RAINBOW NUMBERS OF $\mathbb{Z}_n$ FOR $a_1x_1 + a_2x_2 + a_3x_3 = b$

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Abstract

An exact r-coloring of a set S is a surjective function $c : S \to [r]$. The rainbow number of a set S for equation eq is the smallest integer r such that every exact r-coloring of S contains a rainbow solution to eq. In this paper, the rainbow number of $\mathbb{Z}_p$, for p prime, and the equation $a_1x_1+a_2x_2+a_3x_3 = b$, is determined. The rainbow number of $\mathbb{Z}_n$, for a natural number n, is determined under certain conditions.

1. Introduction

Let c be a coloring of set S. A subset $X \subseteq S$ is rainbow if each element of X is colored distinctly. For example, color $[n] = \{1, 2, \ldots, n\}$ and consider solutions to

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the equation $x_1 + x_2 = x_3$. If each element of a solution $\{a, b, a+b\} \subseteq [n]$ is colored distinctly, that solution is rainbow. One of the first papers to investigate rainbow arithmetic progressions is [7], where Jungić et al. showed that colorings with each color used equally yield rainbow arithmetic progressions. In [7], only 3-term arithmetic progressions are considered which are also solutions to $x_1 + x_2 = 2x_3$. In [1], Axenovich and Fon-Der-Flaass showed that no 5-colorings avoid rainbow 3-term arithmetic progressions. A few articles investigated the anti-van der Waerden number, which is the fewest number of colors needed to guarantee a rainbow arithmetic progression. For example, Butler et al. established, in [4], bounds for the anti-van der Waerden number when coloring $[n]$ and some exact values when coloring $\mathbb{Z}_n$. Later, Berikkyzy, Schulte, and Young determined, in [2], the anti-van der Waerden number for $[n]$ in the case of 3-term arithmetic progressions.

Some of this work was generalized to graphs and abelian groups. Montejano and Serra investigated, in [9], rainbow-free colorings of abelian groups when considering arithmetic progressions. Similarly, rainbow arithmetic progressions in finite abelian groups were studied by co-author Young, in [11], where the anti-van der Waerden numbers were connected to the order of the group. When arithmetic progressions were extended to graphs, Rehm, Schulte, and Warnberg showed, in [10], the anti-van der Waerden number of graph products is either 3 or 4.

Generalizing the equation $x_1 + x_2 = 2x_3$, Bevilacqua et al., in [3], considered $x_1 + x_2 = kx_3$ on $\mathbb{Z}_n$. The rainbow number of $\mathbb{Z}_n$ was determined for these equations when $k = 1$ or $k = p$ where p is prime. These results served as motivation for this paper where the equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ will be considered over $\mathbb{Z}_p$ and $\mathbb{Z}_n$ with p prime. From now on $a_1x_1 + a_2x_2 + a_3x_3 = b$ will be denoted by $\text{eq}(a_1, a_2, a_3, b)$. This paper establishes the rainbow number, also known as the anti-van der Waerden number, of $\mathbb{Z}_n$ for $\text{eq}(a_1, a_2, a_3, b)$ for some equations. One important result that will be used is Huicochea and Montejano's characterization, in [6], of all rainbow-free exact 3-colorings of $\mathbb{Z}_p$ for $\text{eq}(a_1, a_2, a_3, b)$ for all primes p.

1.1. Preliminaries

An *r-coloring* of a set S is a function $c : S \to [r]$ and an r-coloring is *exact* if c is surjective. Note that an exact r-coloring yields a partition of S into r disjoint color classes. This paper will focus on the linear equation $\text{eq}(a_1, a_2, a_3, b)$ given by

$$a_1x_1 + a_2x_2 + a_3x_3 = b \tag{1}$$

and r-colorings of $\mathbb{Z}_n$. An ordered set (s_1, s_2, s_3) is called a *solution* to $\text{eq}(a_1, a_2, a_3, b)$ in $\mathbb{Z}_n$ if $a_1s_1 + a_2s_2 + a_3s_3 \equiv b \bmod n$. Throughout the paper $=$ will be used instead of $\equiv$, and the mod n will not be used unless the context requires clarification.

If c is an r-coloring of $\mathbb{Z}_n$, then a *rainbow solution* in $\mathbb{Z}_n$ to $\text{eq}(a_1, a_2, a_3, b)$ is a solution such that $|\{c(s_1), c(s_2), c(s_3)\}| = 3$, i.e., each member of the solution has been assigned a distinct color by c. A coloring c of $\mathbb{Z}_n$ is *rainbow-free* for $\text{eq}(a_1, a_2, a_3, b)$ if there are no rainbow solutions.

