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# ON GENERALIZED EULER NUMBERS AND POLYNOMIALS RELATED TO VALUES OF THE LERCH ZETA FUNCTION

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**Abstract**
In this paper we mainly study special types of recurrence relations for the generalized Euler numbers and polynomials introduced and investigated by Goro Shimura that are closely related to the values of the Lerch transcendent and the alternating Hurwitz zeta function at non-positive integers.

## 1. Introduction

Let $\zeta(s;a,\gamma)$ be a Dirichlet series of the form

$$\zeta(s;a,\gamma)=\sum_{n=0}^{\infty}\frac{\gamma^n}{(n+a)^s},\quad s\in\mathbb{C},$$

where $0<a\in\mathbb{R}$ and $\gamma\in\mathbb{C}$ with $0<|\gamma|\le 1$. This is a generalization of the Hurwitz zeta function first studied by Lerch [10] in 1887 and usually called the Lerch transcendent (denoted by $\Phi$ in many cases), although some authors call it the Lerch zeta function all at once including the case $\gamma=e^{2\pi iz}$ for $z\in\mathbb{C}$. Clearly this series converges in the half-plane $\mathrm{Re}(s)>1$. When $\gamma=1$ and $-1$, this function reduces to the ordinary Hurwitz and the alternating Hurwitz zeta functions, namely

$$\begin{aligned}&\text{(i)}\qquad \zeta(s;a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\quad \mathrm{Re}(s)>1;\\ &\text{(ii)}\qquad \zeta^*(s;a)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^s},\quad \mathrm{Re}(s)>0,\end{aligned}\tag{1.1}$$

respectively.

According to Shimura's definition written in [18, 19], putting $\boldsymbol{e}(z):=\exp(2\pi iz)$ for $z\in\mathbb{C}$, define the generalized Euler numbers $E_{v,n}$ and polynomials $E_{v,n}(x)$, depending on $c := -\boldsymbol{e}(\alpha)$ with $\alpha \in \mathbb{R}\backslash\mathbb{Z}$, by the generating functions

$$\begin{aligned} &\text{(i)} \qquad \mathbb{E}_c(t) := \frac{(c+1)e^t}{e^{2t}+c} = \sum_{n=0}^{\infty} \frac{E_{c,n}}{n!} t^n; \\ &\text{(ii)} \qquad \mathbb{E}_c(t,x) := \frac{(c+1)e^{xt}}{e^t+c} = \sum_{n=0}^{\infty} \frac{E_{c,n}(x)}{n!} t^n, \end{aligned} \tag{1.2}$$

respectively. Here it requires $|t| < \pi$ if $c = 1$ and $|t| < |\log(-c)|$ if $c \neq 1$.

We wish now to reproduce faithfully Shimura's two theorems from his paper [19], in which some important analytic properties of $\zeta(s; a, \gamma)$ and an evaluation formula for the values at non-positive integers by means of the generalized Euler polynomials as defined above are accurately mentioned. Let $\Gamma$ denote the gamma function.

**Theorem 1.1.** *For $a$ and $\gamma$ as above, the product $\left(e^{2\pi i s} - 1\right)\Gamma(s)\zeta(s; a, \gamma)$ can be continued to an entire function in $s$. In addition, there exists a holomorphic function in $(s, a, \gamma) \in \mathbb{C}^3$, defined for* $\mathrm{Re}(a) > 0$ *and $\gamma \notin \{x \in \mathbb{R} \mid x \geq 1\}$ with no condition on $s$, that coincides with the product when* $\mathrm{Re}(s) > 1$, $0 < a \in \mathbb{R}$, *and* $0 < |\gamma| \leq 1$.

**Theorem 1.2.** *For $0 < k \in \mathbb{Z}$,* $\mathrm{Re}(a) > 0$, *and $\gamma \notin \{x \in \mathbb{R} \mid x \geq 1\}$, the value $\zeta(1-k; a, \gamma)$ is a polynomial function of $a$ and $(\gamma - 1)^{-1}$. More precisely, we have*

$$\zeta(1-k; a, \gamma) = \frac{E_{c,k-1}(a)}{1 + c^{-1}} \tag{1.3}$$

*for such $k, a,$ and $\gamma$, where $c = -\gamma^{-1}$.*

Let us denote by $B_n(x)$ and $E_n(x)$ the classical Bernoulli and Euler polynomials defined by the generating functions

$$\begin{aligned} \mathbb{B}(t,x) &:= \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} \quad (|t| < 2\pi); \\ \mathbb{E}(t,x) &:= \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n \quad (|t| < \pi), \end{aligned}$$

respectively. The special values $B_n := B_n(0)$ and $E_n := 2^n E_n(1/2)$ are called the classical Bernoulli and Euler numbers, respectively. In the case when $c = 1$, one sees that $E_{1,n} = E_n$ and $E_{1,n}(x) = E_n(x)$.

As is well-known, the Hurwitz zeta function defined in (1.1) (i) can be continued meromorphically to the whole $s$-plane with a simple pole at $s = 1$ of residue 1. Based on the fact that $-\zeta(-n; a)/n!$ equals the residue of $\mathbb{B}(z, a)/z^{n+2}$ at $z = 0$, it is possible to evaluate the values of $\zeta(s; a)$ at non-positive integers by means of classical Bernoulli polynomials. Precisely, letting $\mathbb{N} := \{1, 2, 3, \ldots\}$,

$$\zeta(1-n; a) = -\frac{B_n(a)}{n}, \quad n \in \mathbb{N}, \tag{1.4}$$

which is essentially due to Hurwitz [8, p.92]. Similarly, a quasi formula of (1.4) for the alternating Hurwitz zeta function defined in (1.1) (ii) can be stated by

$$\zeta^*(1-n;a) = \frac{E_{n-1}(a)}{2}, \quad n \in \mathbb{N}, \tag{1.5}$$

which is the special case of Shimura's formula (1.3) for $c = 1$; thus $\gamma = -1$.

The main purpose of this paper is to study special types of recurrence relations for the above generalized Euler numbers and polynomials that are closely related to the values of the Lerch zeta function at non-positive integers through (1.3).

In Section 2 we present basic properties of these numbers and polynomials that will be needed in our later discussion. In Section 3 we derive certain shortened recurrence relations, in which some of the preceding numbers or polynomials are excluded. In Section 4 we discuss special forms of recurrence relations that involve different kinds of sums (an ordinary sum and a binomial sum). We close this paper, in Section 5, with some additional remarks on relationships between the generalized Euler polynomials and the Apostol-Euler polynomials as defined below.

