RAINBOW NUMBERS FOR $x_1 + x_2 = kx_3$ IN $\mathbb{Z}_n$

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Abstract

In this work, we investigate the fewest number of colors needed to guarantee a rainbow solution to the equation $x_1 + x_2 = kx_3$ in $\mathbb{Z}_n$. This value is called the rainbow number and is denoted by $rb(\mathbb{Z}_n, k)$ for positive integer values of n and k. We find that $rb(\mathbb{Z}_p, 1) = 4$ for all primes greater than 3 and that $rb(\mathbb{Z}_n, 1)$ can be determined from the prime factorization of n. Furthermore, when k is prime, $rb(\mathbb{Z}_n, k)$ can be determined from the prime factorization of n.

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1. Introduction

Let $\mathbb{Z}_n$ be the cyclic group of order n, and let an *r-coloring* of $\mathbb{Z}_n$ be a function $c : \mathbb{Z}_n \to [r]$ where $[r] := \{1, ..., r\}$. In this paper, we assume that each r-coloring is *exact* (surjective). Given an exact r-coloring, we define r *color classes* $C_i = \{x \in \mathbb{Z}_n \mid c(x) = i\}$ for $1 \leq i \leq r$. Occasionally, when convenient, we will use R, G, B, and Y to denote the colors or the color classes red, green, blue, and yellow, respectively. Furthermore, we will use $\mathrm{im}(c)$ to denote the set of colors used by c.

Fix an integer k. Let a *triple* (x_1, x_2, x_3) be any three elements in $\mathbb{Z}_n$ which are a solution to $x_1 + x_2 \equiv kx_3 \mod n$. When $k = 1$, we will call these triples *Schur triples*. Such a triple is called a *rainbow triple* under a coloring c when $c(x_1) \neq c(x_2)$, $c(x_1) \neq c(x_3)$, and $c(x_2) \neq c(x_3)$. Consequently, a coloring will be called *rainbow-free* when there does not exist a rainbow triple in $\mathbb{Z}_n$ under c.

The *rainbow number* of $\mathbb{Z}_n$ given $x_1+x_2 = kx_3$, denoted $rb(\mathbb{Z}_n, k)$, is the smallest positive integer r such that any r-coloring of $\mathbb{Z}_n$ admits a rainbow triple. By convention, if such an integer does not exist, we set $rb(\mathbb{Z}_n, k) = n+1$. A *maximum* coloring is a rainbow-free r-coloring of $\mathbb{Z}_n$ where $r = rb(\mathbb{Z}_n, k) - 1$.

For a coloring c of $\mathbb{Z}_{st}$, the i^{th} *residue class* modulo t is the set of all the elements in $\mathbb{Z}_{st}$ which are congruent to $i \mod t$. Denote each residue class as $R_i = \{j \in \mathbb{Z}_{st} | j \equiv i \mod t\}$. We say the i^{th} *residue palette* modulo t is the set of colors which appear in the i^{th} *residue class*, and we will denote each palette as $P_i = \{c(j) | j \equiv i \mod t\}$.

Rainbow numbers for the equation $x_1 + x_2 = 2x_3$, for which the solutions are 3-term arithmetic progressions, have been studied in [1], [2], [3], and [5]. These problems are historically rooted in Roth's Theorem, Szemerédi's Theorem, and van der Waerden's Theorem. The first half of our paper explores the rainbow numbers of $\mathbb{Z}_n$ given the Schur equation, $x_1 + x_2 = x_3$. We rely on the work of Llano and Montenjano in [4], Jungić et al. in [3], and Butler et al. in [2] to prove exact values for $rb(\mathbb{Z}_n, 1)$ in terms of the prime factorization of n. Our results are an extension to the results in [1], [3], and [5].

The motivation for our results is captured in the idea that the rainbow number of $\mathbb{Z}_n$ given $x_1 + x_2 = kx_3$ to the prime factors of n. Theorems 1 and 2 confirm that rainbow numbers of depend on the prime factorization.

Theorem 1. *For a prime $p \geq 5$, $rb(\mathbb{Z}_p, 1) = 4$.*

Remark 1. It can be deduced through inspection that $rb(\mathbb{Z}_2, 1) = rb(\mathbb{Z}_3, 1) = 3$.

Theorem 1 gives exact values for $rb(\mathbb{Z}_p, 1)$ where p is prime. Therefore, Theorems 1 and 2 give exact values for $rb(\mathbb{Z}_n, 1)$. The proof for Theorem 2 is at the end of Section 2.3.

Theorem 2. *For a positive integer n with prime factorization $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m}$,*

$$rb(\mathbb{Z}_n, 1) = 2 + \sum_{i=1}^{m} \Big(\alpha_i(rb(\mathbb{Z}_{p_i}, 1) - 2)\Big).$$

We continue by considering the equation $x_1 + x_2 = px_3$ for any prime p. Many of the techniques for the $k = 1$ case generalize. However, there are complications. If we let the prime factorization of n be $n = p^{\alpha} \cdot q_1^{\alpha_1} \cdots q_m^{\alpha_m}$, then we can produce a recursive formula for $rb(\mathbb{Z}_n, p)$ detailed in Theorem 5. To obtain exact results from the recursive formula, we need to know the value of $rb(Z_q, p)$ for prime p and q. These values are determined in Theorems 3 and 4.

We would like to note that Theorem 3 resembles Theorem 3.5 in [3]. In essence, we use the ideas from Theorem 3.5 in [3] to construct a lower bound. The upper bound is obtained by using structural information from Theorem 2 in [4], which we restate as Theorem 6.

Theorem 3. *Let p, q be distinct and prime. Then $rb(\mathbb{Z}_q, p) = 4$ if and only if p, q do not satisfy either of the following conditions:*

1. *p generates $\mathbb{Z}_q^*$,*
2. *$|p| = (q-1)/2$ in $\mathbb{Z}_q^*$ and $(q-1)/2$ is odd.*

Otherwise, $rb(\mathbb{Z}_q, p) = 3$.

Theorem 4. *For $p \geq 3$ prime and $\alpha \geq 1$,*

$$rb(\mathbb{Z}_{p^{\alpha}}, p) = \begin{cases} 3 & p = 3, \alpha = 1 \\ 4 & p = 3, \alpha \geq 2 \\ \frac{p+1}{2} + 1 & p \geq 5 \end{cases}.$$

The values for $rb(\mathbb{Z}_{2^{\alpha}}, 2)$ are resolved in [1]. In conjunction with Theorems 3 and 4, Theorem 5 determines exact values for $rb(\mathbb{Z}_n, p)$. The proof for Theorem 5 is at the end of Section 3.4.

Theorem 5. *Let n be a positive integer, and let p be prime. Let n have prime factorization $n = p^{\alpha} \cdot q_1^{\alpha_1} \cdots q_m^{\alpha_m}$. Then*

$$rb(\mathbb{Z}_n, p) = rb(\mathbb{Z}_{p^{\alpha}}, p) + \sum_{i=1}^{m} \Big(\alpha_i(rb(\mathbb{Z}_{q_i}, p) - 2)\Big).$$

2. Schur Triples

Section 1 is dedicated to proving Theorem 2. In Section 2.1 we introduce the idea of a dominant color to describe the structural properties of colorings of $\mathbb{Z}_p$.

Additionally, we prove Proposition 1, the Schur triple counterpart of Theorem 3.2 in [3]. We use Proposition 1 to prove Theorem 1, concluding Section 1.1. In Section 2.2 we show that the lower bound of $rb(\mathbb{Z}_n, 1)$ can be determined by the prime factorization of n. The equivalent upper bound is proved in 2.3. Combining Sections 2.2 and 2.3 proves Theorem 2.

2.1. Schur Triples in $\mathbb{Z}_p$, p Prime

Let c be a coloring of $\mathbb{Z}_n$. We say a sequence $S_1, S_2, \ldots, S_k$ of colors *appears at position* i if $c(i) = S_1, c(i+1) = S_2, \ldots, c(i+k-1) = S_k$. A sequence is *bichromatic* if it contains exactly two colors. A color R is *dominant* if for $S = \{c(x) : i \leq x \leq j, i < j\}$, $|S| = 2$ implies $R \in S$. That is, R appears in every bichromatic string. Using dominant colors to derive a contradiction is used in [3]. We also use this idea to describe the structure of rainbow-free colorings of $\mathbb{Z}_p$. However, we must show that a dominant color exists.

Lemma 1. *There exists a dominant color in every rainbow-free coloring of* $\mathbb{Z}_n$. *Furthermore,* $c(1)$ *is dominant.*

Proof. Let c be a rainbow-free coloring of $\mathbb{Z}_n$. Note that $(1, i, i+1)$ is a Schur triple for all $i \notin \{0, 1\}$. Since c is rainbow-free, either $c(i) = c(i+1)$, $c(1) = c(i)$, or $c(1) = c(i+1)$. Thus, if $c(i) \neq c(i+1)$, then $c(1)$ must appear on either i or $i+1$. This implies that $c(1)$ is dominant. □

An immediate result from this lemma is that any color which doesn't appear on 1 must be adjacent to itself or the dominant color. Now we can relate the structure of our coloring to the presence of a rainbow triple. Without loss of generality, let $c(1) = R$ be dominant.

Lemma 2. *Let* c *be an* r*-coloring of* $\mathbb{Z}_n$ *with* $r \geq 3$*. If* BB *and* GG *appears in* c*, then there exists a rainbow Schur triple in* c*.*

Proof. Let c be an r-coloring of $\mathbb{Z}_n$ with $r \geq 3$ such that BB and GG appears in c. Without loss of generality, assume R is dominant, and c contains BB and GG. Then, the sequence BBR must appear at some position i and the sequence GGR must appear at some position j.

Consider the Schur triple $(i, j+2, i+j+2)$. Since $c(i) = B$, and $c(j+2) = R$, then either c contains a rainbow Schur triple, or $c(i+j+2)$ is R or B. Assume the second case, and consider the Schur triple $(i+2, j, i+j+2)$. Since $c(i+2) = R$, and $c(j) = G$ then either c contains a rainbow Schur triple or $c(i+j+2)$ is R. Again, assume the second case, and finally consider the triple $(i+1, j+1, i+j+2)$. Since $c(i+1) = B$, $c(j+1) = G$, and $c(i+j+2) = R$, this triple is rainbow. Therefore, c contains a rainbow Schur triple. □

Therefore, if c is a rainbow-free coloring of $\mathbb{Z}_n$ with R dominant, either GG or BB can appear in c, but not both. Next we show that there are ways to re-order colorings while maintaining whether or not Schur triples are rainbow.

