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A REMARK ON PERIODS OF PERIODIC SEQUENCES MODULO m

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Abstract

Let $\{G_n\}$ be a periodic sequence of integers modulo m and let $\{SG_n\}$ be the partial sum sequence defined by $SG_n := \sum_{k=0}^{n} G_k \pmod m$. We give a formula for the period of $\{SG_n\}$. We also show that for a generalized Fibonacci sequence $F(a,b)_n$ such that $F(a,b)_0 = a$ and $F(a,b)_1 = b$, we have

$$S^i F(a,b)_n = S^{i-1}F(a,b)_{n+2} - \binom{n+i}{i-2}a - \binom{n+i}{i-1}b$$

where $S^i F(a,b)_n$ is the i-th partial sum sequence successively defined by $S^i F(a,b)_n := \sum_{k=0}^{n} S^{i-1}F(a,b)_k$. This is a generalized version of the well-known formula

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1$$

of the Fibonacci sequence F_n.

1. Introduction

We denote a sequence $\{a_n\}$ simply by a_n, without brackets, unless some confusion is possible. Given a sequence a_n, its partial sum sequence Sa_n is defined by

$$Sa_n := \sum_{k=0}^{n} a_k.$$

The Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}\,(n \geq 2)$[1] :

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \dots.$$

[1]Sometimes the Fibonacci sequence is defined to start at $n = 1$ instead of at $n = 0$, namely, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}\,(n \geq 3)$, e.g., see [1, Example 4.27].

It is well-known (e.g., see [3, 6, 7, 8, 9]) that, for any integer $m \geq 2$, the Fibonacci sequence F_n modulo m is a periodic sequence. For example, for $m = 2$ and $m = 6$, the Fibonacci sequences F_n modulo 2 and 6, and the partial sum sequences SF_n modulo 2 and 6 are the following:

- $m = 2$:

$$\begin{aligned} F_n &: \underbrace{011}\underbrace{011}\underbrace{011}\underbrace{011}, \cdots\cdots \\ SF_n &: \underbrace{010}\underbrace{010}\underbrace{010}\underbrace{010}, \cdots\cdots \end{aligned}$$

- $m = 6$:

$$\begin{aligned} F_n &: \underbrace{011235213415055431453251}\underbrace{011235213415055431453251}\cdots\cdots \\ SF_n &: \underbrace{012410230454432034214050}\underbrace{012410230454432034214050}\cdots\cdots \end{aligned}$$

The periods of F_n modulo 2 and SF_n modulo 2 are the same number, 3, and also the periods of F_n modulo 6 and SF_n modulo 6 are the same number, 24. In fact, this holds for any integer $m \geq 2$, as we will see. Furthermore, if we consider $S^2F_n := S(SF_n)$ (which will be discussed in Section 4), then the period of S^2F_n is not necessarily the same as that of F_n.

Now, let us consider the following same periodic sequence G_n modulo $m = 2, 3, 4, 5, 6, 7, 8$ (its period is 3):

$$G_n : \underbrace{011}\underbrace{011}\underbrace{011}\cdots\cdots$$

Then its partial sum sequence SG_n becomes as follows:

1. $m = 2 : SG_n : \underbrace{010}\underbrace{010}\underbrace{010}\underbrace{010}\cdots$; period is $3 = 3 \times 1$
2. $m = 3 : SG_n : \underbrace{012201120}\underbrace{012201120}\underbrace{012201120}\underbrace{012201120}\cdots$; period is $9 = 3 \times 3$.
3. $m = 4 : SG_n : \underbrace{012230}\underbrace{012230}\underbrace{012230}\underbrace{012230}\cdots$; period is $6 = 3 \times 2$.
4. $m = 5 : SG_n : \underbrace{012234401123340}\underbrace{012234401123340}\cdots$; period is $15 = 3 \times 5$.
5. $m = 6 : SG_n : \underbrace{012234450}\underbrace{012234450}\underbrace{012234450}\cdots$; period is $9 = 3 \times 3$.
6. $m = 7 : SG_n : \underbrace{012234456601123345560}\cdots$; period is $21 = 3 \times 7$.
7. $m = 8 : SG_n : \underbrace{012234456670}\underbrace{012234456670}\cdots$; period is $12 = 3 \times 4$.

In this note, given any periodic sequence G_n modulo m (not necessarily the Fibonacci sequence modulo m), we give a formula of the period of the partial sum sequence SG_n in terms of the period of the original sequence G_n.