## 2. Basic Properties

In this section we present some basic properties for the generalized Euler numbers and polynomials by based on the generating function methods. We wish to point out beforehand that this section includes not only new but also some known results mainly quoted from [18, pp.25–26].

Here and in what follows, we use the following symbolic notation. For a sequence $\{T_n\}_{n\geq 9}$ of numbers or polynomials, and integers $n, k \geq 0$, we write

$$(pT_k + q)^n := \sum_{j=0}^{n} \binom{n}{j} p^j T_{k+j} q^{n-j},$$

where $p$ and $q$ are any fixed numbers or polynomials. In other words, expand the left-hand side in full by the binomial theorem and then replace $(T_k)^j$ by $T_{k+j}$ for each $j = 0, 1, \ldots, n$. This is a modified version of Lucas' notation $T^k(pT + q)^n$ that was used in his classic book [11, p.252].

As is easily seen, $E_{c,n}$ can be given by a special value of $E_{c,n}(x)$. Indeed, using the identity $\mathbb{E}_c(t) = \mathbb{E}_c(2t, 1/2)$, we have

$$E_{c,n} = 2^n E_{c,n}(1/2) \ \text{ for all } \ n \geq 0. \tag{2.1}$$

In particular, if $\alpha = 1/2$; thus $c = -\boldsymbol{e}(1/2) = 1$, then $E_{c,n}(x)$ is reduced to the classical Euler polynomial $E_n(x)$, i.e., $E_{1,n}(x) = E_n(x)$.

As a result, we see that $E_{c,n}(x)$ is a monic polynomial in $x$ of degree $n$ with polynomial coefficients in $b := -(c+1)^{-1}$. For example, we have

$$\begin{aligned}
&E_{c,0}(x) = 1;\;\; E_{c,1}(x) = x + b;\;\; E_{c,2}(x) = x^2 + 2bx + b(2b+1);\\
&E_{c,3}(x) = x^3 + 3bx^2 + 3b(2b+1)x + b(6b^2 + 6b + 1);\\
&E_{c,4}(x) = x^4 + 4bx^3 + 6b(2b+1)x^2 + 4b(6b^2 + 6b + 1)x\\
&\qquad\qquad + b(24b^3 + 36b^2 + 14b + 1),
\end{aligned}$$

and thus by (2.1),

$$\begin{aligned}
&E_{c,0} = 1;\;\; E_{c,1} = 2b + 1;\;\; E_{c,2} = 8b^2 + 8b + 1;\;\; E_{c,3} = 48b^3 + 72b^2 + 26b + 1;\\
&E_{c,4} = 384b^4 + 768b^3 + 464b^2 + 80b + 1.
\end{aligned}$$

As will be mentioned in detail below, they can be obtained recursively with the initial condition $E_{c,0}(x) = E_{c,0} = 1$.

For integers $n, k$ with $0 \le k \le n$ let $A(n,k)$ be the Eulerian numbers defined by

$$A(n,k) := \sum_{r=0}^{k} (-1)^r \binom{n+1}{r} (k-r)^n.$$

These numbers appear in the so-called Worpitzky identity

$$x^n = \sum_{k=0}^{n} \binom{x+n-k}{n} A(n;k), \quad n = 0, 1, 2, \dots,$$

and satisfy the recurrence relations $A(n,k) = A(n, n+1-k)$ and $A(n+1,k) = kA(n,k) + (n+2-k)A(n,k-1)$, assuming that $A(0,0) = 1$, $A(n,0) = 0$ $(n \ge 1)$, and $A(n,k) = 0$ if $n < k$ by convention.

We present here a list of the first few Eulerian numbers $A(n,k)$ for small positive integers $n$ and $k$ (cf. the OEIS, Sequence A008292, in [22]):

| $n\backslash k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | |
| 2 | 1 | 1 | | | | | | | |
| 3 | 1 | 4 | 1 | | | | | | |
| 4 | 1 | 11 | 11 | 1 | | | | | |
| 5 | 1 | 26 | 66 | 26 | 1 | | | | |
| 6 | 1 | 57 | 302 | 302 | 57 | 1 | | | |
| 7 | 1 | 120 | 1191 | 2416 | 1191 | 120 | 1 | | |
| 8 | 1 | 247 | 4293 | 15619 | 15619 | 4293 | 247 | 1 | |
| 9 | 1 | 502 | 14608 | 88234 | 156190 | 88234 | 14608 | 502 | 1 |

Using these Eulerian numbers, it can be shown that (cf. [5, Lemma 2.3])

$$\frac{d^n}{dt^n} \frac{1}{e^t + c}\bigg|_{t=0} = \frac{(-1)^n}{(c+1)^{n+1}} \sum_{j=1}^{n} A(n,j)(-c)^{j-1} \quad (n \ge 1). \tag{2.2}$$

Let us now define the polynomials $f_n(x)$ in $x$ of degree $n$ by

$$f_0(x) := 1, \quad f_n(x) := \sum_{j=1}^{n} A(n,j)(x+1)^{j-1}x^{n+1-j} \text{ for } n \geq 1.$$

Since $b = -(c+1)^{-1}$ means $c = -(b+1)/b$, we see from (2.2) that $E_{c,n}(x)$ is represented in terms of $f_k(b)$, $k = 0, 1, \ldots, n$, as follows:

$$E_{c,n}(x) = (f_0(b) + x)^n = \sum_{k=0}^{n} \binom{n}{k} f_k(b)x^{n-k}. \tag{2.3}$$

Taking, in particular, $x = 1/2$ in (2.3) and multiplying both sides by $2^n$, we get

$$E_{c,n} = 2^n E_{c,n}(1/2) = (2f_0(b) + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k f_k(b). \tag{2.4}$$

In the case for $c = 1$ (i.e., $b = -1/2$), this gives the expression for classical Euler numbers. Indeed, noting that $E_{1,n} = E_n$ and $f_0(-1/2) = 1$,

$$E_n = 1 + \sum_{k=1}^{n} \binom{n}{k} 2^k f_k(-1/2) = 1 + \sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{k} (-1)^{k+1-j} A(k,j).$$

As an expression of $E_{c,n}(x)$ in terms of generalized Euler numbers, we find from the identity $\mathbb{E}_c(2t, x) = \mathbb{E}_c(t)e^{(2x-1)t}$ that

$$E_{c,n}(x) = \frac{1}{2^n} \left(E_{c,0} + (2x-1)\right)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} E_{c,k} \cdot (2x-1)^{n-k}. \tag{2.5}$$