Lemma 3. *Let c be an r-coloring of $\mathbb{Z}_n$. If m is relatively prime to n, then c has a rainbow Schur triple if and only if $\hat{c}(x) := c(mx)$ contains a rainbow Schur triple. Additionally, the cardinality of each color class will be maintained.*

Proof. Let (x_1, x_2, x_3) be a triple in c. By definition, $x_1+x_2 = x_3$ in $\mathbb{Z}_n$ is equivalent to

$$\begin{aligned} x_1 + x_2 &= sn + r \\ x_3 &= tn + r, \end{aligned}$$

as equations in the integers for some $s, t \in \mathbb{Z}$. Multiply both equations by m to get

$$\begin{aligned} mx_1 + mx_2 &= msn + mr \\ mx_3 &= mtn + mr \end{aligned}$$

Therefore, $mx_1+mx_2 \equiv mr \mod n$, and $mx_3 \equiv mr \mod n$, so $mx_1+mx_2 \equiv mx_3 \mod n$. Thus, (mx_1, mx_2, mx_3) is rainbow in $\hat{c}$ if and only if (x_1, x_2, x_3) is rainbow in c.

Finally, the last statement of Lemma 3 follows from the fact that if m is relatively prime to n, then the map $F : x \mapsto mx$ is a bijection. □

Our next result is the Schur equation counterpart to Theorem 3.2 in [3].

Proposition 1. *Let p be prime. Then every 3-coloring c of $\mathbb{Z}_p$ with $\min(|R|, |G|, |B|) > 1$ contains a rainbow Schur triple.*

Proof. For the sake of contradiction, assume that c is a rainbow-free 3-coloring of $\mathbb{Z}_p$ and $\min(|R|, |G|, |B|) > 1$. Without loss of generality, assume that $|R| = \min(|R|, |G|, |B|)$. Since there are at least two elements of $\mathbb{Z}_p$ colored R, there exists a minimal element $1 \leq i \leq p-1$ such that $c(i) = R$. Because p is prime, i is relatively prime to p and i has a multiplicative inverse. Let $\hat{c}(x) := c(ix)$ so that $\hat{c}(1) = R$. Therefore, by Lemma 1, R is dominant in $\hat{c}$. By Lemma 2, BB and GG cannot both appear in $\hat{c}$. Without loss of generality, assume that GG does not appear in $\hat{c}$. Because R is dominant, R must follow each G, so $|R| \geq |G|$. Furthermore, BR must appear in $\hat{c}$. This implies that $|R| \geq |G| + 1$ in $\hat{c}$ which implies $|R| \geq |G| + 1$ in c by Lemma 3. This contradicts our assumption that $|R| = \min(|R|, |G|, |B|)$. □

Lemma 4. *If c is a rainbow-free r-coloring of $\mathbb{Z}_p$ for a prime p with $r > 2$, then $c(x) = c(-x)$.*

Proof. Let c be a rainbow-free r-coloring of $\mathbb{Z}_p$. For the sake of contradiction, assume that there exists $i, -i$ with $c(i) \neq c(-i)$. Without loss of generality, let $c(i) = R$ and $c(-i) = G$. Now, let $\hat{c}(x) := c(ix)$ and let $\bar{c}(x) := c(-ix)$. By Lemma 3, $\hat{c}$ and $\bar{c}$ are both rainbow-free. Since $\hat{c}(1) = c(i) = R$ and $\bar{c}(1) = c(-i) = G$, R is dominant in $\hat{c}$, and G is dominant in $\bar{c}$. Notice that $\hat{c}(x) = \bar{c}(-x)$, so if two colors are adjacent at some position in $\hat{c}$, then they are also adjacent at some position in $\bar{c}$. Thus, since G is dominant in $\bar{c}$, G must also appear in every bichromatic sequence in $\hat{c}$, and, consequently, G is also dominant in $\hat{c}$. If both R and G are dominant in $\hat{c}$, then $\hat{c}$ must only contain R and G, and $r = 2$; this is a contradiction. □

Note that this lemma shows that the coloring from 1 to $p-1$ must be symmetric in a rainbow-free coloring of $\mathbb{Z}_p$.

Remark 2. For any prime $p \geq 5$, $\mathbb{Z}_p$ can be colored with three colors by coloring zero uniquely and coloring 1 to $p-1$ with two colors in any way such that $c(x) = c(-x)$ for all x. This coloring is rainbow-free since any three group elements which witness three colors must contain 0, and in order to make a Schur triple of three distinct elements where one of the elements is 0 the other two elements must be x and $-x$ for some x (see also Corollary 2 in [4]).

Now we have enough information about the structure of rainbow-free colorings to prove Theorem 1. A color class C is *singleton* if $|C| = 1$.

Proof of Theorem 1. For the sake of contradiction, suppose that $r+1 = rb(\mathbb{Z}_p, 1) > 4$ for a prime $p \geq 5$, and let c be a rainbow-free r-coloring of $\mathbb{Z}_p$ with $r > 3$. Note that since c is rainbow-free, at least one of the color classes in c must contain more than one element. Partition the color classes of c into three sets to define $\hat{c}$, an exact 3-coloring of $\mathbb{Z}_p$. We use the union of the color classes within each part of the partition as the color classes for $\hat{c}$. Since we are concatenating colors, $\hat{c}$ is also rainbow-free. By Proposition 1, regardless of how the color classes of c are partitioned, there exists some color class in $\hat{c}$ with exactly one element. If $r \geq 5$, then there exists a partition of the five or more color classes such that each color class has more than one element. Therefore, $r = 4$.

Furthermore, if two or more color classes are not singleton, then there would exist a partition of the color classes that yields no singleton color classes in $\hat{c}$. Therefore, all but one of the four color classes in c must be singleton.

If there are three singleton color classes in c, then there exists an $x \neq 0$ such that $c(x) \neq c(-x)$. This contradicts Lemma 4, and c cannot be rainbow-free.

Thus, there does not exist an exact rainbow-free r-coloring of $\mathbb{Z}_p$ for $r > 3$ and $p \geq 5$. □

2.2. Lower Bound

In order to prove the lower bound for $rb(\mathbb{Z}_n, 1)$, we examine the relationship between Schur triples in $\mathbb{Z}_n$ and $\mathbb{Z}_{\frac{n}{m}}$ where m divides n.

Lemma 5. *If there exists a Schur triple of the form (x_1, x_2, x_3) in $\mathbb{Z}_n$ where $m|x_1, x_2, x_3$ for some $m|n$, $m, n \in \mathbb{Z}$, then there exists a Schur triple of the form $(x_1/m, x_2/m, x_3/m)$ in $\mathbb{Z}_{\frac{n}{m}}$.*

Proof. By definition, $x_1 + x_2 = x_3$ in $\mathbb{Z}_n$ implies that in the integers

$$\begin{aligned} x_1 + x_2 &= qn + r \\ x_3 &= tn + r, \end{aligned}$$

for some $q, t \in \mathbb{Z}$. Divide both equations by m to get

$$\begin{aligned} \frac{x_1}{m} + \frac{x_2}{m} &= q\frac{n}{m} + \frac{r}{m} \\ \frac{x_3}{m} &= t\frac{n}{m} + \frac{r}{m}. \end{aligned}$$

Now we must check that $\frac{r}{m}$ is an integer. Since $m|(x_1 + x_2 - qn)$, we know $m|r$.

By definition, this means that there exists a Schur triple of the form $(x_1/m, x_2/m, x_3/m)$ in $\mathbb{Z}_{\frac{n}{m}}$. $\square$

This shows that Schur triples can be "projected" from the cyclic group $\mathbb{Z}_n$ to a subgroup $\mathbb{Z}_{\frac{n}{m}}$. Next, we will show another property of Schur triples related to the divisibility of a triple's elements by a prime.

Lemma 6. *For a positive integer n and a prime p, if $x_1 + x_2 \equiv x_3 \mod np$, then p cannot divide exactly two of (x_1, x_2, x_3).*

Proof. If $x_1 + x_2 \equiv x_3 \mod np$, then there exist integers c_1, c_2, and r_0 such that $x_1 + x_2 = c_1np + r_0$ and $x_3 = c_2np + r_0$.

Assume that p divides x_1 and x_2. Then there exist integers c_3 and c_4 such that $x_1 = c_3p$ and $x_2 = c_4p$. We know there exist integers c_5 and r_1 with $0 \le r_1 < p$ such that $x_3 = c_5p + r_1$, so we want to show $r_1 = 0$. Immediately, we see that $c_3p + c_4p = c_1np + r_0$ and $c_5p + r_1 = c_2np + r_0$, which, after substituting for r_0, shows us $c_3p + c_4p = c_1np + c_5p + r_1 - c_2np$. Solving for r_1 gives us

$$\begin{aligned} r_1 &= c_3p + c_4p - c_1np - c_5p + c_2np \\ &= p(c_3 + c_4 - c_1n - c_5 + c_2n) \end{aligned}$$

This means that p divides r_1, forcing $r_1 = 0$. Thus, p divides x_3.

Now assume p divides x_1 and x_3, i.e. there exist integers c_6 and c_7 such that $x_1 = c_6p$ and $x_3 = c_7p$. We know there exist integers c_8 and r_2 with $0 \le r_2 < p$

such that $x_2 = c_8p + r_2$, so we want to show $r_2 = 0$. Immediately, we see that $c_6p + c_8p + r_2 = c_1np + r_0$ and $c_7p = c_2np + r_0$, which, after substituting for r_0, shows us $c_6p + c_8p + r_2 = c_1np + c_7p - c_2np$. Solving for r_2 gives us

$$\begin{aligned} r_2 &= c_1np + c_7p - c_2np - c_6p - c_8p \\ &= p(c_1n + c_7 - c_2n - c_6 - c_8). \end{aligned}$$

This means that p divides r_2, forcing $r_2 = 0$. Thus, p divides x_2. By symmetry, this case is identical to the case where p divides x_2 and x_3.