Theorem 1. *Let p be the period of a periodic sequence G_n modulo m. Then the period of the partial sum sequence SG_n modulo m is equal to*

$$p \times \operatorname{ord}\bigl(SG_{p-1} \bmod m\bigr),$$

where $\operatorname{ord}(x)$ *denotes the order of an element x of $\mathbb{Z}_m$. In particular, if m is a prime, then the period of the partial sum sequence SG_n modulo m is equal to*

$$\begin{cases} p & \text{if } SG_{p-1} \equiv 0 \mod m, \\ p \times m & \text{if } SG_{p-1} \not\equiv 0 \mod m. \end{cases}$$

For example, in the above examples, $SG_2 \bmod m$ is equal to 2 for each m, and the order $\operatorname{ord}\bigl(SG_{p-1} \bmod m\bigr)$ is as follows:

1. $m = 2$; $\operatorname{ord}\bigl(SG_2 \bmod 2\bigr) = 1$.
2. $m = 3$; $\operatorname{ord}\bigl(SG_2 \bmod 3\bigr) = 3$.
3. $m = 4$; $\operatorname{ord}\bigl(SG_2 \bmod 4\bigr) = 2$.
4. $m = 5$; $\operatorname{ord}\bigl(SG_2 \bmod 5\bigr) = 5$.
5. $m = 6$; $\operatorname{ord}\bigl(SG_2 \bmod 6\bigr) = 3$.
6. $m = 7$; $\operatorname{ord}\bigl(SG_2 \bmod 7\bigr) = 7$.
7. $m = 8$; $\operatorname{ord}\bigl(SG_2 \bmod 8\bigr) = 4$.

2. General Fibonacci Sequence $F(a, b)_n$ Modulo m

In this section we observe that the Fibonacci sequence F_n modulo m and its partial sum sequence SF_n modulo m have the same period.

Definition 1. A sequence $\widetilde{F}_n$ satisfying

$$\widetilde{F}_n = \widetilde{F}_{n-1} + \widetilde{F}_{n-2} \quad (n \geq 2)$$

is called *a general Fibonacci sequence*.

For such a general Fibonacci sequence $\widetilde{F}_n$, $\widetilde{F}_0 = a$ and $\widetilde{F}_1 = b$ can be any integers. Therefore, we denote such a general Fibonacci sequence $\widetilde{F}_n$ by $F(a, b)_n$. In [4] it is denoted by $G(a, b, n)$, and the sequence $F(2, 1)_n$ is the *Lucas sequence* and is denoted by L_n.

For a positive integer $m \geq 2$, a general Fibonacci sequence modulo m is defined as follows.

Definition 2. Let $a, b \in \mathbb{Z}_m = \{0, 1, 2, \cdots m-1\}$. The sequence $\widetilde{F}_n$ is defined by

- $\widetilde{F}_0 = a, \widetilde{F}_1 = b$:
- $\widetilde{F}_n \equiv \widetilde{F}_{n-1} + \widetilde{F}_{n-2} \bmod m$ for all $n \geq 2$.

Such a sequence modulo m may be denoted by $F(a,b)_n \pmod{m}$, but to avoid messy notation, we use the same symbol $F(a,b)_n$. The following periodicity of a general Fibonacci sequence modulo m is well-known (e.g., see [6, 7, 8, 9].)

Theorem 2. *For any integer $m \geq 2$, a general Fibonacci sequence $F(a,b)_n$ modulo m is a periodic sequence.*

The period of a general Fibonacci sequence $F(a,b)_n$ modulo m shall be denoted by $\Pi(F(a,b)_n, m)$. For the Fibonacci sequence F_n modulo m, the period $\Pi(F_n, m)$ is called the *Pisano*[2] *period* of m and is denoted by $\pi(m)$. For some interesting results of the Pisano period $\pi(m)$, see, e.g., [6]. For example, one can see a list of the periods $\pi(m)$ for $2 \leq m \leq 2001$; for instance,

$$\pi(10) = 60, \pi(25) = 100, \pi(98) = 336, \pi(250) = 1500, \pi(500) = 1500,$$
$$\pi(625) = 2500, \pi(750) = 3000, \pi(987) = 32, \pi(1250) = 7500,$$
$$\pi(1991) = 90, \pi(2000) = 3000, \pi(2001) = 336.$$

As to relations between $\pi(m)$ and m, the following are known.