The rainbow number of $\mathbb{Z}_n$ for equation $eq = \text{eq}(a_1, a_2, a_3, b)$, denoted $\text{rb}(\mathbb{Z}_n, eq)$, is the smallest positive integer r such that every exact r-coloring of $\mathbb{Z}_n$ has a rainbow solution for eq. If there are no rainbow solutions to eq in an exact n-coloring of $\mathbb{Z}_n$, then the convention will be that $\text{rb}(\mathbb{Z}_n, eq) = n + 1$. Since rainbow solutions to eq require three colors, then $\text{rb}(\mathbb{Z}_n, eq) \geq 3$, for all $n \geq 2$.

The following tools will be used throughout the paper. Given a set $S \subseteq \mathbb{Z}_n$ and $d, t \in \mathbb{Z}_n$, the sets $S + t = \{s + t \mid s \in S\}$ and $dS = \{ds \mid s \in S\}$ are called the t-*translation* and d-*dilation* of S, respectively. If the multiplicative inverse of $a \in \mathbb{Z}_n$ exists, denote the inverse by a^{-1}. The set of all these invertible elements forms a group under multiplication, and it is denoted by $\mathbb{Z}_n^*$. For $d \in \mathbb{Z}_n^*$, let $\langle d \rangle$ be the multiplicative subgroup of $\mathbb{Z}_n^*$ generated by d and $\langle d_1, \ldots, d_k \rangle$ be multiplicatively generated by the d_i's. A subset $S \subseteq \mathbb{Z}_n$ is $\langle d \rangle$-*periodic* if $S = dS$ and a set is called *symmetric* if it is $\langle -1 \rangle$-periodic. For ease of reading, the related results from [6] are referenced below.

Theorem 1. [6, Theorem 3] *Let A, B and C be the color classes of an exact 3-coloring of $\mathbb{Z}_p$ such that $1 \leq |A| \leq |B| \leq |C|$. The coloring is rainbow-free for $\text{eq}(1, 1, -c, 0)$ if and only if, under dilation, one of the following holds true:*

1) $A = \{0\}$, with both B and C symmetric $\langle c \rangle$-periodic subsets.

2) $A = \{1\}$ for

a) $c = 2$, with $(B - 1)$ and $(C - 1)$ symmetric $\langle 2 \rangle$-periodic subsets;

b) $c = -1$, with $(B \backslash \{-2\}) + 2^{-1}$ and $(C \backslash \{-2\}) + 2^{-1}$ symmetric subsets.

3) $|A| \geq 2$, for $c = -1$, with A, B and C arithmetic progressions with difference 1, such that $A = \{i\}_{i=t_1}^{t_2-1}$, $B = \{i\}_{i=t_2}^{t_3-1}$, and $C = \{i\}_{i=t_3}^{t_1-1}$, where $(t_1 + t_2 + t_3) = 1$ or 2.

Theorem 2. [6, Theorem 6] *Let A, B and C be the color classes of an exact 3-coloring of $\mathbb{Z}_p$ such that $1 \leq |A| \leq |B| \leq |C|$. The coloring is rainbow-free for $\text{eq}(a_1, a_2, a_3, b)$, with some $a_i \neq a_j$, if and only if $A = \{s\}$ with $s(a_1 + a_2 + a_3) = b$, and both B and C are sets invariant under six specific transformations.*

Corollary 1. [6, Corollary 8] *Every exact 3-coloring of $\mathbb{Z}_p$ contains a rainbow solution of $\text{eq}(a_1, a_2, a_3, b)$, with some $a_i \neq a_j$, if and only if one of the following holds true:*

1) $a_1 + a_2 + a_3 = 0 \neq b$,

2) $|\langle d_1, \ldots, d_6 \rangle| = p - 1$, *where* $d_1 = -a_3a_1^{-1}$, $d_2 = -a_2a_1^{-1}$, $d_3 = -a_1a_2^{-1}$, $d_4 = -a_3a_2^{-1}$, $d_5 = -a_1a_3^{-1}$, *and* $d_6 = -a_2a_3^{-1}$.

Note that Theorem 3 is the same as the case when $b = 0$ and $c = -1$ in Theorem 1. It is included for completion.