Therefore, we can see that if p divides two elements in (x_1, x_2, x_3), then p must also divide the third. $\square$

Lemma 7. *Let p, t be positive integers with p prime. If there exists a rainbow-free r-coloring of $\mathbb{Z}_t$, then there exists a rainbow-free $r + rb(\mathbb{Z}_p, 1) - 2$-coloring of $\mathbb{Z}_{pt}$.*

Proof. Let t, p be positive integers such that p is a prime. Assume $\hat{c}$ is a rainbow-free r-coloring of $\mathbb{Z}_t$. Then let c be an exact $(r + rb(\mathbb{Z}_p, 1) - 2)$-coloring (if $p = 2$ or $p = 3$, then c is an exact $(r + 1)$-coloring. Otherwise, c is an exact $r + 2$ coloring) of $\mathbb{Z}_{pt}$ as follows:

$$c(x) := \begin{cases} \hat{c}(x/p) & x \equiv 0 \mod p \\ r + 1 & x \equiv 1 \text{ or } p - 1 \mod p \\ r + 2 & \text{otherwise} \end{cases}.$$

Notice that if (x_1, x_2, x_3) is a Schur triple in $\mathbb{Z}_{pt}$, then there are three cases by Lemma 6: p divides exactly one of (x_1, x_2, x_3), p divides each of (x_1, x_2, x_3), or p divides none of (x_1, x_2, x_3).

`Case 1:` The two terms x_i, x_j where $i, j \in \{1, 2, 3\}$ that are not divisible by p are either additive inverses modulo p or are equal modulo p. Thus, $c(x_i) = c(x_j)$ and (x_1, x_2, x_3) does not form a triple.

`Case 2:` The coloring of each x_i is inherited from $\hat{c}$. Since $\hat{c}$ does not admit rainbow triples, we know that this triple will not be rainbow by Lemma 5.

`Case 3:` The three integers in the triple will be colored from $\{r + 1, r + 2\}$, so the triple will not be rainbow. In each case, c is a rainbow-free $r + rb(\mathbb{Z}_p, 1) - 2$-coloring of $\mathbb{Z}_{pt}$. $\square$

Proposition 2. *For any positive integer $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$,*

$$rb(\mathbb{Z}_n, 1) \geq 2 + \sum_{i=1}^{m} \Big(\alpha_i(rb(\mathbb{Z}_{p_i}, 1) - 2)\Big).$$

Proof. If n is prime, there is nothing to show. Suppose that the claim holds true for n where n has N prime factors.

Assume that $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ where $\alpha_1 + \cdots + \alpha_m = N + 1$. By the induction hypothesis, there exists a rainbow-free r-coloring of $\mathbb{Z}_{n/p_1}$ where

$$r = 1 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big) - rb(\mathbb{Z}_{p_1}, 1) + 2.$$

Therefore, by Lemma 7, there exists a rainbow-free $r + rb(\mathbb{Z}_{p_1}, 1) - 2$ coloring of $\mathbb{Z}_n$. Thus, by induction

$$rb(\mathbb{Z}_n, 1) \geq 2 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

□

2.3. Upper Bound

To establish the upper bound for $rb(\mathbb{Z}_n, 1)$, we consider residue classes and their corresponding residue palettes under c. Lemma 8 lets us create a well-defined reduction of a coloring of $\mathbb{Z}_{st}$ to a coloring of $\mathbb{Z}_t$. We use the coloring described in Lemma 9 to prove an upper bound for $rb(\mathbb{Z}_{st}, 1)$.

Lemma 8. *Let $R_0, R_1, \ldots, R_{t-1}$ be the residue classes modulo t for $\mathbb{Z}_{st}$, and let $P_0, P_1, \cdots, P_{t-1}$ be the corresponding residue palettes under rainbow-free c. Then $|P_i \setminus P_0| \leq 1$ for $1 \leq i \leq t-1$.*

Proof. Assume that $|P_i \setminus P_0| \geq 2$. Then R_i must contain at least two elements which receive colors that do not appear in P_0. Without loss of generality, let G and B denote two colors in $P_i \setminus P_0$. Then there exists two integers m and n such that $c(mt+i) = G$ and $c(nt+i) = B$. Consider the Schur triple $(mt-nt, nt+i, mt+i)$. Notice that $mt - nt \equiv 0 \mod t$, $c(mt-nt) \neq G, B$. Thus, we have a rainbow triple under c in $\mathbb{Z}_{st}$, which is a contradiction. Therefore, $|P_i \setminus P_0| \leq 1$ for $1 \leq i \leq t-1$. □

Lemma 9. *Let s and t be positive integers. Let $R_0, R_1, \ldots, R_{t-1}$ be the residue classes modulo t for $\mathbb{Z}_{st}$ with corresponding residue palettes P_i. Suppose c is a coloring of $\mathbb{Z}_{st}$ where $|P_i \setminus P_0| \leq 1$. Let $\hat{c}$ be a coloring of $\mathbb{Z}_t$ given by*

$$\hat{c}(i) := \begin{cases} P_i \setminus P_0 & \text{if } |P_i \setminus P_0| = 1 \\ \alpha & \text{otherwise} \end{cases}$$

where $\alpha \notin P_i$ for $0 \leq i \leq t$. If $\hat{c}$ contains a rainbow Schur triple, then c contains a rainbow Schur triple.

Proof. Suppose (x_1, x_2, x_3) is a rainbow Schur triple in $\hat{c}$. Then, at least two of x_1, x_2, x_3 must receive a color other than α. We consider the following two cases.

`Case 1:` Neither x_1 nor x_2 receive color α.

Without loss of generality, assume that $c(x_1) = G$ and $C(x_2) = B$. This implies that there exist n, m such that $c(nt+x_1) = G$ and $c(mt+x_2) = B$. There is a Schur triple of the form $(nt+x_1, mt+x_2, (n+m)t+(x_1+x_2))$ in $\mathbb{Z}_{st}$. Since $x_1+x_2 \equiv x_3$ mod t, $(n+m)t+(x_1+x_2)$ is in the residue class R_{x_3}. As $\hat{c}(x_3) \neq G, B$, we have $G, B \notin P_{x_3}$. Therefore, the triple $(nt+x_1, mt+x_2, (n+m)t+(x_1+x_2))$ is rainbow.

`Case 2:` One of x_1 or x_2 is colored α.

Without loss of generality, assume that $c(x_1) = \alpha$, $c(x_2) = B$, and $c(x_3) = G$. Then $c(nt+x_2) = B$ for some n, and $c(mt+x_3) = G$ for some m. There is a Schur triple of the form $((m-n)t+(x_3-x_2), nt+x_2, mt+x_3)$ in $\mathbb{Z}_{st}$. Since $x_1+x_2 \equiv x_3$ mod t, $(m-n)t+(x_3-x_2)$ is in the residue class R_{x_1}. As $\hat{c}(x_1) = \alpha$, we have $G, B \notin P_{x_1}$. Therefore, the triple $((m-n)t+(x_3-x_2), nt+x_2, mt+x_3)$ is rainbow.

Hence, if $\hat{c}$ has a rainbow Schur triple, then c has a rainbow Schur triple. $\square$

Proposition 3. *Let s and t be positive integers. Then $rb(\mathbb{Z}_{st}, 1) \leq rb(\mathbb{Z}_s, 1) + rb(\mathbb{Z}_t, 1) - 2$.*

Proof. Let c be an exact r-coloring of $\mathbb{Z}_{st}$, and let $\hat{c}$ be a coloring constructed from c as in Lemma 9. Notice that the set of colors used in c is comprised of the colors in R_0 and each color used in $\hat{c}$ other than α. Thus, $r = |P_0| + |\hat{c}| - 1$, where $|\hat{c}|$ is the number of colors appearing in $\hat{c}$. If c is a rainbow-free coloring of $\mathbb{Z}_{st}$, then R_0 is a rainbow-free coloring of $\mathbb{Z}_s$. Thus, $|P_0| \leq rb(\mathbb{Z}_s, 1) - 1$. Also, $\hat{c}$ is a rainbow-free coloring of $\mathbb{Z}_t$, so $|\hat{c}| \leq rb(\mathbb{Z}_t, 1) - 1$. Thus, $r \leq rb(\mathbb{Z}_s, 1) + rb(\mathbb{Z}_t, 1) - 3$. If we let c be the maximum rainbow-free coloring of $\mathbb{Z}_{st}$, then $r = rb(\mathbb{Z}_{st}, 1) - 1$. This shows that $rb(\mathbb{Z}_{st}, 1) \leq rb(\mathbb{Z}_s, 1) + rb(\mathbb{Z}_t, 1) - 2$. $\square$

Using both the upper bound we just established and the lower bound established in Proposition 2 of Section 2.2, we prove Theorem 2.

Proof of Theorem 2. Recursively applying Proposition 3 to prime factors of n yields

$$rb(\mathbb{Z}_n, 1) \leq 2 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

Since this is identical to the lower bound from Proposition 2 in Section 2.2, we can conclude

$$rb(\mathbb{Z}_n, 1) = 2 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

$\square$

3. Triples for $x_1 + x_2 = px_3$, p Prime

Section 3 is dedicated to proving Theorem 5. In Section 3.1, we establish exact values for $rb(\mathbb{Z}_q, p)$ where $p \neq q$ are prime. Finding an exact value for $rb(\mathbb{Z}_p, p)$ is more difficult, and is the subject of Section 3.2. Some properties of rainbow-free colorings of $\mathbb{Z}_q$ are used in the construction of the general lower bound in Section 3.3. The equivalent upper bound is proved in Section 3.4. Combining Sections 3.3 and 3.4 proves Theorem 5.

3.1. Exact Values for $rb(\mathbb{Z}_q, p)$, $p \neq q$ Prime

Lemmas 10, 11, 12, 13 establish the upper bound $rb(\mathbb{Z}_q, p) \leq 4$. These lemmas are proven by assuming that there exists a rainbow-free r-coloring c with $r \geq 4$, and reducing c to a 3-coloring $\hat{c}$. In each case, we find that $\hat{c}$ does not conform to the structure of a rainbow-free 3-coloring outlined in Theorem 6 proven in [4]. For convenience, we include Theorem 6 and the necessary definitions from [4].