Equating this and (2.3), and then comparing the coefficients of $x^{n-k}$ on both sides, we can deduce an inversion formula for (2.4) such that, with replacing $k$ by $n$,

$$f_n(b) = \frac{1}{2^n} \left(E_{v,0} - 1\right)^n = \frac{1}{2^n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} E_{c,j}.$$

Using the expression of $E_{c,n}(x)$ given in (2.3) or (2.5), it is easy to show that

$$\frac{d}{dx} E_{c,n}(x) = nE_{c,n-1}(x) \quad (n \geq 1). \tag{2.6}$$

Since $E_{c,0}(x) = 1 \neq 0$, this property tells us that $\{E_{c,n}(x)\}_{n \geq 0}$ forms an Appell sequence. As is well-known, (2.6) is tantamount to the so-called Appell identity

$$E_{c,n}(x+y) = (E_{c,0}(x) + y)^n = \sum_{j=0}^{n} \binom{n}{j} E_{c,n-j}(x)y^j,$$

deduced from the identity $\mathbb{E}_c(t, x+y) = \mathbb{E}_c(t, x)e^{yt}$. Since $E_{c,n}(x+y)$ is symmetric with respect to $x$ and $y$, we see that

$$\sum_{j=0}^{n} \binom{n}{j} E_{c,n-j}(x)y^j = \sum_{j=0}^{n} \binom{n}{j} E_{c,n-j}(y)x^j.$$

More details on equivalence conditions related to Appell and Sheffer sequences can be found, e.g., in [7, 17].

In addition to (2.6), it is also easy to show the integral property

$$\int_x^y E_{c,n}(t)dt = \frac{E_{c,n+1}(y) - E_{c,n+1}(x)}{n+1} \quad (n \geq 0).$$

On the other hand, using the functional identity

$$\mathbb{E}_c(t, x+1) = \mathbb{E}_c(t, x)e^t = -c\mathbb{E}_c(t, x) + (c+1)e^{xt},$$

we obtain the most basic recurrence relation for these polynomials, namely

$$E_{c,n}(x+1) = \sum_{i=0}^{n} \binom{n}{i} E_{c,i}(x) = -cE_{c,n}(x) + (c+1)x^n, \tag{2.7}$$

or equivalently, gathering the terms involving $E_{c,n}(x)$ in one place and multiplying the whole by $b$,

$$E_{c,n}(x) = b \sum_{i=0}^{n-1} \binom{n}{i} E_{c,i}(x) + x^n.$$

We now extend (2.7) to a more general recurrence relation. Letting $q \geq 1$ be an arbitrary integer, remember the obvious factorization formula

$$X^q + (-1)^{q-1}Y^q = (X+Y) \sum_{j=0}^{q-1} (-Y)^j X^{q-1-j}.$$

Using this for $X = e^t$ and $Y = c$, we find the functional identity

$$\mathbb{E}_c(t, x) \left(e^{qt} + (-1)^{q-1}c^q\right) = (c+1) \sum_{j=0}^{q-1} (-c)^j e^{(x+q-1-j)t}.$$

Expanding both sides individually into the Taylor power series and then comparing the coefficients of $t^n/n!$ on both sides, we obtain

$$(E_{c,0}(x) + q)^n + (-1)^{q-1}c^q E_{c,\,n}(x) = (c+1) \sum_{j=0}^{q-1} (-c)^j (x+q-1-j)^n, \tag{2.8}$$

or equivalently, in the plain form,

$$\begin{aligned}&\left(1+(-1)^{q-1}c^q\right)E_{c,n}(x)+\sum_{i=0}^{n-1}\binom{n}{i}q^{n-i}E_{c,\,i}(x)\\&\quad=(c+1)\sum_{j=0}^{q-1}(-c)^j(x+q-1-j)^n,\end{aligned}$$

valid for all $n\geq 0$. When $q=1$, this is reduced to (2.7).

Take $x=1/2$ in (2.8) and multiply both sides by $2^n$. Then, from (2.1) we get

$$\begin{aligned}2^n(E_{c,\,0}(1/2)+q)^n&=(E_{c,\,0}+2q)^n=\sum_{i=0}^{n}\binom{n}{i}(2q)^{n-i}E_{c,\,i}\\&=(-1)^qc^qE_{c,\,n}+(c+1)\sum_{j=0}^{q-1}(-c)^j(2q-1-2j)^n.\end{aligned}$$

Furthermore, multiplying this by $b^q$ and gathering the terms involving $E_{c,\,n}$ in one place, one gets the following recurrence relation for the sequence $\{E_{c,n}\}_{n\geq 0}$:

$$\begin{aligned}&\left((b+1)^q-b^q\right)E_{c,\,n}-b^q\sum_{i=0}^{n-1}\binom{n}{i}(2q)^{n-i}E_{c,\,i}\\&\quad=\sum_{j=0}^{q-1}(-1)^jb^{q-1-j}(b+1)^j(2q-1-2j)^n.\end{aligned}$$

As is obvious, if $q=1$, then this reduces to

$$E_{c,n}-b\sum_{i=0}^{n-1}\binom{n}{i}2^{n-i}E_{c,i}=1,$$

and the well-known formula $(E_0+2)^n+E_n=2$ for classical Euler numbers is given immediately by setting $c=1$ (i.e., $b=-1/2$).

On the other hand, based on the identity $\mathbb{E}_{c^{-1}}(t,x)=\mathbb{E}_c(-t,1-x)$, we can prove the symmetry of the generalized Euler polynomials, namely

$$E_{c^{-1},n}(x)=(-1)^nE_{c,n}(1-x)\quad\text{(cf. [19, (0.3a)])},\tag{2.9}$$

which yields $E_{c^{-1},n}=(-1)^nE_{c,n}$ from (2.1) upon taking $x=1/2$. Furthermore, since $\mathbb{E}_c(t,1)=c+1-c\mathbb{E}_c(t,0)$, it can be shown that

$$E_{c,n}(1)=\begin{cases}c+1-cE_{c,0}(0)=1 & \text{if } n=0;\\ E_{c,n}(1)=-cE_{c,n}(0) & \text{if } n\geq 1.\end{cases}\tag{2.10}$$

Hence, taking $x = 0$ in (2.9), we have $E_{c^{-1},0}(0) = E_{c,0}(1) = 1$ and for $n \geq 1$,

$$E_{c^{-1},n}(0) = (-1)^n E_{c,n}(1) = (-1)^{n+1} c E_{c,n}(0).$$

As is clear from (2.3), if $n \geq 1$, then (2.10) implies $(f_0(b) + 1)^n = -c f_n(b)$. Multiplying both sides of this by $b$ after gathering $f_n(b)$ in one place, we get

$$f_n(b) = b \sum_{k=0}^{n-1} \binom{n}{k} f_k(b) \quad (n \geq 1).$$

Using this identity, all terms of the sequence $\{f_n(b)\}_{n \geq 0}$ can be obtained recursively with $f_0(b) = 1$ as an initial condition. Just to be sure, note that $f_n(b)$ is a polynomial in $b$ of degree $n$ with the leading coefficient $\prod_{k=1}^{n} \binom{k}{k-1} = n!$.

