n + 1$
1	11 prime	$\times$	3 prime
2	$\times$	101 prime	5 prime
3	41 prime	$\times$	$\times$
4	$\times$	401 prime	17 prime
5	$\times$	$\times$	$\times$

$\times$=composite, $5 \equiv 2 \pmod 3$

As $2^4 \mid L$, by the minimality argument (Theorem 3), there must be an n-pair $n < 4$ or an n-prime for $n = 4, 5$. From Table 5, there is no such n-pair or 5-prime. There are two 4-primes: 401 and 17.

8.1. Case 1: $r_1 = 401$.

Let $r_1 = 401$. Then we have

$$m = 3 \cdot 11 \cdot 401 \prod_{i \geq 2} v_i = (2^4 \cdot 827 + 1) \prod_{i \geq 2} v_i.$$

By the minimality argument, there must be the other 4-prime 17. So we have

$$m = 3 \cdot 11 \cdot 17 \cdot 401 \prod_{i \geq 3} v_i = (2^6 \cdot 5 \cdot 19 \cdot 37 + 1) \prod_{i \geq 3} v_i.$$

There is no n-pair for $n < 4$ and there is no n-prime with $n = 4, 5$. Thus, we must have a unique 6-prime: 1601. So we have

$$m = 3 \cdot 11 \cdot 17 \cdot 401 \cdot 1601 \prod_{i \geq 4} v_i = (2^8 \cdot 5 \cdot 53 \cdot 5309 + 1) \prod_{i \geq 4} v_i.$$

By the minimality argument, we must have an n-prime with $n = 5, 6, 7, 8$. The only possible case is a 7-prime: 641. We have

$$\begin{aligned} m &= 3 \cdot 11 \cdot 17 \cdot 401 \cdot 1601 \cdot 641 \prod_{i \geq 5} v_i \\ &= (2^7 \cdot 5^2 \cdot 53 \cdot 72145063 + 1) \prod_{i \geq 5} v_i. \end{aligned}$$

If the product is empty, then

$$\boxed{3 \cdot 11 \cdot 17 \cdot 401 \cdot 641 \cdot 1601}$$

is a Carmichael number since $L = 2^7 \cdot 5^2$ divides $m - 1 = 2^7 \cdot 5^2 \cdot 53 \cdot 72145063$. In fact, the product must be empty since we used all n-pairs with $n < 4$ and n-primes for $4 \leq n \leq 7$.

8.2. Case 2: $p_2 = 17$.

Let us now consider the case $p_2 = 17$. Then we have

$$m = 3 \cdot 11 \cdot 17 \prod_{i \geq 2} v_i = (2^4 \cdot 5 \cdot +1) \prod_{i \geq 2} v_i.$$

Since there is no possible pair, we must have the other 4-prime: 401. Hence this case reduces to the previous case.

This proves:

Theorem 10. *When $P = 5$, there is exactly one Carmichael number m with $L = 2^\alpha \cdot 5^2$ in Case A. The factorization of m is given by*

$$m = 3 \cdot 11 \cdot 17 \cdot 401 \cdot 641 \cdot 1601.$$

9. Case: $P = 3$

In this section, we consider Case A ($k_1 = l_1 \leq s_1$) with $P = 3$. From Table 2, there are four combinations of Type 1 primes for $P = 3$:

$$5 \cdot 17, \qquad 5 \cdot 257, \qquad 5 \cdot 65537, \qquad 5 \cdot 17 \cdot 257 \cdot 65537.$$

In any case, we have $k_1 = l_1 = 2$, and thus $q_1 = 2^2 \cdot 3 + 1 = 13$. We also have $2^4 \mid 2^{k_2} \mid L$. Recall that since $P = 3$, the exponents l_i, s_i cannot be restricted by their parities.

We first deal with $5 \cdot 17$ and $5 \cdot 17 \cdot 257 \cdot 65537$ together, and we consider the rest two cases individually.

9.1. Case 1: $17 \mid m$

Suppose that $17 \mid m$. Let

$$m = 5 \cdot 17 \cdot 13 \prod_{i \geq 1} v_i = (2^4 \cdot 3 \cdot 23 + 1) \prod_{i \geq 1} v_i,$$

where each v_i is a prime of Type 1, 2, or 3.

Table 6: Primality for $2^l \cdot 3 + 1$, $2^s \cdot 3^2 + 1$

n	$2^n \cdot 3 + 1$	$2^n \cdot 3^2 + 1$	$2^n + 1$
1	7 prime	19 prime	3 prime
2	13 prime	37 prime	5 prime
3	$\times$	73 prime	$\times$
4	$\times$	$\times$	17 prime
5	97 prime	$\times$	$\times$
6	193 prime	577 prime	$\times$
7	$\times$	1153 prime	$\times$
8	769 prime	$\times$	257 prime
9	$\times$	$\times$	$\times$
10	$\times$	$\times$	$\times$
11	$\times$	18433 prime	$\times$
12	12289 prime	$\times$	$\times$
13	$\times$	$\times$	$\times$
14	$\times$	147457 prime	$\times$
15	$\times$	$\times$	$\times$
16	$\times$	$\times$	65537 prime
17	$\times$	1179649 prime	$\times$
18	786433 prime	$\times$	$\times$
$\mid$	$\times$	$\times$	$\times$
30	3221225473 prime	$\times$	$\times$

$\times$=composite

After removing the used primes from Table 6, we see that there is no n-pair with $2 \leq n < 4$. Also, since 17 is already used, there is no 4-prime. Thus, in this case there is no Carmichael number.

9.2. Case 2: Type 1 Combination is $5 \cdot 257$

We next consider the case when the Type 1 combination is $5 \cdot 257$. Let

$$m = 5 \cdot 257 \cdot 13 \prod_{i \geq 1} u_i = (2^6 + 1)(2^8 + 1) \prod_{i \geq 1} u_i,$$

where each u_i is a prime of Type 2 or 3. We have $2^8 \mid L$. By the minimality argument, we must have an n-pair with $n < 6$ or a 6-prime. From Table 6, we find that there is no n-pair with $n < 6$. There are two 6-primes: 193 and 577.

9.2.1. Subcase 1: $q_2 = 193$

Suppose that $q_2 = 193$ with $l_2 = 6$. Then we have

$$m = 5 \cdot 257 \cdot 13 \cdot 193 \prod_{i \geq 2} u_i = (2^9 \cdot 3 \cdot 2099 + 1) \prod_{i \geq 2} u_i.$$

By the minimality argument, there is either an n-pair $n < 8$ or an 8- or 9-prime. According to Table 6, we only have the 8-prime 769. Then

$$m = 5 \cdot 257 \cdot 13 \cdot 193 \cdot 769 \prod_{i \geq 3} u_i = (2^8 \cdot 3 \cdot 102 \cdot 31963 + 1) \prod_{i \geq 3} u_i.$$

There must be an n-pair with $n < 8$ or an 8-prime by the minimality argument. However, there is no such pair or 8-prime. Hence $q_2 \neq 193$.

9.2.2. Subcase 2: $r_1 = 577$

Suppose that $r_1 = 577$ with $s_1 = 6$. Then we have

$$m = 5 \cdot 257 \cdot 13 \cdot 577 \prod_{i \geq 2} u_i = (2^7 \cdot 293 + 1)(2^8 + 1) \prod_{i \geq 2} u_i.$$

It follows from the minimality arguments that we must have a 7-prime as there is no n-pair $n < 7$. Thus we have $r_2 = 1153$ with $s_2 = 7$, and

$$m = 5 \cdot 257 \cdot 13 \cdot 577 \cdot 1153 \prod_{i \geq 3} u_i = (2^{11} \cdot 3^2 \cdot 602947 + 1) \prod_{i \geq 3} u_i.$$

If the product is empty, then

$$\boxed{m = 5 \cdot 13 \cdot 257 \cdot 577 \cdot 1153}$$

is a Carmichael number since $L = 2^8 \cdot 3^2$ divides $m - 1 = 2^{11} \cdot 3^2 \cdot 602947$. If the product is nonempty, then there is an n-prime with $n = 7, 8, 9, 10, 11$. From Table 6, there are two cases to consider: 8-prime 769 and 11-prime 18433.