For a subset $X \subseteq \mathbb{Z}_q^*$ and $a \in \mathbb{Z}_q^*$ define $aX := \{ax \mid x \in X\}$, $X + a := \{x + a \mid x \in X\}$, and $X - a := X + (-a)$. We say the set aX is the *dilation* of X by a. Let $\langle x \rangle \leq \mathbb{Z}_q^*$ denote the subgroup multiplicatively generated by x. A subset $X \in \mathbb{Z}_q^*$ is H-*periodic* if X is the union of cosets of H, where $H \leq \mathbb{Z}_p^*$. In the case that X is $\langle -1 \rangle$-periodic, we say that X is *symmetric*. This coincides with the notion that X is symmetric if and only if $X = -X$.

Theorem 6. *[[4], Theorem 2] A 3-coloring $\mathbb{Z}_q = A \cup B \cup C$ with $1 \leq |A| \leq |B| \leq |C|$ is rainbow-free for $x_1 + x_2 = kx_3$ if and only if, up to dilation, one of the following holds.*

1. *$A = \{0\}$ and both B and C are symmetric and $\langle k \rangle$-periodic subsets.*
2. *$A = \{1\}$ for*
   - *(i) $k = 2 \mod q$, $(B-1)$ and $(C-1)$ are symmetric and $\langle 2 \rangle$-periodic subsets.*
   - *(ii) $k = -1 \mod q$, $(B \setminus \{2\}) + 2^{-1}$, $(C \setminus \{2\}) + 2^{-1}$ are symmetric subsets.*
3. *$|A| \geq 2$, for $k = -1 \mod q$ and A, B, and C are arithmetic progressions with difference 1 such that $A = [a_1, a_2 - 1]$, $B = [a_2, a_3 - 1]$, and $C = [a_3, a_1 - 1]$, with $(a_1 + a_2 + a_3) = 1$ or 2.*

Suppose that $q \geq 5$ is prime. Let c be a coloring of $\mathbb{Z}_q$ with color classes $C_1, \dots, C_r$ with $1 \leq |C_1| \leq |C_2| \leq \dots \leq |C_r|$ and $r \geq 4$. Theorem 6 tells us that rainbow-free colorings have very particular structure, up to the rearrangement of the elements of $\mathbb{Z}_q$. The overarching goal of Lemmas 10, 11, 12, and 13, is to show that if a coloring has too many colors, then some color classes can be combined to violate the structure of a rainbow-free coloring.

Observation 7. If $C_1 = \{0\}$ and $C_2 = \{x\}$, then $(x, -x, 0)$ is a rainbow triple for $x \neq 0$.

Therefore, if c has two or more singleton color classes, we can assume that $\{0\}$ is not a color class. Furthermore, since dilation preserves the rainbow-free property, we can assume that if $|C_2| = 1$, then $C_1 = \{1\}$.

Lemma 10. *If $p \not\equiv -1 \mod q$ and $|C_2| = 1$, then c admits a rainbow triple.*

Proof. Consider the coloring $\hat{c}$ given by the color classes $C_1, C_2, \bigcup_{i=3}^{r} C_i$. If $\hat{c}$ admits a rainbow triple, then c also admits a rainbow triple and we are done. If $\hat{c}$ does not admit a rainbow triple, then $\hat{c}$ must conform to case 2.(i) in Theorem 6. Therefore, $p \equiv 2 \mod q$. In this case, triples satisfying $x_1 + x_2 = kx_3$ in $\mathbb{Z}_q$ are 3-term arithmetic progressions. In [2], Proposition 3.5 establishes that $rb(\mathbb{Z}_q, 2) \leq 4$. Therefore, there exists a rainbow triple under c. □

Lemma 11. *If $p \equiv -1 \mod q$ and $|C_3| = 1$, then c admits a rainbow triple.*

Proof. Let $C_2 = \{x\}, C_3 = \{y\}$. For the sake of contradiction, assume that c is rainbow free.

If $x = 2$, then $(x, -3, 1)$ is a rainbow triple. The same argument for y shows that $x, y \neq 2$.

Consider the coloring $\hat{c}$ given by the color classes $C_1, C_2, \bigcup_{i=3}^{r} C_i$. Then by Theorem 6 we must have $C_2 \setminus \{2\} + 2^{-1}$ is symmetric and so $x + 2^{-1} = -2^{-1} - x$. Solving for x gives that $x = -2^{-1}$. Considering the coloring given by $C_1, C_3, C_2 \cup \bigcup_{i=4}^{r} C_i$ gives that $y = -2^{-1}$, which is a contradiction. □

Lemma 12. *If $p \not\equiv -1 \mod q$, and $|C_2| \geq 2$, then c admits a rainbow triple.*

Proof. For the sake of contradiction, suppose that c does not admit a rainbow triple. Consider the coloring $\hat{c}$ given by $C_1 \cup C_2, C_3, \bigcup_{i=4}^{r} C_i$. Since $|C_3| \geq |C_2| \geq 2$, notice that $\hat{c}$ does not have a singleton color class and is rainbow-free. This contradicts Theorem 6. □

Lemma 13. *If $p \equiv -1 \mod q$ and $|C_3| \geq 2$, then c admits a rainbow triple.*

Proof. For the sake of contradiction, suppose that c does not admit a rainbow triple. There are two cases: $|C_2| \geq 2$, or $|C_2| = 1$.

Case 1: Assume that $|C_2| \geq 2$ and $C_1 = \{x\}$. By Theorem 6, the coloring $C_1 \cup C_2, C_3, \bigcup_{i=4}^{r} C_i$ is of the form

$$C_1 \cup C_2 = [a_1, a_2 - 1],$$

$$C_3 = [a_2, a_3 - 1],$$

$$\bigcup_{i=4}^{r} C_i = [a_3, a_1 - 1].$$

Notice that x is not adjacent to at least one of C_3 or $\bigcup_{i=4}^{r} C_i$. Without loss of generality, assume x is not adjacent to C_3 (the other case follows the same argument). Consider the coloring $\hat{c}$ given by $C_2, C_1 \cup C_3, \bigcup_{i=4}^{r} C_i$. Notice that $\hat{c}$ can only be dilated by 1 or -1 to preserve the interval structure of $\bigcup_{i=4}^{r} C_i$. However, dilating by 1 or -1 will not make $C_1 \cup C_3$ an arithmetic progression with difference 1. This is a contradiction.

`Case 2:` Assume that $|C_2| = 1$. Consider the coloring $\hat{c}$ given by

$$C_1 \cup C_2, C_3, \bigcup_{i=4}^{r} C_i.$$

By Theorem 6, $\hat{c}$ is of the form

$$C_1 \cup C_2 = [a_1, a_2 - 1],$$

$$C_3 = [a_2, a_3 - 1],$$

$$\bigcup_{i=4}^{r} C_i = [a_3, a_1 - 1]$$

with $a_1 + a_2 + a_3 \in \{1, 2\}$. Since every set is an arithmetic progression with difference 1, we have $a_2 - 1 = a_1 + 1$. This implies that $a_3 \in \{-2a_1 - 1, -2a_1\}$. This implies that $c(-2a_1 - 1) \neq c(a_1), c(a_1 + 1)$. Therefore, triple $(-2a_1 - 1, a_1, a_1 + 1)$ is rainbow, which is a contradiction. □

Lemmas 10, 11, 12, and 13 form a case analysis and road map for finding rainbow solutions. In particular, these lemmas translate the structural properties of rainbow-free 3-colorings, to an upper bound on the rainbow number. The rest of the work in the proof of Theorem 3 determines the relationship between p and q that makes rainbow-free 3-colorings possible.

Proof of Theorem 3. By Lemmas 10, 11, 12, and 13, we know that $rb(\mathbb{Z}_q, p) \leq 4$. Therefore, it suffices to show that there exists a rainbow-free 3-coloring of $\mathbb{Z}_q$ if and only if p, q do not satisfy either condition 1 or 2. First we will prove that if there exists a rainbow-free 3-coloring, then p, q do not satisfy conditions 1 and 2.

Let c be a rainbow-free 3-coloring. There are two cases, $p \not\equiv -1 \mod q$ or $p \equiv -1 \mod q$.

`Case 1:` By Theorem 6, either 0 is uniquely colored, or $p \equiv 2 \mod q$.

Suppose 0 is uniquely colored and $c(1) = R$. Notice that if $c(x) = R$, then $c(px) = R$ and $c(-x) = R$. If p, q satisfy either 1 or 2, then $\{p^i, -p^i \mid i \in \mathbb{Z}\} = \mathbb{Z}_q^*$, which contradicts the fact that c is a 3-coloring.

Suppose $p \equiv 2 \mod q$. Then neither 1 nor 2 are satisfied by Theorem 3.5 in [3].

`Case 2:` Suppose $p \equiv -1 \mod q$. Then $|p| = 2$. If $(q-1)/2$ is odd, then $(q-1)/2 \neq 2$. Therefore, neither 1 nor 2 are satisfied.

To prove the reverse direction, suppose that p, q do not satisfy either 1 or 2. Let c be given by

$$C_1 = \{0\}, C_2 = \{p^i, -p^i \mid i \in \mathbb{Z}\}, C_3 = \mathbb{Z}_q^* \setminus C_2.$$

Since p, q do not satisfy either 1 or 2, C_3 is non-empty. Notice that any rainbow triple must contain 0 and some element $y \in C_2$. However, if $0, y, z$ is a triple, then $z \in C_2$. Therefore, c is rainbow-free. □

The following corollary is used in Section 3.3 to prove a general lower bound for $rb(\mathbb{Z}_n, p)$.