Theorem 3 (Freyd and Brown [2]). *The following hold:*

1. $\pi(m) \leq 6m$ *with equality if and only if* $m = 2 \times 5^k (k = 1, 2, 3, \cdots)$.
2. *For the Lucas number L_n, $\Pi(L_n, m) \leq 4m$ with equality if and only if $m = 6$.*

Given a sequence a_n, the partial sum sequence b_n defined by $b_n := \sum_{k=0}^{n} a_k$ shall be called *the first derived sequence*, instead of "partial sum sequence". The first derived sequence of a general Fibonacci sequence $F(a,b)_n$ shall be denoted by $SF(a,b)_n := \sum_{k=0}^{n} F(a,b)_k$.

Lemma 1. *For a general Fibonacci sequence $F(a,b)_n$ the following holds:*

$$SF(a,b)_n = F(a,b)_{n+2} - b. \tag{3}$$

Proof. We have $SF(a,b)_0 = F(a,b)_0 = F(a,b)_2 - F(a,b)_1 = F(a,b)_2 - b$. So, we suppose that $SF(a,b)_n = F(a,b)_{n+2} - b$. Then

$$\begin{aligned}
SF(a,b)_{n+1} &= SF(a,b)_n + F(a,b)_{n+1} \\
&= F(a,b)_{n+2} - b + F(a,b)_{n+1} \\
&= F(a,b)_{n+2} + F(a,b)_{n+1} - b \\
&= F(a,b)_{n+3} - b \\
&= F(a,b)_{(n+1)+2} - b.
\end{aligned}$$

[2]Pisano is another name for Fibonacci (cf. Wikipedia [10]).

□

Equation (3) is a generalized version of the following well-known formula of the Fibonacci sequence [5]:

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1.$$

From Equation (3), we get the following

Corollary 1. *For any positive integer $m \geq 2$, a general Fibonacci sequence $F(a,b)_n$ modulo m and its first derived Fibonacci sequence $SF(a,b)_n$ modulo m have the same period:*

$$\Pi(F(a,b)_n, m) = \Pi(SF(a,b)_n, m).$$

Lemma 2. *If $\Pi(F(a,b)_n, m) = p$, then we have*

$$SF(a,b)_{p-1} = \sum_{k=0}^{p-1} F(a,b)_k \equiv 0 \mod m. \tag{4}$$

Proof. There are two ways to prove this.

(I) It follows from Lemma 1 that $SF(a,b)_{p-1} = F(a,b)_{p+1} - b$. Since the period of the general Fibonacci sequence $F(a,b)_n$ modulo m is p, we have $F(a,b)_{p+1} \equiv F(a,b)_1$. Hence,

$$\begin{aligned} SF(a,b)_{p-1} &= F(a,b)_{p+1} - b \\ &\equiv F(a,b)_1 - b = 0 \mod m. \end{aligned}$$

(II) By Corollary 1 $\Pi(SF(a,b)_n, m) = p$. Hence, $SF(a,b)_0 \equiv SF(a,b)_p$, namely,

$$F(a,b)_0 \equiv \sum_{k=0}^{p} F(a,b)_k = \sum_{k=0}^{p-1} F(a,b)_k + F(a,b)_p.$$

Since $F(a,b)_0 \equiv F(a,b)_p \mod m$, we get

$$SF_{p-1} = \sum_{k=0}^{p-1} F(a,b)_k \equiv 0 \mod m.$$

□

Indeed, in the following examples we do see the above mod formula (4).

Example 5. For the Fibonacci sequence F_n modulo m

1. $m = 2$: $\pi(2) = 3$. $\underbrace{011}\underbrace{011}\underbrace{011}\underbrace{011}\cdots\cdots$

2. $m = 3$: $\pi(3) = 8$. $\underbrace{01120221}\underbrace{01120221}\underbrace{01120221}\underbrace{01120221}\cdots\cdots$

3. $m = 4$: $\pi(4) = 6$. $\underbrace{011231}\underbrace{011231}\underbrace{011231}\underbrace{011231}\cdots\cdots$

4. $m = 5$: $\pi(5) = 20$. $\underbrace{01123033140443202241}\underbrace{01123033140443202241}\cdots\cdots$

5. $m = 6$: $\pi(6) = 24$. $\underbrace{011235213415055431453251}\underbrace{011235213415055431453251}\cdots$