## 3. Shortened Recurrence Relations

In this section, applying a characteristic property of Appell sequences, we derive certain shortened recurrence relations for generalized Euler polynomials, in which some of the preceding polynomials are completely excluded.

To begin with, we introduce an elementary identity related to an alternating sum of products of binomial coefficients. Note that this identity is not new; but for the sake of completeness we wish to give its short and concise proof.

**Lemma 3.1.** *For any integers $n, m, k \geq 0$ we have*

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{m+i}{n+m-k} = \binom{m}{k}, \tag{3.1}$$

*where $\binom{x}{y} = 0$ if $y < 0$ or $x < y$ by convention. In particular,*

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{m+i}{n} = 1. \tag{3.2}$$

*Proof.* Since (3.2) is nothing but the special case of (3.1) for $m = k$, we give only the proof of (3.1). For integers $n, m \geq 0$ consider two kinds of binomial expansions of $x^n (x+1)^m$ such that

$$x^n (x+1)^m = ((x+1) - 1)^n (x+1)^m = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (x+1)^{m+i}$$
$$= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \sum_{r=0}^{m+i} \binom{m+i}{r} x^r$$

and

$$x^n(x+1)^m = \sum_{j=0}^{m} \binom{m}{j} x^{n+j}.$$

Equating these and comparing the coefficients of $x^{n+m-k}$ for any fixed $k \geq 0$ on both sides, we obtain

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\binom{m+i}{n+m-k} = \binom{m}{m-k} = \binom{m}{k},$$

which is just (3.1), as desired. □

The following is our first main result based on the fact that $\{E_{c,n}(x)\}_{n\geq 0}$ forms an Appell sequence as mentioned in the previous section.

**Theorem 3.2.** *For integers $l, m \geq 0$ we have*

$$(E_{c,l}(x)+y)^m = (E_{c,m}(x+y)-y)^l. \tag{3.3}$$

*Proof.* For simplicity, denoting the left-hand and right-hand sides of (3.3) by $P(x,y)$ and $Q(x,y)$, respectively, we regard them as polynomials in $y$ of degree $l+m$ with polynomial coefficients $p_k(x), q_k(x) \in \mathbb{C}[x]$, namely

$$P(x,y) = \sum_{k=0}^{l+m} p_k(x) y^k \quad \text{and} \quad Q(x,y) = \sum_{k=0}^{l+m} q_k(x) y^k.$$

For (3.3), it suffices to prove that $p_k(x) = q_k(x)$ for all $k \in I := \{0, 1, \ldots, l+m\}$. Expanding the left-hand side of (3.3) using the binomial theorem, we see that

$$p_k(x) = \binom{m}{k} E_{c,l+m-k}(x) \quad \text{for all} \quad k \in I. \tag{3.4}$$

Next differentiate $Q(x,y)$ $k$ times with respect to $y$. Then, based on (2.4) and using Leibniz's rule, we obtain

$$\begin{aligned}
\frac{d^k}{dy^k}Q(x,y) &= \sum_{i=0}^{l}\binom{l}{i}\frac{d^k}{dy^k}\left(E_{c,m+i}(x+y)(-y)^{l-i}\right) \\
&= \sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}\sum_{a=0}^{k}\binom{k}{a}\frac{d^{k-a}}{dy^{k-a}}E_{c,m+i}(x+y)\frac{d^a}{dy^a}y^{l-i} \\
&= \sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}\sum_{0\leq a\leq \min\{k,l-i\}}\binom{k}{a}\left\{(m+i)_{k-a}E_{c,m+i-(k-a)}(x+y)\right\} \\
&\qquad \times\left\{(l-i)_a y^{l-i-a}\right\},
\end{aligned}$$

where $(z)_k = z(z-1)\cdots(z-k+1)$ is the falling factorial of $z$. Set here $y = 0$. Then the only non-vanishing term is $a_{l,m,k}(i)E_{c,l+m-k}(x)$ (the term corresponding to $a = l - i$), where $a_{l,m,k}(i)$ is given by

$$\binom{k}{l-i}(m+i)_{k-l+i}(l-i)_{l-i} = \binom{k}{l-i}\frac{(m+i)!(l-i)!}{(l+m-k)!0!} = k!\binom{m+i}{l+m-k}.$$

Therefore, making use of (3.1), it can be shown that

$$\begin{aligned} q_k(x) = \frac{1}{k!}\frac{d^k}{dy^k}Q(x,y)\Big|_{y=0} &= \left(\frac{1}{k!}\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}a_{l,m,k}(i)\right)E_{c,l+m-k}(x) \\ &= \binom{m}{k}E_{c,l+m-k}(x) \quad (k \geq 0), \end{aligned}$$

which proves in view of (3.4) that $p_k(x) = q_k(x)$ holds true for all $k \in I$. □

**Theorem 3.3.** *For integers $l, m \geq 0$ and $q \geq 1$ we have*

$$\begin{aligned} &(E_{c,\,l}(x) + q)^m + (-1)^{q-1}c^q\,(E_{c,\,m}(x) - q)^l \\ &\quad = (c+1)\sum_{j=0}^{q-1}(-c)^j(x+q-1-j)^m(x-1-j)^l. \end{aligned} \tag{3.5}$$