Corollary 1. *There exists a maximum rainbow-free coloring of $\mathbb{Z}_q$ where 0 is uniquely colored and the color classes are symmetric.*

3.2. Exact Values for $rb(\mathbb{Z}_{p^\alpha}, p)$, p Prime

In order to determine the rainbow numbers for equations of the form $x_1 + x_2 = px_3$ for prime $p \geq 3$, we still need to determine $rb(\mathbb{Z}_{p^\alpha}, p)$ for $\alpha \geq 1$. We will prove Theorem 4 using induction. Observation 8 and Propositions 4, 5, and 6 provide the lower bound and base case for our induction argument. Lemmas 14 and 15 provide the basic structure of a rainbow-free coloring of $\mathbb{Z}_{p^\alpha}$. Lastly, Lemmas 16, and 17 exploit the structure to derive a contradiction by forcing a rainbow triple. Throughout this section, for $0 \leq k \leq p-1$, recall that the k^{th} residue class mod p is the set $R_k = \{j \in \mathbb{Z}_{p^\alpha} : j \equiv k \mod p\}$ and that the k^{th} residue palette P_k is the set of colors which appear on R_k.

Observation 8. Notice $rb(\mathbb{Z}_3, 3) = 3$ and $rb(\mathbb{Z}_9, 3) = 4$.

Proposition 4. *Let $p \geq 3$ be prime. Then $rb(\mathbb{Z}_p, p) = \frac{p+1}{2} + 1$.*

Proof. To prove the lower bound, consider the following coloring:

$$c(x) = \begin{cases} x & 0 \leq x \leq \frac{p+1}{2} \\ -x & \text{otherwise} \end{cases}.$$

Notice that $c(x) = c(-x)$ for all $x \in \mathbb{Z}_p$. Furthermore, if (x_1, x_2, x_3) is a triple, then $x_1 = -x_2$. Thus, c is a rainbow-free $\frac{p+1}{2}$ coloring, and $rb(\mathbb{Z}_p, p) > \frac{p+1}{2}$.

To prove the upper bound, assume that c is an $\frac{p+1}{2} + 1$ coloring of $\mathbb{Z}_p$. By the pigeonhole principle, there exists $x \in \mathbb{Z}_p$ such that $x \neq 0$ and $c(x) \neq c(-x)$. Since $p \geq 3$, $x \neq -x$, and there exist $y \neq x, -x$ such that $c(y) \neq c(x), c(-x)$. Therefore, $(x, -x, y)$ is a rainbow-triple, and $rb(\mathbb{Z}_p, p) \leq \frac{p+1}{2} + 1$. □

For the rest of the section, we will assume that $\alpha \geq 2$.

Proposition 5. *For $\alpha \geq 2$,*

$$rb(\mathbb{Z}_{3^\alpha}, 3) > 3.$$

Proof. Suppose that $\alpha \geq 3$ and $\bar{c}$ is a rainbow-free 3-coloring of $\mathbb{Z}_9$. Let c be a 3-coloring of $\mathbb{Z}_{p^\alpha}$ given by $c(i) := \bar{c}(i \mod 9)$. Assume that x_1, x_2, x_3 is a triple in $\mathbb{Z}_{3^\alpha}$. Then x_1, x_2, x_3 is a triple in $\mathbb{Z}_9$ and cannot be rainbow. □

Proposition 6. *For prime $p \geq 5$ and $\alpha \geq 1$,*

$$rb(\mathbb{Z}_{p^\alpha}, p) \geq \frac{p+1}{2} + 1.$$

Proof. Color all of R_i, R_{p-i} color i for $0 \leq i \leq \frac{p+1}{2}$. Suppose $x_1 + x_2 = px_3$ and $x_1 \equiv j \mod p$ for $0 \leq j \leq p-1$. Then $x_2 \equiv p - j \mod p$, and x_1, x_2, x_2 is not rainbow. □

Lemma 14. *If c does not admit a rainbow triple, then*

$$P_i = P_{p-i}$$

when $0 < i < p$.

Proof. For the sake of contradiction, suppose that there exists $0 < i < p$ with $G \in P_i \setminus P_{p-i}$. Then there exists an element $px+i$ with color G in R_i. Let $py+p-i$ be an element in R_{p-i}. Notice that

$$\begin{aligned}
x_1 &= p(py - x + p - 1 - i) + p - i \\
x_2 &= px + i \\
x_3 &= py + p - i
\end{aligned}$$

is a triple. Since $G \notin P_{p-i}$, we have $c(x_3) = c(x_1)$. Furthermore, $x_1 - x_3 = p(py - x + p - 1 - i) + p - i - py - p + i = p(y(p-1) - x + p - 1)$. Since $py + p - i$ was arbitrary, we can choose y so that $y(p-1) - x + p - 1 \not\equiv 0 \mod p$. Since $y(p-1) - x + p - 1 \not\equiv 0 \mod p$, we know that $y(p-1) - x + p - 1$ is an additive generator of $\mathbb{Z}_{p^{\alpha-1}}$. This implies that $P_{p-i} = \{B\}$.

Let $pz + j$ be an element with $c(pz + j) \notin \{G, B\}$. Then

$$\begin{aligned}
x_1 &= p(pz - x + j - 1) + p - i \\
x_2 &= px + i \\
x_3 &= pz + j
\end{aligned}$$

is a rainbow triple, which is a contradiction. □

Notice that by Lemma 14, it is sufficient to only consider the structure of R_i for $0 < i < \frac{p+1}{2}$.

Lemma 15. *Suppose c does not admit a rainbow triple. If there exists $0 < i < p$ such that $|P_i \setminus P_0| \geq 1$, then $|P_0| = 1$.*

Proof. Since c does not admit a rainbow triple, $P_i = P_{p-i}$. Without loss of generality, suppose that $G \in P_i \setminus P_0$ and let $c(pa_1 + i) = c(pa_2 + p - i) = G$. Let $pb \in R_0$ be arbitrary. Consider the following triple:

$$\begin{aligned} x_1 &= pb \\ x_2 &= p(pa_1 + i - b) \\ x_3 &= pa_1 + i. \end{aligned}$$

Since c is rainbow-free, $c(x_1) = c(x_2)$. Next, consider the following triple:

$$\begin{aligned} x_1' &= p(pa_1 + i - b) \\ x_2' &= p(pa_2 + p - i - pa_1 - i + b) \\ x_3' &= pa_2 + p - i. \end{aligned}$$

Since c is rainbow-free, $c(x_1') = c(x_2')$. This implies that

$$c(pb) = c(p(pa_2 + p - i - pa_1 - i + b)).$$

Notice that difference in position between x_2' and pb, given by $pa_2 + p - i - pa_1 - i + b - b$, does not depend on b. Furthermore, $pa_2 + p - i - pa_1 - i + b - b$ is relatively prime to $p^{\alpha-1}$. Therefore, all elements in R_0 receive the same color. $\square$

Lemma 16. *Let p be prime with $p \geq 5$. If there exists $0 < i < \frac{p+1}{2}$ such that $|P_i \setminus P_0| \geq 2$ and $G \notin P_i \cup P_0$, then c admits a rainbow triple.*

Proof. For the sake of contradiction, suppose that c does not admit a rainbow triple. Since $p \geq 5$ and $|P_0| = 1$, there exists $j \neq i$ such that $0 < j < p$ and $G \in P_j \setminus (P_i \cup P_0)$. By Lemma 14, $P_j = P_{p-j}$ and $P_i = P_{p-i}$. Let $c(pa_1 + j) = c(pa_2 + p - j) = G$. Let $pb + i \in R_i$ be arbitrary. Consider the following triple:

$$\begin{aligned} x_1 &= pb + i \\ x_2 &= p(pa_1 + j - b - 1) + p - i \\ x_3 &= pa_1 + j. \end{aligned}$$

Then $c(x_1) = c(x_2)$. Next consider the following triple:

$$\begin{aligned} x_1' &= p(pa_1 + j - b - 1) + p - i \\ x_2' &= p(pa_2 + p - j - pa_1 - j + b) + i \\ x_3' &= pa_2 + p - j \end{aligned}$$

Then $c(x_1') = c(x_2')$. This implies that

$$c(pb + i) = c(p(pa_2 + p - j - pa_1 - j + b) + i).$$

Notice that the difference in position between x_2' and $pb+i$, given by $pa_1 + p - j - pa_1 - j + b - b$, does not depend on b. Furthermore, $pa_2 + p - j - pa_1 - j + b - b$ is relatively prime to $p^{\alpha-1}$. Therefore, all elements in R_i receive the same color. This is a contradiction, since $|P_i| \geq 2$. $\square$

Lemma 17. *If $p \geq 5$, $\mathbb{Z}_{p^\alpha}$ is colored with at least 4 colors, and there exists $0 < i < \frac{p+1}{2}$ with $im(c) = P_i \cup P_0$ and $|P_i \setminus P_0| \geq 2$, then c admits a rainbow triple.*

Proof. For the sake of contradiction, suppose that c does not admit a rainbow triple. By Lemma 15, let $P_0 = \{R\}$. By Lemma 14, $P_i = P_{p-i}$. Since P_i contains all colors except possibly R, there exists a, b, d such that $c(pa+i) = G$, $c(pb+p-i) = B$ and $c(pd + i) = B$. Consider the following triple:

$$\begin{aligned} x_1 &= pa + i \\ x_2 &= p(pb + p - i - a - 1) + p - i \\ x_3 &= pb + p - i. \end{aligned}$$

Then $c(x_2) \in \{B, G\}$. Let $x \in \{a, d\}$ such that $c(px + i) \neq c(x_2)$ and consider the following triple:

$$\begin{aligned} x_1' &= p(pb - p - i - a - 1) + p - i \\ x_2' &= p(px - pb + p + 2i + a) + i \\ x_3' &= px + i. \end{aligned}$$

Notice that $c(x_2') \in \{B, G\}$. Furthermore, the difference in position between x_2' and $pa + i$, given by $px - pb + p + 2i \equiv 2i \mod p$, does not depend on a, b, d modulo p. Therefore, for any $x \in \mathbb{Z}_p$ there exists $a \equiv x$ such that $c(pa + i) \in \{B, G\}$.

Since P_{p-i} contains all colors of c except for possibly R, there exists y such that $c(py + p - i) = Y$. Select $a \equiv -1 - y \mod p$ such that $c(pa + i) \in \{B, G\}$. Then the triple $(py + p - i, pa + i, a + y + 1)$ is rainbow since $a + y + 1 \in R_0$. $\square$

Proof of Theorem 4. Proposition 5 provides the lower bound for $p = 3$, $\alpha \geq 2$. Observation 8 covers the case when $p = 3, \alpha = 1, 2$.