6. $m = 7$: $\pi(7) = 16$. $\underbrace{0112351606654261}\underbrace{0112351606654261}\underbrace{0112351606654261}\cdots$

7. $m = 8$: $\pi(8) = 12$. $\underbrace{011235055271}\underbrace{011235055271}\underbrace{011235055271}\underbrace{011235055271}\cdots$

3. Periods of Periodic Sequences Modulo m

The first derived sequence $SF(a,b)_n$ modulo m of a general Fibonacci sequence $F(a,b)$ modulo m is periodic and has the same period as that of the general Fibonacci sequence $F(a,b)$ modulo m. As seen in the introduction, it is not the case for an arbitrary periodic sequence modulo m. So, in this section we consider periods of the first derived sequence of a periodic sequence modulo m.

Proposition 1. *Let G_n be a periodic sequence modulo m with period p. If $SG_{p-1} \equiv 0 \mod m$, then for any n we have $SG_n \equiv SG_{n+p} \bmod m$.*

Proof. The period of G_n is p, thus $G_0 \equiv G_p$. Since $SG_0 = G_0$, we have $SG_0 \equiv G_p \mod m$. Since $SG_{p-1} = \sum_{k=0}^{p-1} G_k \equiv 0 \mod m$,

$$SG_0 \equiv SG_{p-1} + G_p = SG_p \mod m.$$

So, suppose that for n we have $SG_n \equiv SG_{n+p}$. Then

$$\begin{aligned} SG_{n+1} &\equiv SG_n + G_{n+1} \mod m \\ &\equiv SG_{n+p} + G_{n+1+p} \mod m \text{ (since } G_{n+1} \equiv G_{n+1+p} \mod m) \\ &\equiv SG_{n+1+p} \mod m. \end{aligned}$$

Hence, by induction, we get the statement of the proposition. □

Here it should be noticed that from Proposition 1 one *cannot* automatically claim that the period of the first derived sequence SG_n modulo m is p; one can claim only that the period of SG_n is *a divisor of the period of the original sequence* G_n:

$$\Pi(SG_n, m) \,\Big|\, \Pi(G_n, m).$$

In fact, we can show

Theorem 4. *Let G_n be a periodic sequence modulo m with period p. If $SG_{p-1} \equiv 0 \mod m$, then $\Pi(SG_n, m) = \Pi(G_n, m)$.*

Proof. Suppose that $\Pi(SG_n, m) \neq \Pi(G_n, m)$, namely $\Pi(SG_n, m)$ is a proper divisor of $p = \Pi(G_n, m)$. Let $d = \Pi(SG_n, m) < p$ and $d \mid p$. Thus, we have that for all $n \geq 0$

$$SG_n \equiv SG_{n+d} \mod m. \tag{6}$$

We have that $SG_0 \equiv SG_d \mod m$ implies $G_1 + \cdots + G_d \equiv 0 \mod m$, thus

$$G_1 \equiv -(G_2 + \cdots + G_d) \mod m.$$

Now, $SG_1 \equiv SG_{1+d} \mod m$ implies $G_2 + \cdots + G_d + G_{1+d} \equiv 0 \mod m$, thus

$$G_{1+d} \equiv -(G_2 + \cdots + G_d) \mod m.$$

Therefore we get

$$G_1 \equiv G_{1+d} \mod m.$$

Let $p = d \times p_0$. Continuing this procedure, we get the following congruences:

- $G_1 \equiv G_{1+d} \equiv G_{1+2d} \cdots \equiv G_{1+(p_0-1)d} \mod m.$
- $G_2 \equiv G_{2+d} \equiv G_{2+2d} \cdots \equiv G_{2+(p_0-1)d} \mod m.$
- $G_3 \equiv G_{3+d} \equiv G_{3+2d} \cdots \equiv G_{3+(p_0-1)d} \mod m.$

  . . . .