9.2.2.1. Subsubcase 1: $q_2 = 769$. Suppose that $q_1 = 769$ with $l_2 = 8$. Then

$$\begin{aligned} m &= 5 \cdot 257 \cdot 13 \cdot 577 \cdot 1153 \cdot 769 \prod_{i \geq 3} u_i \\ &= (2^8 \cdot 3 \cdot 2113 \cdot 5266441 + 1) \prod_{i \geq 3} u_i. \end{aligned}$$

The product must be empty otherwise the minimality argument requires more n-pairs with $n < 8$ or 8-primes, but we used all relevant primes from Table 6. If the product is empty, then m is not a Carmichael number as $3^2 \nmid m - 1$.

9.2.2.2. Subsubcase 2: $r_3 = 18433$. Next, suppose that $r_3 = 18433$ with $s_3 = 11$. Then

$$\begin{aligned} m &= 5 \cdot 257 \cdot 13 \cdot 577 \cdot 1153 \cdot 18433 \prod_{i \geq 3} u_i \\ &= (2^{13} \cdot 3^2 \cdot 11 \cdot 29 \cdot 8710127 + 1) \prod_{i \geq 3} u_i. \end{aligned}$$

If the product is empty, then

$$\boxed{m = 5 \cdot 13 \cdot 257 \cdot 577 \cdot 1153 \cdot 18433}$$

is a Carmichael number since $L = 2^{11} \cdot 3^2$ divides $m - 1 = 2^{13} \cdot 3^2 \cdot 11 \cdot 29 \cdot 8710127$.

Suppose the product is nonempty. Since $2^{11} \mid L$, there is an n-prime with $n = 11, 12, 13$. The only remaining choice is the 12-prime 12289 and we have

$$\begin{aligned} m &= 5 \cdot 257 \cdot 13 \cdot 577 \cdot 1153 \cdot 18433 \cdot 12289 \prod_{i \geq 4} u_i \\ &= (2^{12} \cdot 3 \cdot 13477 \cdot 15201615259 \cdot 8710127 + 1) \prod_{i \geq 4} u_i. \end{aligned}$$

As there is no more 12-prime and possible pair, the product is empty. But then m is not a Carmichael number as $3^2 \nmid m - 1$.

In summary, there are exactly two Carmichael numbers

$$\boxed{5 \cdot 13 \cdot 257 \cdot 577 \cdot 1153} \text{ and } \boxed{5 \cdot 13 \cdot 257 \cdot 577 \cdot 1153 \cdot 18433}$$

in Case 2.

9.3. Case 3: Type 1 Combination is $5 \cdot 65537$

Finally, we consider the case when the Type 1 combination is $5 \cdot 65537$. We have $q_1 = 2^2 \cdot 3 + 1 = 13$. Let

$$m = 5 \cdot 65537 \cdot 13 \prod_{i \geq 1} u_i = (2^6 + 1)(2^{16} + 1) \prod_{i \geq 1} u_i,$$

where each u_i is a prime of Type 2 or 3. We have $2^{16} \mid L$.

By the minimality argument, the product $\prod_{i \geq 1} u_i$ must contain either an n-pair with $n < 5$ or a 6-prime. We see from Table 6 that there is no such pair. There are two 6-primes: 193 and 577. Now we consider these two cases individually.

9.3.1. Subcase 1: $q_2 = 193$

Let us consider the case $q_2 = 193$ with $l_2 = 6$. Then we have

$$m = 5 \cdot 65537 \cdot 13 \cdot 193 \prod_{i \geq 2} u_i = (2^8 \cdot 7^2 + 1)(2^{16} + 1) \prod_{i \geq 2} u_i.$$

As there is no available n-pair with $n < 7$, we must have an 8-prime: $q_3 = 769$ with $l_3 = 8$. Then we have

$$m = 5 \cdot 65537 \cdot 13 \cdot 193 \cdot 769 \prod_{i \geq 3} u_i = (2^{10} \cdot 9421 + 1)(2^{16} + 1) \prod_{i \geq 3} u_i.$$

Since there is no available n-pair with $n < 10$ and there is no 10-prime for $\prod_{i \geq 3} u_i$, this case does not happen.

9.3.2. Subcase 1: $r_1 = 577$

Next, suppose that $r_1 = 577$ with $s_1 = 6$. Then we have

$$m = 5 \cdot 65537 \cdot 13 \cdot 577 \prod_{i \geq 2} u_i = (2^7 \cdot 293 + 1)(2^{16} + 1) \prod_{i \geq 2} u_i.$$

Note that $2^{16} \mid L$. By the minimality argument with Table 6 yields that the 7-prime 1153 must be used. Thus $r_2 = 1153$ with $s_2 = 7$, and we have

$$\begin{aligned} m &= 5 \cdot 65537 \cdot 13 \cdot 577 \cdot 1153 \prod_{i \geq 3} u_i \\ &= (2^8 \cdot 31 \cdot 5449 + 1)(2^{16} + 1) \prod_{i \geq 3} u_i. \end{aligned}$$

Again the minimality argument implies that there is an 8-prime. So $q_2 = 769$ with $l_2 = 8$, and we have

$$\begin{aligned} m &= 5 \cdot 65537 \cdot 13 \cdot 577 \cdot 1153 \cdot 769 \prod_{i \geq 4} u_i \\ &= (2^9 \cdot 11 \cdot 29 \cdot 97 \cdot 2099 + 1)(2^{16} + 1) \prod_{i \geq 4} u_i. \end{aligned}$$

Since there is no available n-pair with $n < 8$ and there is no 9-prime from Table 6, m is not a Carmichael number in this case. In conclusion, we obtain:

Theorem 11. *Let m be a Carmichael number with $L = 2^\alpha 3^2$ for some $\alpha \in \mathbb{N}$. Assume that m has at least one known Fermat prime factor and $k_1 = l_1 \leq s_1$. Then m is one of the following two Carmichael numbers:*

$$\boxed{5 \cdot 13 \cdot 257 \cdot 577 \cdot 1153} \text{ and } \boxed{5 \cdot 257 \cdot 13 \cdot 577 \cdot 1153 \cdot 18433}.$$

This exhausts all the possible $P = 3, 5, 7, 11, 41, 43, 127, 19661$, and we complete the classification of Carmichael numbers for Case A ($k_1 = l_1 \leq s_1$).

10. Case B: $k_1 = s_1 < l_1$

We next consider Case B. So we assume that $k_1 = s_1 < l_1$ if a Type 2 factor exists. If m has no Type 2 factor, then we simply assume that $k_1 = l_1$. The first step is to reduce the number of possible pairs (P, k_1) from Table 1. We remove those (P, k_1) such that $2^{k_1}P^2 + 1$ is not prime. Note that s_1 is even if $P \neq 3$ by Lemma 1. Thus, we remove all $(P, 1)$ from Table 1.

The pairs (P, k_1) such that $2^{k_1}P^2 + 1$ are prime numbers are

$$(P, k_1) = (3, 2), (7, 2), (47, 2), (29, 4).$$

We list all the possible pairs (P, k_1) together with combinations of Type 1 primes in Table 7.