Theorem 3. [6, Theorem 5] *Let* A, B *and* C *be the color classes of an exact* 3-*coloring of* $\mathbb{Z}_p$ *with* $p \geq 3$ *and* $1 \leq |A| \leq |B| \leq |C|$. *The coloring is rainbow-free for* $\mathrm{eq}(1, 1, 1, b)$ *if and only if one of the following holds true:*

1) $A = \{s\}$ *with both* $(B \backslash \{b - 2s\}) + (s - b)2^{-1}$ *and* $(C \backslash \{b - 2s\}) + (s - b)2^{-1}$ *symmetric sets.*

2) $|A| \geq 2$, *and all* A, B *and* C *are arithmetic progressions with the same common difference* d, *so that* $d^{-1}A = \{i\}_{i=t_1}^{t_2-1}$, $d^{-1}B = \{i\}_{i=t_2}^{t_3-1}$, *and* $d^{-1}C = \{i\}_{i=t_3}^{t_1-1}$ *satisfy* $t_1 + t_2 + t_3 \in \{1 + d^{-1}b, 2 + d^{-1}b\}$.

Lemma 1 will be used to extend results for rainbow numbers of equations where $b = 0$ to equations where $b \neq 0$.

Lemma 1. *For* $a_1, a_2, a_3 \in \mathbb{Z}_n$, *let* $a = a_1 + a_2 + a_3$ *and suppose that* $a \in \mathbb{Z}_n^*$. *There exists a rainbow-free* k-*coloring of* $\mathbb{Z}_n$ *for* $a_1x_1 + a_2x_2 + a_3x_3 = b$ *if and only if there exists a rainbow-free* k-*coloring of* $\mathbb{Z}_n$ *for* $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

Proof. Define $T : \mathbb{Z}_n \to \mathbb{Z}_n$ by $T(x) = x - ba^{-1}$. Suppose (s_1, s_2, s_3) is a solution to $a_1x_1 + a_2x_2 + a_3x_3 = b$. Applying the one-to-one transformation T to (s_1, s_2, s_3) gives:

$$\begin{aligned} a_1T(s_1) + a_2T(s_2) + a_3T(s_3) &= a_1(s_1 - ba^{-1}) + a_2(s_2 - ba^{-1}) + a_3(s_3 - ba^{-1}) \\ &= a_1s_1 + a_2s_2 + a_3s_3 + (a_1 + a_2 + a_3)(-ba^{-1}) \\ &= b + a(-ba^{-1}) \\ &= 0. \end{aligned}$$

Similarly, if $(T(s_1), T(s_2), T(s_3))$ is a solution to $a_1x_1 + a_2x_2 + a_3x_3 = 0$, then (s_1, s_2, s_3) is a solution to $a_1x_1 + a_2x_2 + a_3x_3 = b$. This gives a one-to-one correspondence between solutions of the two equations. □

This paper is organized as follows. First, $\mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a_1, a_2, a_3, b))$, is determined in Section 2. The main result in this section, Theorem 4, states that the rainbow number of $\mathbb{Z}_p$ is either 3 or 4. In Section 3, the rainbow number of $\mathbb{Z}_n$ is computed for a natural number n. The main result in this section, Theorem 8, shows that for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$

$$\mathrm{rb}(\mathbb{Z}_n, \mathrm{eq}(a_1, a_2, a_3, b)) = 2 + \sum_{k=1}^{\ell} \left[\alpha_k(\mathrm{rb}(\mathbb{Z}_{p_k}, \mathrm{eq}(a_1, a_2, a_3, b) - 2)\right],$$

under certain conditions. To prove this result, Section 3 includes leading lemmas and theorems such as finding the rainbow number $\text{rb}(\mathbb{Z}_{2^\alpha}, \text{eq}(a_1, a_2, a_3, b))$ in Theorem 6, establishing the right hand side of the above equation as a lower bound in Corollary 4, and as an upper bound in Corollary 7.

2. Rainbow Numbers of $\mathbb{Z}_p$

This section establishes the rainbow number for the equation $\text{eq}(a_1, a_2, a_3, b)$ over $\mathbb{Z}_p$ where p is a prime. Under certain conditions, Lemma 2 establishes that if two elements in a solution are the same, then all three are the same. This fact was mentioned in [6] without proof and has been included for completion.

Lemma 2. *If $a_1s_1 + a_2s_2 + a_3s_3 = 0$ over $\mathbb{Z}_p$ with $|\{s_1, s_2, s_3\}| < 3$, $a_1 + a_2 + a_3 = 0$ and $a_1a_2a_3 \in \mathbb{Z}_p^*$, then $s_1 = s_2 = s_3$.*

Proof. If $s_1 = s_2 = s_3$ the proof is complete. Without loss of generality, assume $s_1 = s_2$. Observe that $a_1 + a_2 + a_3 = 0$ implies $a_3 = -a_1 - a_2$, and therefore $a_1s_1 + a_2s_1 + a_3s_3 = (a_1 + a_2)(s_1 - s_3) = 0$.