We will proceed by induction on α. Suppose that $rb(\mathbb{Z}_{p^{\alpha-1}}, 3) = 4$ for some $\alpha \geq 3$. Let c be a 4 coloring of $\mathbb{Z}_{3^\alpha}$. For the sake of contradiction, suppose that c does not admit a rainbow triple. If $|P_0| = 4$, then c admits a rainbow triple by the induction hypothesis. Therefore, $|P_0| \leq 3$ and there exits $0 < i < p$ such that $|P_i \setminus P_0| \geq 1$. By Lemma 15, $|P_0| = 1$. This implies that $\text{im}(c) = |P_i \setminus P_0|$. By Lemma 17, c admits a rainbow triple. This completes the case when $p = 3$.

Let $p \geq 5$. With Proposition 4 as the base case, we will proceed by induction on α. Suppose that $rb(\mathbb{Z}_{p^{\alpha-1}}, p) = \frac{p+1}{2} + 1$ for some $\alpha \geq 2$. For the sake of contradiction, suppose that c does not admit a rainbow triple. If $|P_0| = \frac{p+1}{2} + 1$, then c admits a rainbow triple by the induction hypothesis. Therefore, $|P_0| \leq \frac{p+1}{2}$ and there exists $0 < j < p$ such that $|P_j \setminus P_0| \geq 1$. By Lemma 15, $P_0 = \{R\}$. By the pigeon hole principle, there exists $0 < i < \frac{p+1}{2}$ such that $|P_i \setminus P_0| \geq 2$. Notice that one of the following must hold:

1. $G \notin P_i \cup P_0$ for some color $G \neq R$,

2. $\mathrm{im}(c) = P_i \cup P_0$.

Therefore, by Lemmas 16 and 17, c must admit a rainbow triple. This completes the case when $p \geq 5$. □

3.3. Lower Bound for $rb(\mathbb{Z}_n, p)$, p Prime

Since p is the coefficient of the equation that we are considering, we will use q to denote a prime other than p. Using values for $rb(\mathbb{Z}_q, k)$, we establish a lower bound for $rb(\mathbb{Z}_n, p)$. In order to proceed in a similar manner as with the Schur equation, Lemmas 18 and 19 are about the structure of triples. Lemma 20 exploits this structural information to construct a coloring that witnesses the lower bound for Proposition 7.

Lemma 18. *If $x_1 + x_2 = kx_3$ is a triple in $\mathbb{Z}_n$ where $m|x_1, x_2, x_3$ for some $m|n$, $m, n \in \mathbb{Z}$, then there exists a triple of the form $x_1/m + x_2/m = kx_3/m$ in $\mathbb{Z}_{\frac{n}{m}}$.*

Proof. By definition $x_1 + x_2 = kx_3$ in $\mathbb{Z}_n$ implies:

$$\begin{aligned} x_1 + x_2 &= qn + r \\ kx_3 &= tn + r \end{aligned}$$

Divide both equations by m to get:

$$\begin{aligned} \frac{x_1}{m} + \frac{x_2}{m} &= q\frac{n}{m} + \frac{r}{m} \\ k\frac{x_3}{m} &= t\frac{n}{m} + \frac{r}{m} \end{aligned}$$

Now we must check that $\frac{r}{m}$ is an integer. Since $m|(x_1 + x_2 - qn)$, we know $m|r$. By definition, this means there exists a triple of the form $x_1/m + x_2/m = kx_3/m$ in $\mathbb{Z}_{\frac{n}{m}}$. □

Next, we show that q cannot divide exactly two terms of a triple.

Lemma 19. *Let (x_1, x_2, x_3) be a triple of the form $x_1 + x_2 = kx_3$ in $\mathbb{Z}_{qn}$. If q is relatively prime to k and q divides two of the terms in (x_1, x_2, x_3) then q must divide the third term in (x_1, x_2, x_3).*

Proof. We consider the case where q divides x_1, x_2 and the case where q divides x_1, x_3.

`Case 1:` Assume q divides x_1, x_2. By definition the equation $x_1 + x_2 = kx_3$ in $\mathbb{Z}_{qn}$ means:

$$\begin{aligned} x_1 + x_2 &= c_1 qn + r \\ k \cdot x_3 &= c_2 qn + r \end{aligned}$$

We rearrange the first equation to get q divides $x_1 + x_2 - c_1 qn$, which implies that q divides r. Thus, q divides $c_2 qn + r$, which implies q divides kx_3. We know q and k are relativity prime, therefore, q must divide x_3.

`Case 2:` Similarly, assume q divides x_1, x_3. By definition the equation $x_1 + x_2 = kx_3$ in $\mathbb{Z}_{qn}$ means:

$$\begin{aligned} x_1 + x_2 &= c_1 qn + r \\ k \cdot x_3 &= c_2 qn + r \end{aligned}$$

From the second equation we get q divides $kx_3 - c_2 qn$, which implies that q divides r. Thus, q divides $x_1 - c_1 \cdot qn - r$, which implies q divides x_2. □

Notice that Lemmas 18 and 19 are stated for the equation $x_1 + x_2 = kx_3$ without the stipulation that k is prime. We can use the above lemmas to find our lower bound.

Lemma 20. *Let q, t be positive integers with q prime, and $q \neq p$. If there exists a rainbow-free r-coloring of $\mathbb{Z}_t$, then there exists a rainbow-free $(r + rb(\mathbb{Z}_q, p) - 2)$-coloring of $\mathbb{Z}_{qt}$.*

Proof. Let $q, t \in \mathbb{Z}$ such that q is prime, and $q \neq p$. Let $\hat{c}$ be a rainbow-free r-coloring for $\mathbb{Z}_t$ and let $\bar{c}$ be a maximum coloring of $\mathbb{Z}_q$ such that 0 is uniquely colored and the other color classes are symmetric subsets, as described in Corollary 1. Let c be an exact $(r+1)$-coloring of $\mathbb{Z}_{qt}$ if $rb(\mathbb{Z}_q, p) = 3$ or an exact $(r+2)$-coloring of $\mathbb{Z}_{qt}$ if $rb(\mathbb{Z}_q, p) = 4$ as follows:

$$c(x) = \begin{cases} \hat{c}(\frac{x}{q}) & x \equiv 0 \mod q \\ r + \bar{c}(x \mod q) & \text{otherwise.} \end{cases}$$

Since q and p are distinct primes, q and p are relatively prime. By Lemma 19, since q is relatively prime to p, q cannot divide exactly two of the terms in

(x_1, x_2, x_3) for the equation $x_1 + x_2 = px_3$. Therefore, for all triples in $\mathbb{Z}_{qt}$, q can divide all three elements, no elements, or exactly one element of the triple.

`Case 1:` If q divides all three terms in (x_1, x_2, x_3), then by the constructions of c, the triple has the same colors as the triple $(\frac{x_1}{q}, \frac{x_2}{q}, \frac{x_3}{q})$ in $\hat{c}$. By Lemma 18, if (x_1, x_2, x_3) is a triple in $\mathbb{Z}_{qt}$ and $q|x_1, x_2, x_3$, then $(\frac{x_1}{q}, \frac{x_2}{q}, \frac{x_3}{q})$ is a triple in $\mathbb{Z}_t$. Thus, since $\hat{c}$ is a rainbow-free coloring, triples where all three elements are divisible by q cannot be rainbow in c.

`Case 2:` Suppose q divides none of the terms in (x_1, x_2, x_3). There is a maximum of two colors added on terms not divisible by q. Thus, there are at most two colors coloring the elements in any such triple, and triples of the form (x_1, x_2, x_3) with each x_i not divisible by q are not rainbow.

`Case 3:` Suppose q divides exactly one of (x_1, x_2, x_3). First assume q divides x_1. Notice that if $x_1 + x_2 \equiv px_3 \mod qt$ then $x_1 + x_2 \equiv px_3 \mod q$. Since 0 is uniquely colored in $\bar{c}$, the rainbow-free coloring of $\mathbb{Z}_q$, any triple in $\mathbb{Z}_q$ of the form $0+x_2 \equiv px_3 \mod q$ is colored so that x_2 and x_3 receive the same color. In this case, $c(x_2) = r+\bar{c}(x_2 \mod q)$ and $c(x_3) = r+\bar{c}(x_3 \mod q)$, so (x_1, x_2, x_3) is not rainbow under c. If q divides either x_2 or x_3 the argument proceeds the same way. □

Proposition 7. *Let p be prime and let n be an integer with prime factorization $n = p^\alpha \cdot q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdots q_m^{\alpha_m}$ where q_i is prime, $q_i \neq q_j$ for $i \neq j$ and $\alpha_i \geq 0$. Then,*

$$rb(\mathbb{Z}_n, p) \geq rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{q_i}, p) - 2)\Big)$$

Proof. If n is a power of p, then there is nothing to show. Suppose that the claim holds true for n where n has N prime factors that are not p.

Assume that $n = p^\alpha \cdot q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdots q_m^{\alpha_m}$ where $\alpha_1 + \cdots + \alpha_m = N + 1$. By the induction hypothesis, there exists a rainbow-free r-coloring of $\mathbb{Z}_{n/q_1}$ where

$$r = rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{q_i}, p) - 2)\Big) - rb(\mathbb{Z}_{q_1}, p) + 2.$$

Therefore, by Lemma 20 there exists a rainbow-free $r + rb(\mathbb{Z}_{q_1}, p) - 2$ coloring of Z_n. Thus, by induction

$$rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{q_i}, p) - 2)\Big).$$

□

3.4. Upper Bound for $rb(\mathbb{Z}_n, p)$, p Prime

In this section we prove an upper bound matching Proposition 7 for all prime factors q of n. Consider Proposition 8.