Table 7: The possible pairs (P, k_1) for Case B

Combination of Type 1 primes	(P, k_1)
$5 \cdot 17$	$(3, 2), (7, 2)$
$5 \cdot 257$	$(3, 2)$
$5 \cdot 65537$	$(3, 2)$, $(7, 2)$, $(47, 2)$
$5 \cdot 17 \cdot 257 \cdot 65537$	$(3, 2)$, $(7, 2)$
$17 \cdot 257 \cdot 65537$	$(29, 4)$

11. The Impossible Case: $P = 29$

Let us consider the case $P = 29$. We see from Table 7 that $17 \cdot 257 \cdot 65537$ is the only Type 1 combination for $P = 29$. Note that $2^{16} \mid L$. We have $k_1 = s_1 = 4$, and hence $r_1 = 2^4 \cdot 29^2 + 1 = 13457$. Let

$$m = 17 \cdot 257 \cdot 65537 \cdot 13457 \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2 or 3. We have

$$m = (2^5 \cdot 3 \cdot 2383 + 1)(2^8 + 1)(2^{16} + 1) \prod_{i \geq 1} u_i.$$

It follows from $P = 29 \equiv 2 \pmod 3$ that l_i is odd and s_i is even. Since the parities of l_i and s_i are distinct, there is no n-pair for any n. (Note that there is no available Type 1 primes.) Since $2^{16} \mid L$, by the minimality argument, there is a 5-prime (and it must be a Type 2 prime). Thus we have $q_1 = 2^5 \cdot 29 + 1 = 929$ with $l_1 = 5$. Then we have

$$m = 17 \cdot 257 \cdot 65537 \cdot 13457 \cdot 929 \prod_{i \geq 2} u_i$$
$$= (2^6 \cdot 5^2 \cdot 19 \cdot 6991 + 1)(2^8 + 1)(2^{16} + 1) \prod_{i \geq 2} u_i$$

Since $2^{16} \mid L$, the minimality argument requires a 6-prime. As 6 is even, the only possible 6-prime is of Type 3. However, the number $2^6 \cdot 29^2 + 1 = 5^2 \cdot 2153$ is composite. Thus we have:

Theorem 12. *If $P = 29$, then there is no Carmichael number for Case B.*

12. The Impossible Case: $P = 47$

We now consider the case $P = 47$. From Table 7, the Type 1 combination is $5 \cdot 65537$. So $k_1 = s_1 = 2$, and $r_1 = 2^2 \cdot 47^2 + 1 = 8837$. Let

$$m = 5 \cdot 65537 \cdot 8837 \prod_{i \geq 1} u_i = (2^3 \cdot 3 \cdot 7 \cdot 263 + 1)(2^{16} + 1) \prod_{i \geq 1} u_i,$$

where u_i is a Type 2 or 3 prime.

Since $P = 47 \equiv 2 \pmod 3$, Corollary 1 implies that there is no available n-pair. The only possible 3-prime is of Type 2 but $2^3 \cdot 47 + 1 = 13 \cdot 29$ is composite. This proves:

Theorem 13. *If $P = 47$, then there is no Carmichael number for Case B.*

13. The Impossible Case: $P = 7$

We next deal with the case $P = 7$ in Case B. The combinations of Type 1 primes for $P = 7$ are

$$5 \cdot 17, \qquad 5 \cdot 65537, \qquad 5 \cdot 17 \cdot 257 \cdot 65537.$$

We have $k_1 = s_1 = 2$, and thus $r_1 = 2^2 \cdot 7^2 + 1 = 197$. Note that $p_2 = 2^{k_2} + 1$ with $k_2 \geq 4$ in all cases. We have $2^4 \mid 2^{k_2} \mid L$. Let

$$m = 5 \cdot 197 \cdot p_2 \prod_{i \geq 1} v_i = (2^3 \cdot 3 \cdot 41 + 1)(2^{k_2} + 1) \prod_{i \geq 1} v_i,$$

where v_i is a prime of Type 1, 2, or 3, or the product could be empty.

Since $2^4 \mid L$, there must be an n-pair with $n < 3$ or a 3-prime otherwise 3 is the unique 2-power. Since $P = 7 \equiv 1 \pmod 3$, we know l_i, s_i are even. Thus, there is no 3-prime. Since we already used 2-primes p_1, r_1 with $k_1 = s_1 = 2$, there is no 2-pair. This proves:

Theorem 14. *If $P = 7$, then there is no Carmichael number for Case B.*

14. Case: $P = 3$ for Case B

The last possible P for Case B is $P = 3$. It follows from Table 7 that the combinations of Type 1 primes for $P = 3$ are

$$5 \cdot 17, \qquad 5 \cdot 257, \qquad 5 \cdot 65537, \qquad 5 \cdot 17 \cdot 257 \cdot 65537.$$

In all cases, we have $k_1 = s_1 = 2$, $k_2 \geq 4$, and thus $r_1 = 2^2 \cdot 3^2 + 1 = 37$ and $2^4 \mid L$.

14.1. Case 1: $17 \mid m$

Let us fist consider the case when $17 \mid m$. Let

$$m = 5 \cdot 17 \cdot 37 \prod_{i \geq 1} v_i = (2^3 \cdot 23 + 1)(2^4 + 1) \prod_{i \geq 1} v_i,$$

where v_i is a prime of Type 1, 2, or 3. By the minimality argument, there is an n-pair $n < 4$ or a 3-prime. From Table 6, we see that there is no such pair and the only 3-prime is 73. Then we have

$$m = 5 \cdot 17 \cdot 37 \cdot 73 \prod_{i \geq 2} v_i = (2^4 \cdot 3 \cdot 4783 + 1) \prod_{i \geq 2} v_i.$$

Since there are no 4-primes and possible n-pairs, the product must be empty. But if the product is empty, then m is not a Carmichael number as 3^2 does not divide $m - 1$.

14.2. Case 2: The Type 1 Combination is $5 \cdot 257$

We next consider the case when the Type 1 combination is $5 \cdot 257$. Then

$$m = 5 \cdot 257 \cdot 37 \prod_{i \geq 1} u_i = (2^3 \cdot 3 \cdot 7 \cdot 283 + 1) \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2, or 3. Note that $2^8 \mid L$. By the minimality argument, we have the 3-prime 73. So

$$m = 5 \cdot 257 \cdot 37 \cdot 73 \prod_{i \geq 2} u_i = (2^6 \cdot 3 \cdot 18077 + 1) \prod_{i \geq 2} u_i.$$

If the product is empty, then m is not a Carmichael number as 3^2 does not divide $m - 1$. If the product is nonempty, then since there is no n-pair $n < 6$, there must be a 6-prime. There are two 6-primes from Table 6: 193 and 577.

14.2.1. Subcase 1: $q_1 = 193$

Suppose that $q_1 = 193$. Then we have

$$m = 5 \cdot 257 \cdot 37 \cdot 73 \cdot 193 \prod_{i \geq 3} u_i = (2^7 \cdot 3^3 \cdot 13 \cdot 44729 + 1) \prod_{i \geq 3} u_i.$$

If the product is empty, then m is not a Carmichael number as $2^8 \mid L$ but 2^8 does not divide $m-1$. By the minimality argument, there is a 7-prime as no possible n-pair exists. The unique 7-prime is 1153. So we have

$$\begin{aligned} m &= 5 \cdot 257 \cdot 37 \cdot 73 \cdot 193 \cdot 1153 \prod_{i \geq 4} u_i \\ &= (2^8 \cdot 3^3 \cdot 11 \cdot 10158227 + 1) \prod_{i \geq 4} u_i. \end{aligned}$$

If the product is empty, then

$$\boxed{m = 5 \cdot 37 \cdot 73 \cdot 193 \cdot 257 \cdot 1153}$$

is a Carmichael number since $L = 2^8 \cdot 3^2$ divides $m - 1 = 2^8 \cdot 3^3 \cdot 11 \cdot 10158227$.

If the product is nonempty, then there must be an 8-prime, which is 769. Then

$$m = 5 \cdot 257 \cdot 37 \cdot 73 \cdot 193 \cdot 1153 \cdot 769 \prod_{i \geq 5} u_i = (2^9 \cdot 3 \cdot 19 \cdot 20351473151 + 1) \prod_{i \geq 5} u_i.$$

As there is no 9-prime or possible n-pair, the product must be empty. Then m is not a Carmichael number as $3^2 \nmid m - 1$.