  . . . .
- $G_{d-1} \equiv G_{2d-1} \equiv G_{3d-1} \cdots \equiv G_{p_0 d-1} \mod m.$
- $G_d \equiv G_{2d} \equiv G_{3d} \equiv \cdots \equiv G_{p_0 d} = G_p \mod m.$

Since $G_0 \equiv G_p \mod m$, the final congruences $G_d \equiv G_{2d} \equiv G_{3d} \equiv \cdots \equiv G_{p_0 d} = G_p \mod m$ become

- $G_0 \equiv G_d \equiv G_{2d} \equiv G_{3d} \equiv \cdots \equiv G_p \mod m.$

Hence $\Pi(G_n, m) = d < p$, which contradicts the fact that $\Pi(G_n, m) = p$. □

As in the proof (II) of Lemma 2, if $\Pi(G_n, m) = \Pi(SG_n, m) = p$, then we have $SG_{p-1} \equiv 0 \mod m$. Therefore we get the following

Corollary 2. *Let G_n be a periodic sequence modulo m with period p. Then we have $SG_{p-1} \equiv 0 \mod m$ if and only if $\Pi(SG_n, m) = \Pi(G_n, m)$.*

Next, we consider the case when $SG_{p-1} \not\equiv 0 \mod m$.

Theorem 5. *Let G_n be a periodic sequence modulo m with period p. If $SG_{p-1} \not\equiv 0 \mod m$, then*

$$\Pi(SG_n, m) = s \times \Pi(G_n, m),$$

where s is the order of $SG_{p-1} (\text{mod } m)$ in $\mathbb{Z}_m$, i.e., s is the smallest non-zero integer such that

$$s \times SG_{p-1} \equiv 0 \mod m.$$

Proof. First we observe that for all $i \geq 2$

$$SG_{ip-1} \equiv SG_{(i-1)p-1} + SG_{p-1} \mod m$$

from which we obtain that for all $i \geq 1$

$$SG_{ip-1} \equiv i \times SG_{p-1} \mod m$$

Indeed, the first p-tuple is $\{SG_0, SG_1, \cdots, SG_{p-1}\}$ and the second p-tuple is $\{SG_p, SG_{p+1}, \cdots, SG_{2p-1}\}$, which is, modulo m, the same as

$$\{SG_{p-1} + SG_0, SG_{p-1} + SG_1, \cdots, SG_{p-1} + SG_{p-1}\}.$$

Hence, $SG_{2p-1} \equiv SG_{p-1} + SG_{p-1} \mod m$, thus we get

$$SG_{2p-1} \equiv 2 \times SG_{p-1} \mod m. \tag{7}$$

The third p-tuple $\{SG_{2p}, SG_{2p+1}, \cdots, SG_{3p-1}\}$ is, modulo m, the same as

$$\{SG_{2p-1} + SG_0, SG_{2p-1} + SG_1, \cdots, SG_{2p-1} + SG_{p-1}\}.$$

Hence, $SG_{3p-1} \equiv SG_{2p-1} + SG_{p-1}$, thus from Equation (7) we get

$$SG_{3p-1} \equiv 3 \times SG_{p-1} \mod m.$$

Now, let us suppose that

$$SG_{jp-1} \equiv j \times SG_{p-1} \mod m. \tag{8}$$

We see that the $(j+1)$-th p-tuple $\{SG_{jp}, SG_{jp+1}, \cdots, SG_{(j+1)p-1}\}$ is, modulo m, the same as

$$\{SG_{jp-1} + SG_0, SG_{jp-1} + SG_1, \cdots, SG_{jp-1} + SG_{p-1}\}.$$

Hence, $SG_{(j+1)p-1} \equiv SG_{jp-1} + SG_{p-1} \mod m$, and thus from Equation (8) we get

$$SG_{(j+1)p-1} \equiv (j+1) \times SG_{p-1} \mod m..$$

Hence, by induction, we have that for all $i \geq 1$

$$SG_{ip-1} \equiv i \times SG_{p-1} \mod m.$$

Since $s \times SG_{p-1} \equiv 0 \mod m$, we have

$$SG_{ps-1} \equiv \sum_{k=0}^{ps-1} G_k \equiv 0 \mod m.$$

As in the proof of Proposition 1, we see that for any n

$$SG_n \equiv SG_{n+ps} \mod m.$$

Hence the period of SG_n is a divisor of ps. Suppose that such a divisor is a proper one, denoted by δ. Then, as in the proof of Theorem 4, we have

$$G_n \equiv G_{n+\delta} \mod m.$$

Since the period of G_n is p, δ has to be a multiple of p, thus $\delta = \omega p$ for some non-zero integer ω. Since $\delta = \omega p$ is a proper divisor of ps, ω is a proper divisor of s, in particular $\omega < s$. Then, as in the proof (II) of Lemma 2, $SG_n \equiv GS_{n+\omega p} \mod m$ implies that

$$GS_{\omega p-1} = \sum_{k=0}^{\omega p-1} G_k \equiv 0 \mod m.$$

In other words

$$\omega \times SG_{p-1} \equiv 0 \mod m.$$

This contradicts the fact that s is the smallest non-zero integer such that $s \times SG_{p-1} \equiv 0 \mod m$. Hence the period δ of SG_n has to be exactly ps, i.e.,

$$\Pi(SG_n, m) = s \times \Pi(G_n, m).$$

□

Therefore we get the following theorem:

Theorem 6. *Let p be the period of a periodic sequence G_n modulo m. Then the period of the partial sum sequence SG_n modulo m is equal to*

$$\mathrm{ord}\big(SG_{p-1} \bmod m\big) \times p,$$

where $\mathrm{ord}(x)$ denotes the order of an element x of $\mathbb{Z}_m$.

In particular, if m is a prime, then the period of the partial sum sequence SG_n modulo m is equal to

$$\begin{cases} p & \text{if } SG_{p-1} \equiv 0 \mod m, \\ m \times p & \text{if } SG_{p-1} \not\equiv 0 \mod m. \end{cases}$$

We note that for an element $n \in \mathbb{Z}_m$, the order $\mathrm{ord}(n)$ of the element n is given by

$$\mathrm{ord}(n) = \frac{\mathrm{LCM}(n,m)}{n}$$

where $\mathrm{LCM}(n,m)$ is the least common multiple of n and m.

Example 9. Let us consider the following periodic sequence G_n in $\mathbb{Z}_m$ with $m = 15, 30, 36$

$$G_n : \underbrace{20190823}\underbrace{20190823}\underbrace{20190823}\cdots\cdots .$$

Then $\Pi(G_n, m) = 8$ for any m and we have that $2+0+1+9+0+8+2+3 = 25$.

1. $\Pi(SG_n, 15) = 8 \times \frac{\mathrm{LCM}(25,15)}{25} = 8 \times \frac{75}{25} = 8 \times 3 = 24.$
2. $\Pi(SG_n, 30) = 8 \times \frac{\mathrm{LCM}(25,30)}{25} = 8 \times \frac{150}{25} = 8 \times 6 = 48.$
3. $\Pi(SG_n, 36) = 8 \times \frac{\mathrm{LCM}(25,36)}{25} = 8 \times \frac{25\times 36}{25} = 8 \times 36 = 288.$

4. Higher Derived General Fibonacci Sequences $S^iF(a,b)_n$

Given a sequence A_n, for a non-negative integer i we define the i-th derived sequence inductively as follows:

$$S^iA_n := \sum_{k=0}^{n} S^{i-1}A_k, \quad i \geq 1$$

where $S^0A_n := A_n$.

Thus for a general Fibonacci sequence $F(a,b)_n$, we can consider the i-th derived sequence $S^iF(a,b)_n$. By tedious computation we can show the following formulas, which are further extensions of Lemma 1.

Proposition 2. *For all integers a, b and $n \geq 0$, the following formulas hold.*

1. $S^2F(a,b)_n = SF(a,b)_{n+2} - a - (n+2)b.$
2. $S^3F(a,b)_n = S^2F(a,b)_{n+2} - (n+3)a - \dfrac{(n+2)(n+3)}{2}b.$
3. $S^4F(a,b)_n = S^3F(a,b)_{n+2} - \dfrac{(n+3)(n+4)}{2}a - \dfrac{(n+2)(n+3)(n+4)}{2\times 3}b.$

These formulas are re-expressed as follows:

$$\begin{aligned} S^2F(a,b)_n &= SF(a,b)_{n+2} - \binom{n+2}{0}a - \binom{n+2}{1}b \\ &= SF(a,b)_{n+2} - \binom{n+2}{2-2}a - \binom{n+2}{2-1}b. \end{aligned}$$

$$\begin{aligned} S^3F(a,b)_n &= S^2F(a,b)_{n+2} - \binom{n+3}{1}a - \binom{n+3}{2}b \\ &= S^2F(a,b)_{n+2} - \binom{n+3}{3-2}a - \binom{n+3}{3-1}b. \end{aligned}$$

$$\begin{aligned} S^4F(a,b)_n &= S^3F(a,b)_{n+2} - \binom{n+4}{2}a - \binom{n+4}{3}b \\ &= S^3F(a,b)_{n+2} - \binom{n+4}{4-2}a - \binom{n+4}{4-1}b. \end{aligned}$$

So, by these re-expressions, it is natural to think that the following general formula would hold and it turns out that it is the case, i.e., we have the following result.