Proposition 8. *Let n be a positive integer, and let p be prime. Let n have prime factorization $n = p^\alpha \cdot q_1^{\alpha_1} \cdots q_m^{\alpha_m}$. Then*

$$rb(\mathbb{Z}_n, p) \leq rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \left(rb(\mathbb{Z}_{q_i^{\alpha_i}}, p) - 2 \right).$$

Proof. Suppose c is a rainbow-free r-coloring of $\mathbb{Z}_n$. It is a fact from abstract algebra that

$$\mathbb{Z}_n \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{q_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{q_m^{\alpha_m}}.$$

By abuse of notation, we allow $\alpha = 0$. With this fact, we can consider $\mathbb{Z}_{q_i^{\alpha_i}}$ as a subgroup of $\mathbb{Z}_n$. Let c_0 be the coloring of c restricted to $\mathbb{Z}_{p^\alpha}$. Let c_i be the coloring c restricted to $\mathbb{Z}_{q_i^{\alpha_i}}$. Since c is rainbow-free, c_i is also rainbow-free. Thus, $|c_i| \leq rb(\mathbb{Z}_{q_i}^{\alpha_i}, p) - 1$. By accounting for the fact that $c(0)$ is over counted, we can conclude that

$$r \leq rb(\mathbb{Z}_{p^\alpha}, p) - 1 + \sum_{i=1}^{m} \left(rb(\mathbb{Z}_{q_i^{\alpha_i}}, p) - 2 \right).$$

Recall that if $\alpha = 0$, then $rb(\mathbb{Z}_{p^\alpha}, p) = 2$ by convention. The proposition immediately follows by letting $r = rb(\mathbb{Z}_n, p) - 1$. $\square$

All that remains to prove Theorem 5 is to decompose $rb(\mathbb{Z}_{q^\alpha}, p)$ where p, q are prime into $2 + \alpha \cdot (rb(\mathbb{Z}_q, p) - 2)$. There is a nice case that follows the same ideas as for the upper bound of Theorem 2. This case is when the residue class R_0 receives enough colors to guarantee that $|P_i \setminus P_0| \leq 1$. However, this case is not forced. It is possible that a residue class other than R_0 receives the most colors. In this case, we create auxiliary colorings of residue classes that violate Theorem 6. Lemmas 21 and 22 show that the auxiliary colorings we use preserve rainbow solutions. The case where $p \not\equiv 2 \mod q$ culminates in Proposition 9, combining Lemmas 23, 24, and 25. Lemma 26 shows that there exists a coloring that fits Lemma 23. Lemma 27 uses the fact that $p \equiv 2 \mod q$ to repeatedly apply Lemma 26, resolving the last case.

Suppose the following for the rest of this section: Let $q \neq p$ be prime. Let c be a coloring of $\mathbb{Z}_{q^\alpha}$ where $\alpha \geq 2$. Let $R_0, R_1, \cdots, R_{q-1}$ be the residue classes modulo q for $\mathbb{Z}_{q^\alpha}$ with corresponding residue palettes $\{P_i\}$. We will let G, B denote two colors that are not in P_0.

The next two lemmas have the exact same proof.

Lemma 21. *Suppose $\hat{c}$ is a 3-coloring of $\mathbb{Z}_q$ such that $X \subseteq P_i$ implies $c(i) \in X$ where X is a nonempty subset of $\{G, B\}$, and $c(i) \in \{G, B\}$ implies $c(i) \in P_i$, and $c(i) = \beta$ otherwise. If $\hat{c}$ contains a rainbow triple, then c contains a rainbow triple.*

Lemma 22. *Suppose $|P_i \setminus P_j| \leq 1$. Let $\hat{c}$ be a coloring of $\mathbb{Z}_q$ such that:*

$$\hat{c}(i) := \begin{cases} P_i \setminus P_j & \textit{if } |P_i \setminus P_j| = 1 \\ \beta & \textit{otherwise} \end{cases}$$

If $\hat{c}$ contains a rainbow triple then c contains a rainbow triple.

Proof of Lemmas 21 and 22. Suppose that (x_1, x_2, x_3) is a rainbow triple in $\mathbb{Z}_q$ under $\hat{c}$. There are two cases: $\hat{c}(x_3) = \beta$, or $\hat{c}(x_3) \neq \beta$.

`Case 1:` If $\hat{c}(x_3) = \beta$, then $\beta \neq \hat{c}(x_1), \hat{c}(x_2)$. Without loss of generality, suppose that x_1 and x_2 are colored G and B, respectively. This implies that there exists u, v such that $c(qu + x_1) = G$ and $c(qv + x_2) = B$. We must find an integer s such that

$$u + v - ps \equiv \begin{cases} 1 \mod q^{\alpha-1} & x_1 + x_2 \geq q \\ 0 \mod q^{\alpha-1} & x_1 + x_2 < q \end{cases}.$$

Since p and q are relatively prime, we can always solve for s. Therefore, there exists a rainbow triple in $\mathbb{Z}_{q^\alpha}$ under c.

`Case 2:` Assume $\hat{c}(x_3) \neq \beta$. Without loss of generality, $\hat{c}(x_1) \neq \beta$, and there exist u, v such that $c(qu + x_1) = G$ and $c(qv + x_3) = B$ where $G, B \notin P_{x_2}$. Notice that $pqv - qu + px_3 - x_1 \in R_{x_2}$. Therefore, there exists a rainbow triple in $\mathbb{Z}_{q^\alpha}$ under c. □

Lemma 23. *Let c be rainbow-free with $|P_i \setminus P_0| \leq 1$. Then*

$$|c| \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 3.$$

Proof. Suppose c is an r-coloring. Let $\hat{c}$ be a coloring constructed from c as described in Lemma 22. Notice that the set of colors used in c is comprised of the colors in R_j and each color used in $\hat{c}$ other than α. Thus, we know that $r = |P_j| + |\hat{c}| - 1$, where $|\hat{c}|$ is the number of colors appearing in $\hat{c}$.

Since c is a rainbow-free coloring of $\mathbb{Z}_{q^\alpha}$, we know $c|_{R_0}$ must be a rainbow-free coloring of $\mathbb{Z}_q$, so $|P_0| \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) - 1$. Furthermore, $\hat{c}$ is a rainbow-free coloring of $\mathbb{Z}_q$, implying that $|\hat{c}| \leq rb(\mathbb{Z}_q, p) - 1$. Therefore, $r \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 3$. □

From Lemma 23, we can conclude that if c has too many colors, then either c has a rainbow triple, or $|P_i \setminus P_0| \geq 2$ for some i. In the next lemma, we show that if c has too many colors and $|P_j \setminus P_0| \geq 2$ for some j, then we can still conclude that c has a rainbow triple.

Lemma 24. *Suppose c has $r = rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 2$ colors and $p \not\equiv 2, -1 \mod q$. Then c is not rainbow-free.*

Proof. By Lemma 23, there exists P_j such that $|P_j \setminus P_0| \geq 2$, or we are done. For the sake of contradiction, suppose c is rainbow-free. Without loss of generality, $G, B \in P_j \setminus P_0$. Since $p \not\equiv 2 \mod q$, we have a x such that $2j \equiv px \mod q$ and $x \not\equiv j \mod q$. Without loss of generality, this implies that $G \in P_x$. Furthermore, P_{-j} cannot contain either B or G.

Define a coloring $\hat{c}$ of $\mathbb{Z}_q$ by

$$\hat{c}(i) = \begin{cases} B & i = j \\ G & i = x \\ B & B \in P_i, i \neq j, x \\ G & G \in P_i, i \neq j, x \\ \beta & \text{otherwise} \end{cases}$$

where the ambiguity if $G, B \in P_i$ and $i \neq j, x$ is resolved arbitrarily. Notice that $\hat{c}(0) = \beta, \hat{c}(j) = B$, and $\hat{c}(x) = G$. Therefore, $\hat{c}$ is always a 3-coloring. Now, if we find that $\hat{c}$ has a rainbow triple, then we have reached a contradiction via Lemma 21. By Theorem 6, there are three ways $\hat{c}$ can be rainbow-free: 0 is uniquely colored, $p \equiv -1 \mod q$, or $p \equiv 2 \mod q$. However, none of these three situations hold. The element 0 is not uniquely colored since $\hat{c}(-j) = \beta$. Furthermore, $p \not\equiv 2 \mod q$ and $p \equiv -1 \mod q$ by assumption. Thus, we can find a rainbow triple. □

Lemma 25. *Suppose c has $r = rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 2$ and $p \equiv -1 \mod q$. Then c is not rainbow-free.*

Proof. For the sake of contradiction, suppose that c is rainbow-free. By Theorem 3, $rb(\mathbb{Z}_q, p) = 4$. Notice that $|P_0| \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) - 1$, which implies that there are at least 3 colors not represented in P_0. Without loss of generality, we will let these colors be denoted by G, B, and Y. Given an ordering of colors $X_1 < X_2 < X_3$, consider the coloring $\hat{c} : \mathbb{Z}_q \to \{X_1, X_2, X_3, \beta\}$ given by

$$\hat{c}(x) = \begin{cases} \min\{X_i | X_i \in P_x\} & \{X_1, X_2, X_3\} \cap P_x \neq \{\} \\ \beta & \text{otherwise} \end{cases}.$$

Suppose $\hat{c}$ contains a rainbow triple x, y, z with colors $\hat{c}(x) < \hat{c}(y) < \hat{c}(z)$ (where $\beta > X_i$ for all i). Then there exists $sq + x \in R_x$, $tq + y \in R_y$ with $c(sq + x) = \hat{c}(x)$ and $c(tq + y) = c(y)$. By construction of $\hat{c}$, we have $\hat{c}(x), \hat{c}(y) \notin P_z$. This implies completing $sq + x$ and $tq + y$ with an element in R_z will provide a rainbow triple. Therefore, if there exists an ordering of G, B, Y such that $\hat{c}$ is a 4-coloring of $\mathbb{Z}_q$, then we are done.

`Claim:` If $|P_x \setminus P_0| \geq 1$, then $|P_x \setminus P_0| \geq 2$.