14.2.2. Subcase 2: $r_3 = 577$

Suppose now that $r_3 = 577$. Then

$$m = 5 \cdot 257 \cdot 37 \cdot 73 \cdot 577 \prod_{i \geq 3} u_i = (2^{11} \cdot 3 \cdot 325951 + 1) \prod_{i \geq 3} u_i.$$

If the product is empty, then m is not a Carmichael number as $3^2 \nmid m - 1$. If the product is nonempty, then by the minimality argument with $2^8 \mid L$, there is an n-pair with $n < 8$ or an n-prime with $n = 8, 9, 10, 11$. Looking at Table 6, we see that there is no possible such pair. We have the 8-prime 769 and the 11-prime 18433.

14.2.2.1. Subsubcase 1: 8-prime 769 Suppose that 769 is a divisor of m. Then

$$m = 5 \cdot 257 \cdot 37 \cdot 73 \cdot 577 \cdot 769 \prod_{i \geq 4} u_i = (2^8 \cdot 3^4 \cdot 283 \cdot 262433 + 1) \prod_{i \geq 4} u_i.$$

If the product is empty, then

$$\boxed{m = 5 \cdot 37 \cdot 73 \cdot 257 \cdot 577 \cdot 769}$$

is a Carmichael number since $L = 2^8 \cdot 3^2$ divides $m - 1 = 2^8 \cdot 3^4 \cdot 283 \cdot 262433$.

If the product is nonempty, then the minimality argument implies that m has an 8-prime. However, there is no more available 8-primes. Thus, in this case m is not a Carmichael number.

14.2.2.2. Subsubcase 2: 11-prime 18433 Next, suppose that 18433 is a divisor of m. Then

$$m = 5 \cdot 257 \cdot 37 \cdot 73 \cdot 577 \cdot 18433 \prod_{i \geq 4} u_i = (2^{12} \cdot 3 \cdot 3004127393 \cdot +1) \prod_{i \geq 4} u_i.$$

If the product is empty, then m is not a Carmichael number as $3^2 \nmid m - 1$. If the product is nonempty, then the minimality argument yields that there is an n-prime with $n = 8, 9, 10, 11, 12$. From Table 6, we have the 8-prime 769 and the 12-prime 12289. The former case was already dealt with in the previous case. So, we consider the case $12289 \mid m$. Then we have

$$\begin{aligned} m &= 5 \cdot 257 \cdot 37 \cdot 73 \cdot 577 \cdot 18433 \cdot 12289 \prod_{i \geq 5} u_i \\ &= (2^{13} \cdot 3^2 \cdot 2357 \cdot 2610502159 + 1) \prod_{i \geq 5} u_i. \end{aligned}$$

If the product is empty, then

$$\boxed{m = 5 \cdot 257 \cdot 37 \cdot 73 \cdot 577 \cdot 12289 \cdot 18433}$$

is a Carmichael number since $L = 2^{12} \cdot 3^2$ divides

$$m - 1 = 2^{13} \cdot 3^2 \cdot 2357 \cdot 2610502159.$$

Suppose that the product is nonempty. Then since $2^{12} \mid L$, by the minimality argument, there must be an n-pair with $n < 11$ or 12-prime or 13-prime. Table 6 shows that this is not the case.

14.3. Case 3: The Type 1 Combination is $5 \cdot 65537$

The last case for $P = 3$ in Case B is when the Type 1 combination is $5 \cdot 65537$. Then

$$m = 5 \cdot 65537 \cdot 37 \prod_{i \geq 1} u_i = (2^3 \cdot 23 + 1)(2^{16} + 1) \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2 or 3. Note that $2^{16} \mid L$. By the minimality argument, the product cannot be empty, and there must be a 3-prime. The only 3-prime is 73 from Table Table 6. Then

$$m = 5 \cdot 65537 \cdot 37 \cdot 73 \prod_{i \geq 2} u_i = (2^6 \cdot 211 + 1)(2^{16} + 1) \prod_{i \geq 2} u_i.$$

Again, the product cannot be empty, and there must be a 6-prime. There are two 6-primes: 193 and 577.

14.3.1. Subcase 1: 6-prime is 193

Suppose that $193 \mid m$. Then we have

$$m = 5 \cdot 65537 \cdot 37 \cdot 73 \cdot 193 \prod_{i \geq 3} u_i = (2^7 \cdot 7 \cdot 2909 + 1)(2^{16} + 1) \prod_{i \geq 3} u_i.$$

Furthermore, the product cannot be empty, and there must be a 7-prime. The only 7-prime is 1153 and we have

$$m = 5 \cdot 65537 \cdot 37 \cdot 73 \cdot 193 \cdot 1153 \prod_{i \geq 3} u_i = (2^9 \cdot 641 \cdot 9157 + 1)(2^{16} + 1) \prod_{i \geq 3} u_i.$$

And again, by the minimality argument the product is not empty, but there is no 9-prime or admissible pairs.

14.3.2. Subcase 2: 6-prime is 577

Next suppose that $577 \mid m$. Then we have

$$m = 5 \cdot 65537 \cdot 37 \cdot 73 \cdot 577 \prod_{i \geq 3} u_i = (2^8 \cdot 61 \cdot 499 + 1)(2^{16} + 1) \prod_{i \geq 3} u_i.$$

By the minimality argument, the 8-prime 769 divides m. Then

$$m = 5 \cdot 65537 \cdot 37 \cdot 73 \cdot 577 \cdot 769 \prod_{i \geq 3} u_i = (2^9 \cdot 7^2 \cdot 238853 + 1)(2^{16} + 1) \prod_{i \geq 3} u_i.$$

There is no 9-prime or admissible pairs. Hence, there is no Carmichael number in this case. In summary, we obtain:

Theorem 15. *Let m be a Carmichael number with $L = 2^\alpha \cdot 3^2$. In Case B ($k_1 = s_1 < l_1$), the following three are all Carmichael numbers of this type:*

$$m = 5 \cdot 37 \cdot 73 \cdot 193 \cdot 257 \cdot 1153$$
$$m = 5 \cdot 37 \cdot 73 \cdot 257 \cdot 577 \cdot 769$$
$$m = 5 \cdot 37 \cdot 73 \cdot 257 \cdot 577 \cdot 12289 \cdot 18433$$

15. Case C: $l_1 = s_1 < k_1$

Let m be a Carmichael number with $L = 2^\alpha P^2$ as before. In this section, we consider Case C: $l_1 = s_1 < k_1$. Note that $1 \le l_1 < k_2$. From Table 1, the possible values for k_1 are $2, 4, 8$.

16. Case 1: $k_1 = 2$

Consider the case $k_1 = 2$. Then we must have $l_1 = s_1 = 1$. By Lemma 1, this yields that $P = 3$ as s_1 is odd. Then we have $q_1 = 2 \cdot 3 + 1 = 7$ and $r_1 = 2 \cdot 3^2 + 1 = 19$. From Table 1, the Type 1 combinations for $P = 3$ with $k_1 = 2$ are

$$5 \cdot 17, \qquad 5 \cdot 257, \qquad 5 \cdot 65537, \qquad 5 \cdot 17 \cdot 257 \cdot 65537.$$

In all cases, we have $2^4 \mid L$. Let

$$m = 5 \cdot 7 \cdot 19 \prod_{i \ge 1} v_i = (2^3 \cdot 83 + 1) \prod_{i \ge 1} v_i,$$

where v_i is a prime of Type 1, 2, or 3, or it could be empty. As $2^4 \mid L$, by the minimality argument (Theorem 3) there must be a 2-pair or a 3-prime in $\prod_{i \ge 1} v_i$. We see from Table 6 that $(13, 37)$ is the only 2-pair and 73 is the only 3-prime.