*In particular,*

$$(E_{c,\,l}(x) + 1)^m + c\,(E_{c,\,m}(x) - 1)^l = (c+1)x^m(x-1)^l. \tag{3.6}$$

*Proof.* Since (3.6) is the particular case of (3.5) for $q = 1$, it is enough to prove only (3.5). Set $y = q$ in (3.3), that is,

$$(E_{c,l}(x) + q)^m = (E_{c,m}(x+q) - q)^l\,. \tag{3.7}$$

Making use of (2.8), the right-hand side of (3.7) can be rewritten as follows:

$$\begin{aligned} (E_{c,m}(x+q) - q)^l &= \sum_{i=0}^{l}\binom{l}{i}E_{c,m+i}(x+q)(-q)^{l-i} \\ &= \sum_{i=0}^{l}(-q)^{l-i}\binom{l}{i}\left((-c)^qE_{c,m+i}(x) + (c+1)\sum_{j=0}^{q-1}(-c)^j(x+q-1-j)^{m+i}\right) \\ &= (-c)^q\sum_{i=0}^{l}(-q)^{l-i}\binom{l}{i}E_{c,m+i}(x) \\ &\quad + (c+1)\sum_{j=0}^{q-1}(-c)^j(x+q-1-j)^m\sum_{i=0}^{l}\binom{l}{i}(-q)^{l-i}(x+q-1-j)^i \end{aligned}$$

$$= (-c)^q \left(E_{c,m}(x) - q\right)^l + (c+1)\sum_{j=0}^{q-1}(-c)^j(x+q-1-j)^m(x-1-j)^l.$$

Therefore, (3.5) is deduced by substituting this into (3.7). □

**Corollary 3.4.** *For integers $l, m \geq 0$ and $q \geq 1$ we have*

$$\begin{aligned}&\left(E_{c,\,l} + 2q\right)^m + (-1)^{q-1}c^q\left(E_{c,\,m} - 2q\right)^l \\ &\quad = (c+1)\sum_{j=0}^{q-1}(-1)^{l-j}c^j(2q-1-2j)^m(1+2j)^l.\end{aligned}$$

*In particular,*

$$\left(E_{c,l} + 2\right)^m + c\left(E_{c,m} - 2\right)^l = (-1)^l(c+1).$$

*Proof.* Take $x = 1/2$ in (3.5) and (3.6), and then use (2.1). □

Applying Shimura's formula (1.3), we can convert all identities for generalized Euler polynomials into those for the values of $\zeta(s; a, \gamma)$ at non-positive integers. For example, writing formally as $\zeta(s-i; a, \gamma) = \left(\zeta(s; a, \gamma)\right)^i$ for any integers $i \geq 0$, we see that (3.6) is tantamount to the identity

$$\gamma\left(\zeta(-l; a, \gamma) + 1\right)^m - \left(\zeta(-m; a, \gamma) - 1\right)^l = -a^m(a-1)^l,$$

where $\mathrm{Re}(a) > 0$ and $\gamma = -c^{-1} \notin \{x \in \mathbb{R} \mid x \geq 1\}$. When $\gamma = -1$ (i.e., $c = 1$), this identity reduces to, writing similarly as $(\zeta^*(s-i; a)) = \zeta^*(s; a))^i$ for $i \geq 0$,

$$\left(\zeta^*(-l; a) + 1\right)^m + \left(\zeta^*(-m; a) - 1\right)^l = a^m(a-1)^l,$$

which is equivalent to (3.6) with $c = 1$, in view of (1.5).

It is also possible to deduce special types of recurrence relations for the values of $\zeta(s; a)$ at non-positive integers. For example, writing as $\zeta(s-i; a) = (\zeta(s; a))^i$ for $i \geq 0$ and based on (1.4), we get the following identity (cf. [3, (4.5)]):

$$\left(\zeta(-l; a) + 1\right)^m - \left(\zeta(-m; a) - 1\right)^l = a^m(a-1)^l + \frac{(-1)^{l+1}}{l+m+1}\binom{l+m}{m}^{-1}.$$

This is just a translation of the Saalschütz-Gelfand type formula for classical Bernoulli polynomials, namely, defining $\beta_i(a) := B_i(a)/i$ for $i \geq 1$,

$$\left(\beta_{l+1}(a) + 1\right)^m - \left(\beta_{m+1}(a) - 1\right)^l = a^m(a-1)^l + \frac{(-1)^{l+1}}{l+m+1}\binom{l+m}{m}^{-1}.$$

An elementary proof of this formula can be found in [1, Section 3]. See also [4] for such types of recurrence relations in the classical Bernoulli number case.

## 4. Recurrence Relations Involving Different Kinds of Sums

In this section we investigate special forms of recurrence relations for generalized Euler numbers and polynomials involving two different kinds of sums.

At first, we present an elementary lemma that is concerned with the Euler-beta integrals. The proof is quite easy.

**Lemma 4.1.** *It follows that*

$$\begin{aligned} &\text{(i)} \qquad \int_0^1 u^n(1-u)^m du = \frac{n!m!}{(n+m+1)!} \quad (n,m \geq 0); \\ &\text{(ii)} \qquad \frac{1}{2}\int_0^1 \frac{1-u^{n+1}-(1-u)^{n+1}}{u(1-u)}du = \int_0^1 \frac{1-u^n}{1-u}du = H_n \quad (n \geq 1), \end{aligned} \tag{4.1}$$

*where* $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ *is the nth harmonic number.*

By making use of this lemma, we derive the following recurrence relations that combine different kinds of sums (an ordinary sum and a binomial sum).

**Theorem 4.2.** *For an integer* $n \geq 1$ *we have*

$$c\sum_{i=0}^{n} E_{c,i}(x) + \sum_{i=0}^{n}\binom{n+1}{i+1}E_{c,i}(x) = (c+1)\sum_{i=0}^{n} x^i; \tag{4.2}$$

$$c\sum_{i=1}^{n} \frac{E_{c,i}(x)}{i} + \sum_{i=1}^{n}\binom{n}{i}\frac{E_{c,i}(x)}{i} = (c+1)\sum_{i=1}^{n}\frac{x^i}{i} - H_n; \tag{4.3}$$

$$\begin{aligned} &c\sum_{i=0}^{n-1} \frac{E_{c,i}(x)}{n-i} - \sum_{i=1}^{n-1}\binom{n}{i}H_iE_{c,i}(x) \\ &\quad = (c+1)H_n\left(E_{c,n}(x) - x^n\right) + (c+1)\sum_{i=1}^{n}\frac{x^{n-i}}{i}. \end{aligned} \tag{4.4}$$

*Proof.* Letting $u$ be a real parameter, consider the functional identity

$$c\mathbb{E}_c(ut,x)e^{(1-u)t} + \mathbb{E}_c(ut,x)e^t = (c+1)e^{(u(x-1)+1)t},$$

which is easily verified by a direct calculation. Differentiating both sides $n$ times with respect to $t$ based on Leibniz's rule and then setting $t=0$, we obtain

$$\begin{aligned} &c\sum_{i=0}^{n}\binom{n}{i}u^i(1-u)^{n-i}E_{c,i}(x) + \sum_{i=0}^{n}\binom{n}{i}u^iE_{c,i}(x) \\ &\quad = (c+1)(u(x-1)+1)^n. \end{aligned} \tag{4.5}$$

All the proofs of (4.2)–(4.4) given below rely on this parameterized identity.