Theorem 7. *For $i \geq 1$ we have*

$$S^iF(a,b)_n = S^{i-1}F(a,b)_{n+2} - \binom{n+i}{i-2}a - \binom{n+i}{i-1}b. \tag{10}$$

When $i = 1$ we set $\binom{n+1}{1-2} = \binom{n+1}{-1} = 0$.

Proof. The proof is by induction. Since the formula is already proved in the cases when $i = 1, 2, 3, 4$ as above, we suppose that the above formula (10) holds for $i = j$:

$$S^jF(a,b)_n = S^{j-1}F(a,b)_{n+2} - \binom{n+j}{j-2}a - \binom{n+j}{j-1}b \tag{11}$$

and we show the formula for $i = j + 1$:

$$S^{j+1}F(a,b)_n = S^jF(a,b)_{n+2} - \binom{n+j+1}{j-1}a - \binom{n+j+1}{j}b. \tag{12}$$

First we have

$$\begin{aligned} &S^{j+1}F(a,b)_n \\ &= \sum_{k=0}^{n} S^jF(a,b)_k \quad \text{(by the definition of } S^{j+1}F(a,b)_n\text{)} \\ &= \sum_{k=0}^{n} \left\{ S^{j-1}F(a,b)_{k+2} - \binom{k+j}{j-2}a - \binom{k+j}{j-1}b \right\} \quad \text{(by Equation (11))} \\ &= \sum_{k=0}^{n} S^{j-1}F(a,b)_{k+2} - \sum_{k=0}^{n}\binom{k+j}{j-2}a - \sum_{k=0}^{n}\binom{k+j}{j-1}b \end{aligned}$$

$$= \left(\sum_{k=0}^{n+2} S^{j-1}F(a,b)_k - S^{j-1}F(a,b)_0 - S^{j-1}F(a,b)_1\right) \\ - \sum_{k=0}^{n}\binom{k+j}{j-2}a - \sum_{k=0}^{n}\binom{k+j}{j-1}b$$

$$= S^jF(a,b)_{n+2} - S^{j-1}F(a,b)_0 - S^{j-1}F(a,b)_1 - \sum_{k=0}^{n}\binom{k+j}{j-2}a - \sum_{k=0}^{n}\binom{k+j}{j-1}b.$$

Here we observe that for all $j \geq 1$ we have $S^{j-1}F(a,b)_0 = a$, which is obvious, and

$$S^{j-1}F(a,b)_1 = (j-1)a + b. \tag{13}$$

Equation (13) can be seen by induction as follows. If $j = 1$, then $S^{j-1}F(a,b)_1 = S^0F(a,b)_1 = F(a,b)_1 = b$. So, suppose that $S^{j-1}F(a,b)_1 = (j-1)a + b$ holds and we show that $S^jF(a,b)_1 = ja + b$. Indeed,

$$\begin{aligned} S^jF(a.b)_1 &= S^{j-1}F(a,b)_0 + S^{j-1}F(a,b)_1 \quad \text{(by the definition of } S^jF(a.b)_1\text{)} \\ &= a + (j-1)a + b \quad \text{(by the above)} \\ &= ja + b. \end{aligned}$$

Hence we have

$$\begin{aligned} &S^{j+1}F(a,b)_n \\ &= S^jF(a,b)_{n+2} - a - \{(j-1)a + b\} - \sum_{k=0}^{n}\binom{k+j}{j-2}a - \sum_{k=0}^{n}\binom{k+j}{j-1}b \\ &= S^jF(a,b)_{n+2} - \left\{j + \sum_{k=0}^{n}\binom{k+j}{j-2}\right\}a - \left\{1 + \sum_{k=0}^{n}\binom{k+j}{j-1}\right\}b. \end{aligned}$$

We want to show

$$j + \sum_{k=0}^{n}\binom{k+j}{j-2} = \binom{n+j+1}{j-1}, \tag{14}$$

$$1 + \sum_{k=0}^{n}\binom{k+j}{j-1} = \binom{n+j+1}{j}. \tag{15}$$

To show these, we recall the following formula (Pascal's Rule):