16.1. Subcase 1: the 2-pair $(13, 37)$

Suppose that m has the 2-pair $(13, 37)$. Then

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \prod_{i \ge 3} v_i = (2^3 \cdot 39983 + 1) \prod_{i \ge 3} v_i.$$

By the minimality argument, m has the 3-prime 73. Then

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \prod_{i \ge 3} v_i = (2^7 \cdot 182423 + 1) \prod_{i \ge 3} v_i.$$

Note that we have used all possible primes with 2-power less than 4 (See Table 6) and 17 is the unique 4-prime. Thus $17 \nmid m$, otherwise 2^4 is the unique smallest power of 2. It follows that $2^8 \mid L$. Hence by the minimality argument, there is either an n-pair with $n < 7$ or a 7-prime. Thus, there are two cases: the 6-pair $(193, 577)$ and the 7-prime 1153.

16.1.1. Subsubcase 1: the 6-pair $(193, 577)$

Suppose that m has the 6-pair $(193, 577)$. Then we have

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 193 \cdot 577 \prod_{i \ge 5} v_i = (2^7 \cdot 2003 \cdot 10142191 + 1) \prod_{i \ge 5} v_i.$$

As $2^8 \mid L$, the minimality arguments yields that m has the 7-prime 1153. Then

$$\begin{aligned} m &= 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 193 \cdot 577 \cdot 1153 \prod_{i \geq 6} v_i \\ &= (2^8 \cdot 1092373 \cdot 10721143 + 1) \prod_{i \geq 6} v_i. \end{aligned}$$

If the product $\prod_{i \geq 6} v_i = 1$, then m is not a Carmichael number because $3^2 \nmid m - 1$. Thus the product is nonempty. As there is no more admissible pairs, there must be a 8-prime. There are two 8-primes: 257 and 769

16.1.1.1. Subsubsubcase 1: the 8-prime: 257 Suppose that $257 \mid m$. Then we have

$$\begin{aligned} m &= 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 193 \cdot 577 \cdot 1153 \cdot 257 \prod_{i \geq 7} v_i \\ &= (2^{10} \cdot 3^2 \cdot 83607005432809 + 1) \prod_{i \geq 7} v_i. \end{aligned}$$

If the product is empty, then

$$\boxed{m = 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 193 \cdot 257 \cdot 577 \cdot 1153}$$

is a Carmichael number since $L = 2^8 \cdot 3^2$ divides $m - 1 = 2^{10} \cdot 3^2 \cdot 83607005432809$.

Note that $2^8 \mid L$. If the product is nonempty, then there is an n-prime with $n = 8, 9, 10, 11$ as there is no admissible pair. So m has the 8-prime 769, and then

$$\begin{aligned} m &= 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 193 \cdot 577 \cdot 1153 \cdot 257 \cdot 769 \prod_{i \geq 7} v_i \\ &= (2^8 \cdot 3 \cdot 11 \cdot 31 \cdot 2262537965202233 + 1) \prod_{i \geq 7} v_i. \end{aligned}$$

Since there is no more admissible pair or 8-prime, the product is empty, but then m is not a Carmichael number as $3^2 \nmid m - 1$.

16.1.1.2. Subsubsubcase 2: the 8-prime: 769 Next, suppose that $769 \mid m$. Then

$$\begin{aligned} m &= 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 193 \cdot 577 \cdot 1153 \cdot 769 \prod_{i \geq 7} v_i \\ &= (2^9 \cdot 4503066806229347 + 1) \prod_{i \geq 7} v_i. \end{aligned}$$

If the product is empty, then m is not a Carmichael number as $3 \nmid m - 1$. Then by the minimality argument, there is an 8-prime or a 9-prime. Since there is no 9-prime, m must have the 8-prime 257. Then this case reduces to the previous case.

16.1.2. Subsubcase 2: the 7-prime 1153

Now suppose that $r_4 = 1153 \mid m$. Then

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 1153 \cdot 577 \prod_{i \geq 5} v_i = (2^{12} \cdot 11 \cdot 597539 + 1) \prod_{i \geq 5} v_i.$$

Since $2^8 \mid L$, by the minimality argument, there is an n-prime with $8 \leq n \leq 12$. Those are 8-primes 769, 257, the 11-prime 18433, and the 12-prime 12289 from Table 6.

16.1.2.1. Subsubsubcase 1: the 8-prime 769 When $769 \mid m$, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 1153 \cdot 577 \cdot 769 \prod_{i \geq 6} v_i = (2^8 \cdot 89 \cdot 1483 \cdot 612737 + 1) \prod_{i \geq 6} v_i.$$

If the product is empty, then m is not a Carmichael number as $3 \nmid m - 1$. Thus it has the 8-prime 257, and then

$$\begin{aligned} m &= 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 1153 \cdot 577 \cdot 769 \cdot 257 \prod_{i \geq 7} v_i \\ &= (2^{10} \cdot 3^2 \cdot 61 \cdot 9464682529 + 1) \prod_{i \geq 6} v_i. \end{aligned}$$

The product must be empty since there is no n-prime for $n = 8, 9, 10$. Then

$$\boxed{m = 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 257 \cdot 577 \cdot 769 \cdot 1153}$$

is a Carmichael number since $L = 2^8 \cdot 3^2$ divides $m - 1$.

16.1.2.2. Subsubsubcase 2: the 8-prime 257 When $257 \mid m$, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 1153 \cdot 577 \cdot 257 \prod_{i \geq 6} v_i = (2^8 \cdot 3 \cdot 43 \cdot 209518481 + 1) \prod_{i \geq 6} v_i.$$

If the product is empty, then m is not a Carmichael number as $3^2 \nmid m - 1$. Thus m has the 8-prime 769, and this case reduces to the previous case.

16.1.2.3. Subsubsubcase 3: the 11-prime 18433 If $18433 \mid m$, then

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 1153 \cdot 577 \cdot 257 \cdot 18433 \prod_{i \geq 6} v_i = (2^{11} x + 1) \prod_{i \geq 6} v_i,$$

where x is an odd number. (From now on, to save space, I use x for an odd number, especially when m is not a Carmichael number.) Note that $2^{11} \mid L$ but there is no more 11-prime. Thus the product must be empty. If the product is empty, then m is not a Carmichael number since $3^2 \nmid m - 1$.

16.1.2.4. Subsubsubcase 4: 12-prime 12289 When $12289 \mid m$, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 1153 \cdot 577 \cdot 257 \cdot 12289 \prod_{i \geq 6} v_i = (2^{14}x + 1) \prod_{i \geq 6} v_i.$$

Since $2^{12} \mid L$, there is an n-prime with $n = 12, 13, 14$. From Table 6, such a prime must be the 14-prime 147457. Then

$$m = 5 \cdot 7 \cdot 19 \cdot 13 \cdot 37 \cdot 73 \cdot 1153 \cdot 577 \cdot 257 \cdot 12289 \cdot 147457 \prod_{i \geq 7} v_i = (2^{15}x + 1) \prod_{i \geq 7} v_i$$

and $2^{14} \mid L$. Since there is no 14-prime and 15-prime, the product must be empty. But then m is not a Carmichael number as $3 \nmid m - 1$.

16.2. Subcase 2: the 3-prime 73

Next, we consider the case $r_2 = 73$ with $s_2 = 3$. Then

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \prod_{i \geq 2} v_i = (2^5 \cdot 37 \cdot 41 + 1) \prod_{i \geq 2} v_i,$$

Since $2^4 \mid L$, by the minimality argument, there must be either the 2-pair $(13, 37)$ or an n-prime with $n = 4, 5$. The former was already considered in Section 16.1. From Table 6, there are the 4-prime 17 and 5-prime 97.

16.2.1. Subsubcase 1: the 4-prime 17

When $17 \mid m$, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 17 \prod_{i \geq 3} v_i = (2^4 \cdot 3^2 \cdot 11 \cdot 521 + 1) \prod_{i \geq 3} v_i,$$

Since there is no more 4-prime, the product is empty. Then

$$\boxed{m = 5 \cdot 7 \cdot 17 \cdot 19 \cdot 73}$$

is a Carmichael number since $L = 2^4 \cdot 3^2$ divides $m - 1$.