**1.** Integrate both sides of (4.5) from 0 to 1 with respect to $u$. Then, using (4.1) (i) and the easy identity $\binom{n}{i}\frac{i!(n-i)!}{(n+1)!} = \frac{1}{n+1}$ $(0 \le i \le n)$, we get

$$\frac{c}{n+1}\sum_{i=0}^{n} E_{c,i}(x) + \sum_{i=0}^{n}\binom{n}{i}\frac{E_{c,i}(x)}{i+1} = \frac{c+1}{n+1}\sum_{i=0}^{n} x^i,$$

which yields (4.2) by multiplying both sides by $n+1$.

**2.** In (4.5), gathering the terms involving $E_{c,0}(x)(=1)$ in one place, write it as

$$c\sum_{i=1}^{n}\binom{n}{i}u^i(1-u)^{n-i}E_{c,i}(x) + \sum_{i=1}^{n}\binom{n}{i}u^iE_{c,i}(x)$$
$$= (c+1)\sum_{i=1}^{n}\binom{n}{i}u^i(x-1)^i + c(1-(1-u)^n).$$

Dividing the whole by $u$, we get

$$c\sum_{i=1}^{n}\binom{n}{i}u^{i-1}(1-u)^{n-i}E_{c,i}(x) + \sum_{i=1}^{n}\binom{n}{i}u^{i-1}E_{c,i}(x)$$
$$= (c+1)\sum_{i=1}^{n}\binom{n}{i}u^{i-1}(x-1)^i + c\frac{1-(1-u)^n}{u}.$$

Integrating both sides from 0 to 1 with respect to $u$ and using both formulas in (4.1), we can deduce the recurrence relation

$$c\sum_{i=1}^{n}\frac{E_{c,i}(x)}{i} + \sum_{i=1}^{n}\binom{n}{i}\frac{E_{c,i}(x)}{i} = (c+1)\sum_{i=1}^{n}\binom{n}{i}\frac{(x-1)^i}{i} + cH_n. \tag{4.6}$$

Since the sum on the right-hand side can be written as

$$\sum_{i=1}^{n}\binom{n}{i}\frac{(x-1)^i}{i} = \sum_{i=1}^{n}\frac{x^i}{i} - H_n \quad \text{(cf. [2, Lemma 3.1])}, \tag{4.7}$$

we see that (4.6) yields (4.3).

**3.** Recall now (2.6), namely

$$\sum_{i=0}^{n}\binom{n}{i}E_{c,i}(x) = -cE_{c,n}(x) + (c+1)x^n \quad (n \ge 0).$$

Using this identity, rewrite the second sum on the left-hand side of (4.5) as

$$\sum_{i=0}^{n}\binom{n}{i}u^iE_{c,i}(x) = -\sum_{i=0}^{n}\binom{n}{i}(1-u^i)E_{c,i}(x) + \sum_{i=0}^{n}\binom{n}{i}E_{c,i}(x)$$

$$= -\sum_{i=0}^{n}\binom{n}{i}(1-u^i)E_{c,i}(x) - cE_{c,n}(x) + (c+1)x^n.$$

Substituting this identity into (4.5) and gathering the terms involving $E_{c,n}(x)$ in one place, we get

$$\begin{aligned} c\sum_{i=0}^{n-1}\binom{n}{i}u^i(1-u)^{n-i}E_{c,i}(x) - \sum_{i=1}^{n-1}\binom{n}{i}(1-u^i)E_{c,i}(x) \\ = (c+1)(1-u^n)E_{c,n}(x) + (c+1)\left\{(u(x-1)+1)^n - x^n\right\}. \end{aligned} \tag{4.8}$$

Here, the last term on the right-hand side equals $(c+1)$ times

$$\begin{aligned} (u(x-1)+1)^n - x^n &= \{(1-u)(1-x)+x\}^n - x^n \\ &= \sum_{i=1}^{n}\binom{n}{i}(1-u)^i(1-x)^i x^{n-i}. \end{aligned}$$

Hence, substituting this into (4.8) and dividing both sides by $1-u$, we have

$$\begin{aligned} c\sum_{i=0}^{n-1}\binom{n}{i}u^i(1-u)^{n-1-i}E_{c,i}(x) - \sum_{i=1}^{n-1}\binom{n}{i}\frac{1-u^i}{1-u}E_{c,i}(x) \\ = (c+1)\frac{1-u^n}{1-u}E_{c,n}(x) + (c+1)\sum_{i=1}^{n}\binom{n}{i}(1-u)^{i-1}(1-x)^i x^{n-i}. \end{aligned}$$

Integrate both sides of this identity from 0 to 1 with respect to $u$. Then we obtain by using (4.1) (i),

$$\begin{aligned} c\sum_{i=0}^{n-1}\frac{E_{c,i}(x)}{n-i} - \sum_{i=1}^{n-1}\binom{n}{i}H_iE_{c,i}(x) \\ = (c+1)H_nE_{c,n}(x) + (c+1)\sum_{i=1}^{n}\binom{n}{i}\frac{(1-x)^i x^{n-i}}{i}. \end{aligned} \tag{4.9}$$

Making use of (4.7) replaced $x$ by $1/x$, the last summation term on the right-hand side of (4.9) can be written as $(c+1)$ times

$$\begin{aligned} \sum_{i=1}^{n}\binom{n}{i}\frac{(1-x)^i x^{n-i}}{i} &= x^n\sum_{i=1}^{n}\binom{n}{i}\frac{1}{i}\left(\frac{1}{x}-1\right)^i = x^n\left(\sum_{i=1}^{n}\frac{1}{ix^i} - H_n\right) \\ &= \sum_{i=1}^{n}\frac{x^{n-i}}{i} - H_nx^n. \end{aligned}$$

By substituting this into (4.9) we get (4.4). $\square$

Taking $x = 1/2$ in Theorem 4.2 and using (2.1), we can deduce the following recurrence relations for generalized Euler numbers.