Without loss of generality, let $G < B < Y$ be an ordering that maximizes the number of colors used by $\hat{c}$. Since $G \notin P_0$, $\hat{c}$ is at least a 2-coloring. Furthermore, $\hat{c}$ always uses β on 0. We may also assume that $\hat{c}$ is not a 4-coloring. This implies that $\hat{c}$ does not use Y. Now if $Y \in P_x$, then either $G \in P_x$ or $B \in P_x$. By reordering the colors, we can conclude that if $B \in P_x$, then either $G \in P_x$ or $Y \in P_x$, and if $G \in P_x$, then either $B \in P_x$ or $Y \in P_x$. In particular, if $|P_x \setminus P_0| \geq 1$, then $|P_x \setminus P_0| \geq 2$. This concludes the proof of the claim.

Let P_j be such that $|P_j \setminus P_0| \geq 2$. By applying the $\phi(x) = j^{-1}x$ to $\mathbb{Z}_q$, we can assume that $j = 1$. Furthermore, we have that $P_{-1} \subseteq P_0$. By selecting two elements in R_1 with different colors not in P_0, we can conclude that $|P_{-2} \setminus P_0| \geq 2$. In particular, the set $I = \{i : |P_i \setminus P_0| \geq 2\}$ contains $(-2)^k$ for any non-negative integer k. Notice that if $q = 5$, then I contains 4 which gives us a rainbow triple since $P_{-1} \subseteq P_0$. Furthermore, if $q \geq 17$, then $1, -2 = q-2, 4, -8 = q-8$ are distinct modulo q. The fact that $1, -2, 4, -8$ are distinct for primes $7, 11, 13$ is true by inspection. Therefore, $|I| \geq 4$ and $q \geq 7$.

For any four distinct elements in I there exists indices x_1, x_2, x_3, x_4 such that (without loss of generality for the colors) $G \in P_{x_1} \cap P_{x_2}$ and $B \in P_{x_3} \cap P_{x_4}$. Since $|I| \geq 4$ we can define a coloring $\hat{c}$ of $\mathbb{Z}_q$ by

$$\hat{c}(i) = \begin{cases} G & i = x_1, x_2 \\ B & i = x_3, x_4 \\ B & B \in P_i, i \neq x_1, x_2, x_3, x_4 \\ G & G \in P_i, i \neq x_1, x_2, x_3, x_4 \\ \beta & \text{otherwise} \end{cases}$$

where the ambiguity if $G, B \in P_i$ and $i \neq x_1, x_2, x_3, x_4$ is resolved arbitrarily. If $\hat{c}$ has a rainbow solution, then c has a rainbow solution by Lemma 21.

Notice that every color class of $\hat{c}$ has size at least 2. Therefore, by Theorem 6, up to dilation, $\beta = [a_3, a_1 - 1]$, $G = [a_1, a_2 - 1]$, and $B = [a_2, a_3 - 1]$ with $a_1 + a_2 + a_3 \equiv 0, 1 \mod q$ and $0 < a_1 < a_2 < a_3 \leq q$. Furthermore, $[a_1, a_3 - 1]$ must be closed under multiplication by -2, since we can find a rainbow triple otherwise. We will conclude that this structure is impossible. Let $\succ$ be an ordering on $\mathbb{Z}_q$, where $a > b$ if $a' > b'$ where a' (resp. b') is a representative of the equivalence class a (resp. b) such that $0 \leq a' \leq q$.

First, assume that $a_1 = 1$. This implies that $a_3 - 1 \succeq -2$ and $a_3 = -1$. However, this is a contradiction since

$$a_1 + a_2 + a_3 \equiv 1 + a_2 - 1 \equiv 0, 1 \mod q$$

and $q > a_2 > 1$. Therefore, we can conclude that $a_1 > 1$ (and $a_1 \succ 1$). Notice that if $q - \frac{q+1}{2} \in [a_1, a_3 - 1]$, then we have reached a contradiction, since $-2(q - \frac{q+1}{2}) \equiv 1 \mod q$. We will complete the proof by assuming that $a_1 \succ q - \frac{q+1}{2}$ or $q - \frac{q+1}{2} \succ a_3 - 1$, and deriving a contradiction in each case.

`Case 1:` Assume that $q - \frac{q+1}{2} \succ a_3 - 1$. If $a_1 \preceq \lfloor q/4 \rfloor$, then $q - 2a_1 \succeq q - \frac{q+1}{2}$, which is a contradiction. Therefore, $a_3 - 1 \succ \lfloor q/4 \rfloor$. Now if $a_3 - 1 \succeq \lfloor q/3 \rfloor$, then $4(a_3 - 1) \succ a_3 - 1$. Furthermore, if $\lfloor q/3 \rfloor \succ a_3 - 1$, then $-2(a_3 - 1) \succ a_3 - 1$. Either one of the previous two possibilities results in a contradiction since $[a_1, a_3 - 1]$ is closed under multiplication by -2.

`Case 2:` Assume that $q - \frac{q+1}{2} \prec a_1$. If $a_3 - 1 \succeq 3q/4$, then $q - 2(a_3 - 1) \preceq q - \frac{q+1}{2}$, which is a contradiction. Therefore, $a_1 \prec \lfloor 3q/4 \rfloor$. Now if $a_1 \succeq \lceil 2q/3 \rceil$, then $-2a_1 \prec a_1$. Furthermore, if $\lceil 2q/3 \rceil \succ a_1$, then $4a_1 \prec a_1$. Either one of the previous two possibilities results in a contradiction since $[a_1, a_3 - 1]$ is closed under multiplication by -2.

Since both cases result in contradictions, we conclude that c is not rainbow-free. □

Proposition 9. *Let p, q be prime such that $p \not\equiv 2 \mod q$. Then*

$$rb(\mathbb{Z}_{q^\alpha}, p) \leq 2 + \alpha\,(rb(\mathbb{Z}_q, p) - 2)\,.$$

Proof. Combining Lemmas 23, 24, and 25, we know that $rb(\mathbb{Z}_{q^\alpha}, p) - 1 \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 2$ for $\alpha \geq 2$. Recursively applying this inequality gives the desired result. □

Lemma 26. *Suppose c is a rainbow-free coloring of $\mathbb{Z}_{q^\alpha}$ for $x_1 + x_2 = px_3$ where $\alpha \geq 2$ and $q \neq p$ is prime. Let $R_0, \cdots, R_{q-1}$ be the residue classes modulo q of $\mathbb{Z}_{q^\alpha}$, with corresponding color palettes $P_0, \cdots, P_{q-1}$. Let j be an index such that $|P_j| \geq |P_i|$ for all $0 \leq i \leq q-1$. Then $|P_i \setminus P_j| \leq 1$ for all $0 \leq i \leq q-1$.*

Proof. For the sake of contradiction, assume that there exists i such that $|P_i \setminus P_j| \geq 2$. This implies that there exists $qu + i$ and $qv + i$ with colors G and B respectively, that are not in P_j. Without loss of generality, $v > u$.

First suppose that $P_{pi-j} \neq P_j$. There are two cases: either P_{pi-j} has a color that is not in P_j, or P_j has a color that is not in P_{pi-j}.

`Case 1:` Suppose that $c(sq+pi-j) \notin P_j$. Without loss of generality, $c(sq+pi-j) \neq G$. Then

$$\begin{aligned}
x_1 &= qs + pi - j \\
x_2 &= pqu + -qs + j \\
x_3 &= qu + i
\end{aligned}$$

is a rainbow triple.

`Case 2:` Suppose that $c(qs + j) \notin P_{pi-j}$. Then

$$\begin{aligned}
x_1 &= qs + j \\
x_2 &= pqu - qs + pi - j \\
x_3 &= qu + i
\end{aligned}$$

is rainbow.

Since c is assumed to be rainbow-free, both cases result in a contradiction. Therefore, $P_j = P_{pi-j}$.

Let $qs+j \in R_j$. Since c is rainbow-free, $c(pqu-qs+pi-j) = c(qs+j)$. Similarly, the triple

$$\{q(pu-s)+pi-j, q(pv-pu+s)+j, qv+i\}$$

shows that $c(pqv-pqu+qs+j) = c(pqu-qs+pi-j) = c(qs+j)$. Notice that the difference of position between $pqv-pqu+qs+j$ and $qs+j$ in R_j is $p(v-u)$. Since $p \neq q$ is prime and $v > u$, we know that $p(v-u)$ is an additive generator of $\mathbb{Z}_{q^{\alpha-1}}$ (since $p \not| q$). Therefore, R_j is monochromatic; this contradicts the maximality of $|P_j|$. □

Lemma 27. *If $p \equiv 2 \mod q$, then $rb(\mathbb{Z}_{q^\alpha}, p) \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 2$.*

Proof. Suppose c is an r-coloring. By Lemma 26, there exists index j such that $|P_j| \geq |P_i|$ for all $0 \leq i \leq q-1$. Then $|P_i \setminus P_j| \leq 1$ for all $0 \leq i \leq q-1$. Let $\hat{c}$ be a coloring constructed from c as described in Lemma 22. Notice that the set of colors used in c is comprised of the colors in R_j and each color used in $\hat{c}$ other than α. Thus, we know that $r = |P_j| + |\hat{c}| - 1$, where $|\hat{c}|$ is the number of colors appearing in $\hat{c}$.

Notice that $p \equiv 2 \mod q$ implies that any triple with 2 elements in R_j must be completely contained in R_j. Therefore, R_j acts like $Z_{q^{\alpha-1}}$ when we only consider the positions within R_j. Since c is a rainbow-free coloring of $\mathbb{Z}_{q^{\alpha-1}}$, we know $c|_{R_j}$ must be a rainbow-free coloring of $\mathbb{Z}_q$, so $|P_0| \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) - 1$. Furthermore, $\hat{c}$ is a rainbow-free coloring of $\mathbb{Z}_{q^{\alpha-1}}$, implying that $|\hat{c}| \leq rb(\mathbb{Z}_q, p) - 1$. Therefore, $r \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 3$. □

Proof of Theorem 5. Applying Proposition 9 or Lemma 27 for to every prime factor $q_i \neq p$ of n in Proposition 8 gives

$$rb(\mathbb{Z}_n, p) \leq rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \Big(\alpha_i(rb(\mathbb{Z}_{q_i}, p) - 2)\Big).$$

Since this is identical to the lower bound from Proposition 7, we can conclude

$$rb(\mathbb{Z}_n, p) = rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \Big(\alpha_i(rb(\mathbb{Z}_{q_i}, p) - 2)\Big).$$

□

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