16.2.2. Subsubcase 2: the 5-prime 97

When $97 \mid m$, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \prod_{i \geq 3} v_i = (2^9 \cdot 17 \cdot 541 + 1) \prod_{i \geq 3} v_i.$$

If the product is empty, then m is not a Carmichael number as $3 \nmid m - 1$. Since $2^5 \mid L$, by the minimality argument, there is an n-prime with $n = 5, 6, 7, 8$. From Table 6, there are five possible such prime numbers: 6-primes 193, 577, the 7-prime 1153, and 8-primes 769, 257. We consider these cases individually below.

Convention: We use x for an odd number such that $2^a x + 1$ is not a Carmichael number.

16.2.2.1. Subsubsubcase 1: 6-primes 193 When 193 | m, then

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \prod_{i \geq 4} v_i = (2^6 x + 1) \prod_{i \geq 4} v_i.$$

Our convention is that x is an odd number and 2^6x+1 is not a Carmichael number. By the minimality argument, the 6-prime 577 must be used. Then

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \prod_{i \geq 5} v_i = (2^8 x + 1) \prod_{i \geq 5} v_i.$$

Since $2^6 \mid L$, the possible values for v_6 are the 7-prime 1153 and 8-primes 769, 257.

16.2.2.1.1. Case 1153 | m When 1153 | m, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 1152 \prod_{i \geq 6} v_i = (2^7 x + 1) \prod_{i \geq 6} v_i.$$

There is no other 7-prime. No Carmichael number exists in this case.

16.2.2.1.2. Case 769 | m When 769 | m, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 769 \prod_{i \geq 6} v_i = (2^{12} x + 1) \prod_{i \geq 6} v_i.$$

Note that $2^8 \mid L$. There must be an n-prime with $n = 8, 9, 10, 11, 12$. From Table 6, these are the 8-prime 257, the 11-prime 18433, and the 12-prime 12289.

1. When 257 | m, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 769 \cdot 257 \prod_{i \geq 7} v_i = (2^8 x + 1) \prod_{i \geq 7} v_i$$

   and there is no further 8-prime. No Carmichael number exists in this case.

2. When 18433 | m, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 769 \cdot 18433 \prod_{i \geq 7} v_i = (2^{11} x + 1) \prod_{i \geq 7} v_i.$$

   Note that $2^{11} \mid L$ and there is no 11-prime. No Carmichael number exists in this case.

3. When 12289 | m.

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 769 \cdot 12289 \prod_{i \geq 7} v_i = (2^{14} x + 1) \prod_{i \geq 7} v_i.$$

Note that $2^{12} \mid L$. So m must have an n-prime with $n = 12, 13, 14$. Table 6 gives the 14-prime 147457. Then combining this number yields

$$m = (2^{16}x + 1) \prod_{i \geq 8} v_i.$$

Since $2^{14} \mid L$, there is an n-prime with $n = 14, 15, 16$. Table 6 gives the 16-prime 65537. Then including this factor, we have

$$m = (2^{17}x + 1) \prod_{i \geq 9} v_i.$$

Since $2^{16} \mid L$, we must have the 17-prime 1179649. This gives

$$m = (2^{18}x + 1) \prod_{i \geq 10} v_i.$$

Furthermore, m is divisible by the 18-prime 786433. Then m is

$$5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 769 \cdot 12289 \cdot 147457 \cdot 65537 \cdot 1179649 \cdot 786433 \prod_{i \geq 11} v_i$$

$$= (2^{23} \cdot 3^3 \cdot 13 \cdot 14783 \cdot 1020702401040725135124085171 + 1) \prod_{i \geq 11} v_i$$

Since there is no more n-prime for $18 \leq n \leq 23$, the product must be empty. Then

$$\boxed{5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 769 \cdot 12289 \cdot 147457 \cdot 65537 \cdot 1179649 \cdot 786433}$$

is a Carmichael number since $L = 2^{18} \cdot 3^2$ divides $m - 1$.

16.2.2.1.3. Case 257 | m When $257 \mid m$, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 193 \cdot 577 \cdot 257 \prod_{i \geq 6} v_i = (2^9x + 1) \prod_{i \geq 6} v_i.$$

Since $2^8 \mid L$, there is an n-prime with $n = 8, 9$. As there is no 9-prime in Table 6, we have the 8-prime 769. Combining the number 769, we have

$$m = (2^8x + 1) \prod_{i \geq 7} v_i.$$

As there is no more 8-prime, no Carmichael number exists in this case.

16.2.2.2. Subsubsubcase 2: the 6-prime 577 When $577 \mid m$, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 577 \prod_{i \geq 4} v_i = (2^6x + 1) \prod_{i \geq 4} v_i.$$

Since $2^6 \mid L$, m is divisible by the 6-prime 193. Then this reduces to the case dealt in Section 16.2.2.1.

16.2.2.3. Subsubsubcase 3: the 7-prime 1153 When $1153 \mid m$, we have

$$m = (2^7 x + 1) \prod_{i \geq 4} v_i.$$

Since $2^7 \mid L$ but there is no 7-prime, no Carmichael number exists in this case.

16.2.2.4. Subsubsubcase 4: the 8-prime 769 When $769 \mid m$, we have

$$m = (2^8 x + 1) \prod_{i \geq 4} v_i.$$

Since $2^8 \mid L$, the 8-prime 257 divides m. Then including 257, we have

$$m = (2^9 x + 1) \prod_{i \geq 5} v_i.$$

As there is no more 8-prime and 9-prime, no Carmichael number exists in this case.

16.2.2.5. Subsubsubcase 5: the 8-prime 257 When $257 \mid m$, we have

$$m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 257 \prod_{i \geq 4} v_i = (2^8 \cdot 3^2 \cdot 23 \cdot 41 \cdot 557 + 1) \prod_{i \geq 4} v_i.$$

If the product is empty, then

$$\boxed{m = 5 \cdot 7 \cdot 19 \cdot 73 \cdot 97 \cdot 257}$$

is a Carmichael number since $L = 2^8 \cdot 3^2$ divides $m - 1$. Otherwise, since $2^8 \mid L$, the 8-prime 769 divides m, and

$$m = (2^9 x + 1) \prod_{i \geq 5} v_i.$$

As there is no more 8-prime and 9-prime, no other Carmichael number exists in this case.

17. Case 2: $k_1 = 4$

We consider the case $k_1 = 4$ in Case C. We have three cases to consider: $l_1 = s_1 = 1, 2, 3$.

17.1. $l_1 = s_1 = 1$

Suppose that $l_1 = s_1 = 1$. Since s_1 is odd, we must have $P = 3$ by Lemma 1. So $q_1 = 7$ and $r_1 = 19$.

There are two combinations of Type 1 primes: $17 \cdot 257$ and $17 \cdot 65537$ from Table 1. In either case, we have $2^4 \mid L$.

Let

$$m = 17 \cdot (2^{k_2} + 1) \cdot 7 \cdot 19 \prod_{i \geq 1} u_i = (2^2 \cdot 5 \cdot 113 + 1)(2^{k_2} + 1) \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2 or 3 and k_2 is either 4 or 16.

By the minimality argument (Theorem 3), there must be a 2-prime: $q_1 = 13$ or $r_2 = 37$.

17.1.1. Subcase 1: the 2-prime 13

When $13 \mid m$, we have

$$\begin{aligned} m &= 17 \cdot (2^{k_2} + 1) \cdot 7 \cdot 19 \cdot 13 \prod_{i \geq 2} u_i \\ &= (2^4 \cdot 11 \cdot 167 + 1)(2^{k_2} + 1) \prod_{i \geq 2} u_i \\ &= (2^4 x + 1) \prod_{i \geq 2} u_i \end{aligned}$$

for both values of k_2. Here we use the same convention of x as before: x is an odd number and $2^4 x + 1$ is not a Carmichael number. Since $2^4 \mid L$ and there is no more 4-prime and n-pair with $n < 4$, no Carmichael number exists in this case.