**Corollary 4.3.** *For an integer $n \geq 0$ we have*

$$\text{(i)} \qquad c\sum_{i=0}^{n} 2^{n-i}E_{c,i} + \sum_{i=0}^{n}\binom{n+1}{i+1}2^{n-i}E_{c,i} = (c+1)(2^{n+1}-1);$$

$$\text{(ii)} \qquad c\sum_{i=1}^{n} \frac{2^{n-i}E_{c,i}}{i} + \sum_{i=1}^{n}\binom{n}{i}\frac{2^{n-i}E_{c,i}}{i} = (c+1)\sum_{i=1}^{n}\frac{2^{n-i}}{i} - 2^n H_n;$$

$$\text{(iii)} \qquad c\sum_{i=0}^{n-1} \frac{2^{n-i}E_{c,i}}{n-i} - \sum_{i=1}^{n-1}\binom{n}{i}2^{n-i}H_iE_{c,i}$$
$$= (c+1)H_n(E_{c,n}-1) + (c+1)\sum_{i=1}^{n}\frac{2^i}{i}.$$

Similar to the case for Theorems 3.2 and 3.3, all recurrence relations mentioned in Theorem 4.2 can be converted into those for the values of $\zeta(s; a, \gamma)$ at non-positive integers applying Shimura's formula (1.3). For example, (4.2) is translated into

$$\sum_{i=0}^{n}\zeta(-i; a, \gamma) = \gamma\sum_{i=0}^{n}\binom{n}{i}\zeta(-i; a, \gamma) + \sum_{i=0}^{n}a^i,$$

where $\mathrm{Re}(a) > 0$ and $\gamma = -c^{-1}$. In particular, if $\gamma = -1$ (i.e., $c = 1$), then this identity is reduced to

$$\sum_{i=0}^{n}\zeta^*(-i; a) = -\sum_{i=0}^{n}\binom{n}{i}\zeta^*(-i; a) + \sum_{i=0}^{n}a^i,$$

which is obviously equivalent to (4.2) with $x = a$ and $c = 1$.

## 5. Additional Remarks

Letting $\lambda$ be a real or complex parameter, we now define the Apostol-Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$ and Apostol-Euler polynomials $\mathcal{E}_n(x; \lambda)$ associated with $\lambda$ by means of the generating functions

$$\begin{aligned} &\text{(i)} \qquad \mathbb{B}(t, x; \lambda) := \frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty}\frac{\mathcal{B}_n(x; \lambda)}{n!}t^n \\ &\qquad\qquad (|t| < 2\pi \text{ if } \lambda = 1; |t| < |\log\lambda| \text{ if } \lambda \neq 1); \\ &\text{(ii)} \qquad \mathbb{E}(t, x; \lambda) := \frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty}\frac{\mathcal{E}_n(x; \lambda)}{n!}t^n \\ &\qquad\qquad (|t| < \pi \text{ if } \lambda = 1; |t| < |\log(-\lambda)| \text{ if } \lambda \neq 1), \end{aligned} \tag{5.1}$$

respectively. When $\lambda = 1$, these polynomials reduce to the classical Bernoulli and Euler polynomials. Moreover, the numbers defined by $\mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda)$ and $\mathcal{E}_n(\lambda) := 2^n\mathcal{E}_n(1/2;\lambda)$ are called, respectively, the Apostol-Bernoulli and Apostol-Euler numbers. Of course we may define them by using the exponential generating functions $\mathbb{B}(t;\lambda) := t/(\lambda e^t - 1)$ and $\mathbb{E}(t;\lambda) := 2e^t/(\lambda e^{2t} + 1)$, respectively.

In 1951 Apostol [6] first introduced and investigated $\mathcal{B}_n(x;\lambda)$ as a function of $x$ in order to evaluate the values of the Lerch transcendent function $\Phi$ (we use here the notation $\Phi$ for convenience' sake) at non-positive integers, defined by analytic continuation of the series

$$\Phi(s;a,\lambda) := \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+a)^s},$$

where $a \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$. It requires the condition, either $\mathrm{Re}(s) > 1$ if $|\lambda| = 1$ or $s \in \mathbb{C}$ if $|\lambda| < 1$ for convergence. Here note that $\lambda$ is completely independent of $\gamma$ as defined and used in Section 1. As a result, he established the formula

$$\Phi(1-n;a,\lambda) = -\frac{\mathcal{B}_n(a;\lambda)}{n}, \quad n \in \mathbb{N},$$

which includes Hurwitz's formula (1.4) as the special case for $\lambda = 1$.

Later, these functions $\mathcal{B}_n(x;\lambda)$ have been studied quite extensively by many authors under the name "the Apostol-Bernoulli polynomials" mainly from analytic, arithmetic and combinatorial points of view, and generalized in various directions. The higher order Apostol-Bernoulli and Apostol-Euler polynomials first defined and studied by Luo are especially very interesting, possessing many attractive and important properties. For their explicit definitions and details on related matters, see, e.g., the papers by Luo [12, 13] and by Luo and Srivastava [14, 15].

We wish to explore below some basic relationships between the Apostol-Euler and generalized Euler polynomials. Setting $\lambda = 1/c$ in the generating functions for these two kinds of polynomials, we can find the following relation:

$$\mathbb{E}(t,x;1/c) = g(c)\mathbb{E}_c(t,x), \quad g(c) := \frac{2c}{c+1}.$$

Here note that the function $g(c)$ is meaningful because $c = -\boldsymbol{e}(\alpha) \neq 0, -1$ for any $\alpha \in \mathbb{R}\backslash\mathbb{Z}$. Expanding both sides of this individually into the Taylor power series and then comparing the coefficients of $t^n/n!$ on both sides, we get

$$\mathcal{E}_n(x;1/c) = g(c)E_{c,n}(x) \quad (n \geq 0). \tag{5.2}$$

Using this relation, one can rephrase all identities for Apostol-Euler polynomials in words of generalized Euler polynomials, and vice versa. Taking $x = 1/2$ in (5.2) and multiplying both sides by $2^n$, we have

$$\mathcal{E}_n(1/c) = g(c)E_{c,n} \quad (n \geq 0). \tag{5.3}$$

We now present below easy examples to show how to convert from one to the other. As the first example, we observe the following identity expressing $\mathcal{E}_n(x;\lambda)$ in terms of Apostol-Euler numbers:

$$\mathcal{E}_n(x;\lambda) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k(\lambda)(2x-1)^{n-k}, \tag{5.4}$$

as can be deduced from $\mathbb{E}(2t,x;\lambda) = \mathbb{E}(t;\lambda)e^{(2x-1)t}$. Setting here $\lambda = 1/c$, from (5.2) and (5.3) we can get (2.5). Conversely, if we set $c = 1/\lambda$ in (2.5), then it turns back to (5.4). Thus we may state that (2.5) and (5.2) are *practically* the same.