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Then we have

$$
\begin{aligned}
&j+\sum_{k=0}^{n}\binom{k+j}{j-2}\\
&=\binom{j}{1}+\sum_{k=0}^{n}\binom{j+k}{2+k}\\
&=\underbrace{\binom{j}{1}+\binom{j}{2}}+\binom{j+1}{3}+\binom{j+2}{4}+\cdots+\binom{j+n}{2+n}\\
&=\binom{j+1}{2}+\binom{j+1}{3}+\binom{j+2}{4}+\cdots+\binom{j+n}{2+n} \quad \text{(using Pascal's Rule)}\\
&=\underbrace{\binom{j+1}{2}+\binom{j+1}{3}}+\binom{j+2}{4}+\cdots+\binom{j+n}{2+n}\\
&=\binom{j+2}{3}+\binom{j+2}{4}+\binom{j+3}{5}+\cdots+\binom{j+n}{2+n}\\
&=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \text{(using Pascal's Rule step by step)}\\
&=\binom{j+n}{1+n}+\binom{j+n}{2+n}\\
&=\binom{j+n+1}{2+n} \quad \text{(using Pascal's Rule)}\\
&=\binom{j+n+1}{j-1}.
\end{aligned}
$$

Thus we get Equation (14). Similarly, using Pascal's Rule step by step we get

$$
\begin{aligned}
1+\sum_{k=0}^{n}\binom{k+j}{j-1} &= 1+\sum_{k=0}^{n}\binom{j+k}{1+k}\\
&=\binom{j}{0}+\binom{j}{1}+\binom{j+1}{2}+\binom{j+2}{3}+\cdots+\binom{j+n}{1+n}\\
&=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&=\binom{j+n}{n}+\binom{j+n}{1+n}\\
&=\binom{j+n+1}{1+n} \quad \text{(using Pascal's Rule)}\\
&=\binom{j+n+1}{j}.
\end{aligned}
$$

Thus we get Equation (15). $\square$

Example 16. For each case of the above Example 5

1. In the case when $m = 2$:

$$\begin{aligned}
F_n &: \underbrace{011}\,\underbrace{011}\,\underbrace{011}\,\underbrace{011}\cdots\cdots \\
SF_n &: \underbrace{010}\,\underbrace{010}\,\underbrace{010}\,\underbrace{010}\,\underbrace{010}\cdots\cdots \\
S^2F_n &: \underbrace{011100}\,\underbrace{011100}\,\underbrace{011100}\,\underbrace{011100}\cdots\cdots \\
S^3F_n &: \underbrace{010111101000}\,\underbrace{010111101000}\,\underbrace{010111101000}\,\underbrace{010111101000}\cdots\cdots \\
S^4F_n &: \underbrace{011010110000}\,\underbrace{011010110000}\,\underbrace{011010110000}\,\underbrace{011010110000}\cdots\cdots
\end{aligned}$$

2. In the case when $m = 3$:

$$\begin{aligned}
F_n &: \underbrace{01120221}\,\underbrace{01120221}\,\underbrace{01120221}\,\underbrace{01120221}\cdots\cdots \\
SF_n &: \underbrace{01211020}\,\underbrace{01211020}\,\underbrace{01211020}\,\underbrace{01211020}\cdots\cdots \\
S^2F_n &: \underbrace{010122111212002220201100}\,\underbrace{010122111212002220201100}\cdots\cdots \\
S^3F_n &: \underbrace{011210120202221022112000}\,\underbrace{011210120202221022112000}\cdots\cdots \\
S^4F_n &: \underbrace{0121220221102122101211111202001002210200212022222201011211002101102010000}\cdots
\end{aligned}$$

From Theorem 6 we get the following.

Theorem 8. *We let* $p_i = \Pi\Big(S^iF(a,b)_n, m\Big)$ *with* $p_0 = \Pi\Big(F(a,b)_n, m\Big)$. *Then we have*

$$\Pi\Big(S^{i+1}F(a,b)_n, m\Big) = \mathrm{ord}\Big(S^iF(a,b)_{p_i-1} \bmod m\Big) \times \Pi\Big(S^iF(a,b)_n, m\Big).$$

In particular, if m *is a prime, we have*

$$\Pi\Big(S^{i+1}F(a,b)_n, m\Big) = \begin{cases} \Pi\Big(S^iF(a,b)_n, m\Big) & \text{if } S^iF(a,b)_{p_i-1} \equiv 0 \mod m, \\ m \times \Pi\Big(S^iF(a,b)_n, m\Big) & \text{if } S^iF(a,b)_{p_i-1} \not\equiv 0 \mod m. \end{cases}$$

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