17.1.2. Subcase 2: the 2-prime 37

Suppose that $37 \mid m$. Then

$$m = 17 \cdot (2^{k_2} + 1) \cdot 7 \cdot 19 \cdot 37 \prod_{i \geq 2} u_i = (2^3 \cdot 10457 + 1)(2^{k_2} + 1) \prod_{i \geq 2} u_i$$

By the minimality argument, there is a 3-prime: 73. So we have

$$m = 17 \cdot (2^{k_2} + 1) \cdot 7 \cdot 19 \cdot 37 \cdot 73 \prod_{i \geq 2} u_i = (2^4 x + 1) \prod_{i \geq 2} u_i$$

for both values of k_2. Since there is no 4-prime, no Carmichael number exists in this case.

17.2. $l_1 = s_1 = 2$

We now consider the case $l_1 = s_1 = 2$ in Case C. As l_1 is even, this implies that either $P = 3$ or $P \equiv 1 \pmod 3$ by Lemma 1.

17.2.1. Case: $P = 3$

Suppose that $P = 3$. Then we have $q_1 = 13$ and $r_1 = 37$. Then Type 1 prime combinations are either $17 \cdot 257$ or $17 \cdot 65537$ from Table 1. In either case, we have $2^4 \mid L$. Let

$$m = 17 \cdot (2^{k_2} + 1) \cdot 13 \cdot 37 \prod_{i \geq 1} u_i,$$

where u_i is a prime of Type 2 or 3. Here k_2 is either 4 or 16. Then we compute

$$m = (2^4 x + 1) \prod_{i \geq 1} u_i$$

for both values of k_2. Since $2^4 \mid L$, the minimality argument (Theorem 3) implies that the product has either an n-pair with $n < 4$ or a 4-prime. Table 6 shows that this is not the case.

17.2.2. Case: $P \equiv 1 \pmod 3$

Let $P \equiv 1 \pmod 3$. From Table 1, we find primes $P \equiv 1 \pmod 3$ with $k_1 = 4$ such that $2^2 P + 1$ and $2^2 P^2 + 1$ are both prime. See Table 8. The values of such P are

Table 8: Possible $P \equiv 1 \pmod 3$ with $l_1 = s_1 = 2$, $k_1 = 4$

P	$2^2 \cdot P + 1$	$2^2 \cdot P^2 + 1$	Type 1 prime combination
7	29 prime	197 prime	$17 \cdot 257$
13	53 prime	677 prime	$17 \cdot 257$
43	173 prime	$\times$	
127	509 prime	$\times$	
2579	$\times$	$\times$	

$\times$=composite

$P = 7$ and $P = 13$.

17.2.2.1. Subcase 1: $P = 7$ When $P = 7$, we have $q_1 = 29$ and $r_1 = 197$. The only Type 1 combination for $P = 7$ is $17 \cdot 257$.

Let

$$m = 17 \cdot 257 \cdot 29 \cdot 197 \prod_{i \geq 1} u_i = (2^5 \cdot 3^3 \cdot 7 \cdot 4127 + 1) \prod_{i \geq 1} u_i,$$

where u_i is a Type 2 or 3 prime. Note that $2^8 \mid L$.

By the minimality argument, there must be an n-pair with $n < 5$ or a 5-prime. However this is impossible from Table 3. (Recall that l_i, s_i are even as $P \equiv 1 \pmod{3}$.) No Carmichael number exists in this case.

17.2.2.2. Subcase 1: $P = 13$ When $P = 13$, we have $q_1 = 53$, and $r_1 = 677$. The only Type 1 combination is $17 \cdot 257$. Note that $2^8 \mid L$. Let

$$m = 17 \cdot 257 \cdot 53 \cdot 677 \prod_{i \geq 1} u_i = (2^3 \cdot 293 \cdot 79813 + 1) \prod_{i \geq 1} u_i.$$

where u_i is a Type 2 or 3 prime. By the minimality argument, there must be a 3-prime but since l_i, s_i are even as $13 \equiv 1 \pmod{3}$, there is no 3-prime. No Carmichael number exists in this case.

17.3. $l_1 = s_1 = 3$

If $l_1 = s_1 = 3$, then $P = 3$ since s_1 is odd by Lemma 1. However, $2^3 \cdot 3 + 1 = 5^2$ is not a prime. Hence no Carmichael number exists in this case.

18. Case 3: $k_1 = 8$

We consider the case $l_1 = s_1 < k_1 = 8$. There are seven cases according to the value of $l_1 = 1, 2, \ldots, 7$. Note that $257 \cdot 65537$ is the unique Type 1 combination with $k_1 = 8$.

18.1. Subcase 1: $l_1 = s_1 = 1$

When $l_1 = s_1 = 1$, as s_1 is odd, we must have $P = 3$. Then $q_1 = 7$ and $r_1 = 19$. Let

$$m = 257 \cdot 65537 \cdot 7 \cdot 19 \prod_{i \geq 1} u_i = (2^2 \cdot 3 \cdot 11 + 1) \cdot 257 \cdot 65537 \prod_{i \geq 1} u_i,$$

where u_i is a Type 2 or 3 prime. Note that $2^{16} \mid L$.

By the minimality argument (Theorem 3) we must have a 2-prime. Then we have either $q_2 = 13$ or $r_2 = 37$. (See Table 6.)

18.1.1. Subsubcase 1: $q_2 = 13$

When $13 \mid m$, then we have

$$m = 257 \cdot 65537 \cdot 7 \cdot 19 \cdot 13 \prod_{i \geq 2} u_i = (2^6 \cdot 3^3 + 1) \cdot 257 \cdot 65537 \prod_{i \geq 2} u_i.$$

Since $2^{16} \mid L$, there is an n-pair with $n < 6$ or a 6-prime. We see from Table 6 that there is no such pair and we have two 6-primes: 193 and 577.

18.1.1.1. When $q_3 = 193$ When $193 \mid m$, we have

$$m = 257 \cdot 65537 \cdot 7 \cdot 19 \cdot 13 \cdot 193 \prod_{i \geq 3} u_i = (2^8 x + 1) \prod_{i \geq 3} u_i.$$

As there is no admissible pair, we have the 8-prime 769. Combining 769, we have

$$m = (2^{13} x + 1) \prod_{i \geq 4} u_i.$$

As there is no admissible pairs and 13-primes, no Carmichael number exists in this case.

18.1.1.2. When $r_2 = 577$ If $577 \mid m$, then

$$m = 257 \cdot 65537 \cdot 7 \cdot 19 \cdot 13 \cdot 577 \prod_{i \geq 3} u_i = (2^9 x + 1) \prod_{i \geq 3} u_i$$

There is no n-pair with $n < 9$ and there is no 9-prime. Hence no Carmichael number exists in this case.

18.1.2. Subsubcase 2: $r_2 = 37$

When $37 \mid m$, then we have

$$m = 257 \cdot 65537 \cdot 7 \cdot 19 \cdot 37 \prod_{i \geq 2} u_i = (2^3 \cdot 3 \cdot 5 \cdot 41 + 1) \cdot 257 \cdot 65537 \prod_{i \geq 2} u_i.$$

As $2^{16} \mid L$, m is divisible by the 3-prime 73. Combining this yields

$$m = (2^6 x + 1) \cdot 257 \cdot 65537 \prod_{i \geq 3} u_i.$$

Hence we have a 6-prime: 193, 577.