For another example, we next observe the identity

$$\lambda\mathcal{E}_n(x+1;\lambda) = -\mathcal{E}_n(x;\lambda) + 2x^n, \tag{5.5}$$

which is derived from $\lambda\mathbb{E}(t,x+1;\lambda) = -\mathbb{E}(t,x;\lambda) + 2e^{xt}$. Similar to the above, setting $\lambda = 1/c$ and multiplying both sides by $(c+1)/2$, we see that (5.5) can be converted into (2.6). Needless to say, such a conversion operation is invertible.

In the same way as above, based on (5.2), it is possible to convert all recurrence relations in Theorems 3.2, 3.3 and 4.2 into those for Apostol-Euler polynomials. Similarly, by using (5.3) the recurrence relations in Corollaries 3.4 and 4.3 can be rephrased by means of Apostol-Euler numbers.

In connection with the above, define the generalized Apostol-Euler polynomials $\mathcal{E}_{n,\chi}(x;\lambda)$ attached to a Dirichlet character $\chi$ with conductor $f = f_\chi$ by means of the generating function

$$\mathbb{E}_\chi(t,x;\lambda) := \frac{2}{\lambda^f e^{ft}+1}\sum_{a=0}^{f-1}\chi(a)\lambda^a e^{(x+a)t} = \sum_{n=0}^{\infty}\frac{\mathcal{E}_{n,\chi}(x;\lambda)}{n!}t^n,$$

where $\lambda$ is a real or complex parameter with $\lambda^f \neq -1$, and $|t+\log\lambda| < \pi/f$. In particular, if $\chi = \chi^0$ (the principal character), then we have $\mathcal{E}_{n,\chi^0}(x;\lambda) = \mathcal{E}_n(x;\lambda)$. As well as the ordinary Apostol-Euler polynomials as defined and used above, these generalized ones attached to $\chi$ also satisfy similar types of recurrence relations, which we will not discuss here in detail. Many interesting new approaches to a more wide class of generalized Apostol-type numbers and polynomials can be found, e.g., in [9, 16, 20, 21] and the references therein.

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## References

[1] T. Agoh, Recurrences for Bernoulli and Euler polynomials and numbers, *Expo. Math.* **18** (2000), 197–214.

[2] T. Agoh, On Miki's identity for Bernoulli numbers, *Integers* **16** (2016), # 73, 12 pp.

[3] T. Agoh, Shortened recurrence relations for generalized Bernoulli polynomials, *J. Number Theory* **176** (2017), 149–173.

[4] T. Agoh and K. Dilcher, Reciprocity relations for Bernoulli numbers, *Amer. Math. Monthly* **115** (2008), 237–244.

[5] T. Agoh and M. Yamanaka, A study of Frobenius-Euler numbers and polynomials, *Ann. Sci. Math. Québec* **34** (2010), 1–14.

[6] T. M. Apostol, On the Lerch zeta function, *Pacific J. Math.* **1** (1951), 161–167.

[7] A. Di Bucchianico, D. Loeb and G.-C. Rota, Umbral calculus in Hilbert space. In: *Mathematical Essays in Honor of Gian-Carlo Rota* (Cambridge, MA, 1996), Progr. Math., 161, Birkhäuser, Boston, 1998, pp. 213–238.

[8] A. Hurwitz, Einige Eigenschaften der Dirichlet'schen Functionen $F(s) = \sum \left(\frac{D}{n}\right) \frac{1}{n^s}$, die bei der Bestimmung der Classenanzahlen binärer quadratischer Formen auftreten, *Zeitschrift f. Math. u. Physik* **27** (1882), 86–101 (Mathematische Werke Vol. I, pp.72–88, Birkhäuser, Basel, 1923).

[9] T. Kim, On the analogs of Euler numbers and polynomials associated with $p$-adic $q$-integral on $\mathbb{Z}_p$ at $q = 1$, *J. Math. Anal. Appl.* **331** (2007), 779–792.

[10] M. Lerch, Note sur la fonction $\mathfrak{K}(w, x, s) = \sum_{k=0}^{\infty} e^{2k\pi ix}/(w + k)^s$, *Acta Math.* **11** (1887), 19–24.

[11] F. E. A. Lucas, *Théorie de nombres, Vol. 1,* Gauthiers-Villars, Paris, 1891. Reprinted by A. Blanchard, Paris, 1961.

[12] Q.-M. Luo, Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions, *Taiwanese J. Math.* **10** (2006), 917–925.

[13] Q.-M. Luo, Some formulas for Apostol-Euler polynomials associated with Hurwitz zeta function at rational arguments, *Appl. Anal. Discrete Math.* **3** (2009), 336–346.

[14] Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, *J. Math. Anal. Appl.* **308** (2005), 290–302.

[15] Q.-M. Luo and H. M. Srivastava, Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, *Comput. Math. Appl.* **51** (2006), 631–642.

[16] H. Ozden and Y. Simsek, Unified representation of the family of $L$-functions, *J. Inequal. Appl.*, 2013, Paper No. 2013:64, 10 pp.

[17] G.-C. Rota, D. Kahaner and A. Odlyzko, On the foundations of combinatorial theory, VIII. Finite Operator Calculus, *J. Math. Anal. Appl.* **42** (1973), 684–760.

[18] G. Shimura, *Elementary Dirichlet series and modular forms*, Springer Monogr. Math., Springer, New York, 2007.

[19] G. Shimura, The critical values of generalizations of the Hurwitz zeta function, *Doc. Math.* **15** (2010), 489–506.

[20] Y. Simsek, Analysis of the $p$-adic $q$-Volkenborn integrals: An approach to generalized Apostol-type special numbers and polynomials and their applications, *Cogent Math.* **3** (2016), Article ID 1269393, 17 pp.

[21] H. M. Srivastava, T. Kim and Y. Simsek, $q$-Bernoulli numbers and polynomials associated with multiple $q$-zeta functions and basic $L$-series, *Russ. J. Math. Phys.* **12** (2005), 241–268.

[22] N. J. A. Sloane (ed.), Sequence A008292 in *The On-Line Encyclopedia of Integer Sequences* (OEIS), electronically published at https://oeis.org/.