18.1.2.1. When $q_2 = 193$ When $193 \mid m$, combining this we have

$$m = (2^8 x + 1) \prod_{i \geq 4} u_i.$$

Then we have the 8-prime 769, and

$$m = (2^{12} x + 1) \prod_{i \geq 5} u_i,$$

which implies again that the 12-prime 12289 divides m. Hence

$$m = (2^{13} x + 1) \prod_{i \geq 6} u_i,$$

but there is no 13-prime. Thus no Carmichael number exists in this case.

18.1.2.2. When $r_4 = 577$ When $577 \mid m$, we combine this and obtain

$$m = (2^7 x + 1) \prod_{i \geq 4} u_i.$$

Hence we have the 7-prime 1153 and

$$m = (2^8 x + 1) \prod_{i \geq 5} u_i,$$

which implies the existence of the 8-prime 769. Then

$$m = (2^9 x + 1) \prod_{i \geq 5} u_i,$$

but there is no 9-prime. Hence no Carmichael number exists in this case.

18.2. Subcase 2: $l_1 = s_1 = 2$

Consider the case $l_1 = s_1 = 2$. As l_1 is even, $P = 3$ or $P \equiv 1 \pmod 3$.

18.2.1. Subsubcase 1: When $P = 3$

When $P = 3$, we have $q_1 = 13$ and $r_1 = 37$. Note that $2^{16} \mid L$. Then we have by the minimality argument

$$\begin{aligned} m &= 257 \cdot 65537 \cdot 13 \cdot 37 \prod_{i \geq 1} u_i \\ &= (2^5 \cdot 3 \cdot 5 + 1) 257 \cdot 65537 \prod_{i \geq 1} u_i \qquad \text{then the 5-prime 97 divides } m \\ &= (2^6 \cdot 3^6 + 1) 57 \cdot 65537 \prod_{i \geq 2} u_i. \end{aligned}$$

This implies that m is divisible by a 6-prime: 193, 577.

18.2.1.1. the 6-prime 193 When $193 \mid m$, we have

$$\begin{aligned} m &= (2^{11} x + 1) \prod_{i \geq 3} u_i \qquad \text{then the 11-prime 18433 divides } m \\ &= (2^{12} x + 1) \prod_{i \geq 4} u_i \qquad \text{then the 12-prime 12289 divides } m \\ &= (2^{13} x + 1) \prod_{i \geq 5} u_i. \end{aligned}$$

(There were no admissible pairs to consider.) Since there is no 13-prime, no Carmichael number exists in this case.

18.2.1.2. the 6-prime 577 When $577 \mid m$, we have

$$\begin{aligned} m &= 257 \cdot 65537 \cdot 13 \cdot 37 \cdot 97 \cdot 577 \prod_{i \geq 3} u_i \\ &= (2^7 x + 1) \cdot 257 \cdot 65537 \prod_{i \geq 4} u_i \qquad \text{then the 7-prime 1153 divides } m \\ &= (2^9 x + 1)(2^{16} + 1) \prod_{i \geq 4} u_i \end{aligned}$$

(There were no admissible pairs to consider.) Since there is no 9-prime, no Carmichael number exists in this case.

18.2.2. Subsubcase 2: When $P \equiv 1 \pmod 3$

Suppose that $P \equiv 1 \pmod 3$. Then the only possible P are $7, 13, 241$. However, as $2^2 \cdot 241 + 1 = 5 \cdot 193$ is not prime, there are two cases: $P = 7$ and $P = 13$.

18.2.2.1. When $P = 7$ When $P = 7$, we have $q_1 = 29$ and $r_1 = 197$. Note that $2^{16} \mid L$. Then the minimality arguments (see Table 3) gives

$$\begin{aligned} m &= 257 \cdot 65537 \cdot 29 \cdot 197 \prod_{i \geq 1} u_i \\ &= (2^4 x + 1) \cdot 257 \cdot 65537 \prod_{i \geq 1} u_i \qquad \text{then the 4-prime 113 divides } m \\ &= (2^6 x + 1) \cdot 257 \cdot 65537 \prod_{i \geq 2} u_i \end{aligned}$$

and m is divisible by a 6-prime: 449, 3137.

1. When $449 \mid m$, we have
$$m = (2^7 x + 1) \cdot 257 \cdot 65537 \prod_{i \geq 3} u_i$$
but there is no 7-prime.

2. When $3137 \mid m$, we have
$$m = (2^9 x + 1) \cdot 257 \cdot 65537 \prod_{i \geq 3} u_i$$
but there is no 9-prime.

Hence no Carmichael number exists in this case.

18.2.2.2. When $P = 13$ When $P = 13$, then $q_1 = 53$ and $r_1 = 677$. Then we have

$$m = 257 \cdot 65537 \cdot 53 \cdot 677 \prod_{i \geq 1} u_i = (2^3 x + 1) \cdot 257 \cdot 65537 \prod_{i \geq 1} u_i$$

Since l_2, s_2 are even, there is no 3-prime. Hence no Carmichael number exists in this case.

18.3. Subcase 3: $l_1 = s_1 = 3$

When $l_1 = s_1 = 3$, we have $P = 3$ by Lemma 6. But $2^3 \cdot 3 + 1 = 5^2$ is not prime. Hence no Carmichael number exists in this case.

18.4. Subcase 4: $l_1 = s_1 = 4$

When $l_1 = s_1 = 4$, we have either $P = 3$ or $P \equiv 1 \pmod 3$. As $2^4 \cdot 3 + 1 = 7^2$ is not prime, $P \neq 3$. There are three values for $P \equiv 1 \pmod 3$ with $k_1 = 8$: $P = 7, 13, 241$. However, $2^4 \cdot 7^2 + 1 = 5 \cdot 157$, $2^4 \cdot 13 + 1 = 11 \cdot 19$ and $2^4 \cdot 241 + 1 = 7 \cdot 19 \cdot 29$ are not prime. Hence there is no possible P in this case.

18.5. Subcase 5: $l_1 = s_1 = 5$

Consider the case $l_1 = s_1 = 5$. As s_1 is odd, $P = 3$. However $2^5 \cdot 3 + 1 = 17^2$ is not prime. Hence we have no Carmichael number in this case.

18.6. Subcase 6: $l_1 = s_1 = 6$

Suppose $l_1 = s_1 = 6$. As l_1 is even, we have either $P = 3$ or $P \equiv 1 \pmod 3$.

18.6.1. When $P = 3$

Suppose that $P = 3$. Then $q_1 = 2^6 \cdot 3 + 1 = 193$ and $r_1 = 2^6 \cdot 3^2 + 1 = 577$. Recall that $2^{16} \mid L$. Then we have

$$m = 257 \cdot 65537 \cdot 193 \cdot 577 \prod_{i \geq 1} u_i = (2^{10} x + 1) \cdot (2^{16} + 1) \prod_{i \geq 1} u_i$$

There is no n-pair with $6 < n < 10$ and there is no 10-prime (See Table 6). Hence no Carmichael number exists in this case by the minimality argument.

18.6.2. When $P \equiv 1 \pmod 3$

Consider the case $P \equiv 1 \pmod 3$. Then $P = 7, 13, 241$. However $2^6 \cdot 13 + 1 = 7^2 \cdot 17$ and $2^6 \cdot 241 + 1 = 5^2 \cdot 617$ are not prime.

When $P = 7$, we have $q_1 = 449$ and $r_1 = 3137$. Then

$$m = 257 \cdot 65537 \cdot 449 \cdot 3137 \prod_{i \geq 1} u_i = (2^8 x + 1) \prod_{i \geq 1} u_i$$

As there is no n-pair with $6 < n < 8$ and 8-prime (see Table 3) , there is no Carmichael number in this case.

18.7. Subcase 7: $l_1 = s_1 = 7$

If $l_1 = s_1 = 7$, then $P = 3$ as s_1 is odd. However, $2^7 \cdot 3 + 1 = 5 \cdot 7 \cdot 11$ is not prime. Thus, no Carmichael number exists in this case by the minimality argument.

This exhausted all the cases, and this completes the proof of Theorem 1.

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