Since $\mathbb{Z}_p$ has no zero divisors, this gives $a_1 + a_2 = 0$ or $s_1 - s_3 = 0$. Note that $a_1 + a_2 = 0$ along with $a_1 + a_2 + a_3 = 0$ gives $a_3 = 0$ which contradicts $a_1a_2a_3 \neq 0$. Therefore, $s_1 - s_3 = 0$ and so $s_1 = s_3$ and $|\{s_1, s_2, s_3\}| = 1$. □

If $p = 2$, by convention $\text{rb}(\mathbb{Z}_2, \text{eq}) = 3$. The case when $p = 3$ is handled next.

Proposition 1. *For all $a_1, a_2, a_3, b \in \mathbb{Z}_3$,*

$$\text{rb}(\mathbb{Z}_3, \text{eq}(a_1, a_2, a_3, b)) = \begin{cases} 3 & \textit{if } b = 0 \textit{ and } a_i = a_j, \quad \textit{for some } i \neq j \\ & \textit{or } b \neq 0 \textit{ and } a_i \neq a_j, \quad \textit{for some } i \neq j, \\ 4 & \textit{otherwise.} \end{cases}$$

Proof. Note there is only one way (up to isomorphism) to color $\mathbb{Z}_3$ with three distinct colors. Suppose eq has a rainbow solution and, without loss of generality, assume a solution is $(1, 2, 0)$. For $\text{eq}(a_1, a_2, a_3, 0)$, $a_1 + 2a_2 = 0$ implies $a_1 = a_2$. It then follows that a rainbow solution will exist if and only if $a_i = a_j$ for some $i \neq j$, giving $\text{rb}(\mathbb{Z}_3, \text{eq}(a_1, a_2, a_3, 0)) = 3$. If the a_i's are all distinct, by standard convention, $\text{rb}(\mathbb{Z}_3, \text{eq}(a_1, a_2, a_3, 0)) = 4$.

Now consider $\text{eq}(a_1, a_2, a_3, b)$ for $b \neq 0$. The solution $(1, 2, 0)$ gives $a_1 - a_2 = b$. Since $b \neq 0$, then $a_1 \neq a_2$. It then follows that $\text{rb}(\mathbb{Z}_3, \text{eq}(a_1, a_2, a_3, b)) = 3$ if $a_i \neq a_j$ for some $i \neq j$. Otherwise, $\text{rb}(\mathbb{Z}_3, \text{eq}(a_1, a_2, a_3, b)) = 4$. □

Next, the case $p \geq 5$ will be discussed. Theorem 4 shows that the rainbow number of $\text{eq}(a_1, a_2, a_3, b)$ is either 3 or 4 depending on the different variations of

a_1, a_2, a_3 and b. The following theorem also uses notation established in Corollary 1.

Theorem 4. *Let $a_1, a_2, a_3, b \in \mathbb{Z}_p$ with some $a_i \neq a_j$ and $a_1a_2a_3 \in \mathbb{Z}_p^*$ for $p \geq 5$, then*

$$\mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a_1, a_2, a_3, b)) = \begin{cases} 3 & \textit{if } |\langle d_1, d_2, \ldots, d_6 \rangle| = p - 1 \\ & \textit{or } a_1 + a_2 + a_3 = 0 \neq b, \\ 4 & \textit{otherwise.} \end{cases}$$

Proof. The proof follows by case analysis. First, define $eq = \mathrm{eq}(a_1, a_2, a_3, b)$.

`Case 1`: $|\langle d_1, d_2, \ldots, d_6 \rangle| = p - 1$ or $a_1 + a_2 + a_3 = 0 \neq b$. The conditions in this case are the conditions of Corollary 1, thus $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 3$ and $\mathrm{rb}(\mathbb{Z}_p, eq) = 3$.

`Case 2`: $|\langle d_1, d_2, \ldots, d_6 \rangle| < p - 1$ and $a_1 + a_2 + a_3 \neq 0$. By Corollary 1, there exists a rainbow-free 3-coloring which implies $\mathrm{rb}(\mathbb{Z}_p, eq) \geq 4$. Since $a_1 + a_2 + a_3 \neq 0$ there is a unique $s \in \mathbb{Z}_p$ such that $s(a_1 + a_2 + a_3) = b$. Suppose there is a 4-coloring of $\mathbb{Z}_p$ with color classes A, B, C, and D such that $s \in A$. Create a 3-coloring with color classes $A \cup B$, C, and D. By construction, s is not in a color class by itself. Theorem 2 now guarantees there is a rainbow solution in this 3-coloring which corresponds to a rainbow solution in the 4-coloring. Thus, $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 4$ and hence, $\mathrm{rb}(\mathbb{Z}_p, eq) = 4$.

`Case 3`: $|\langle d_1, d_2, \ldots, d_6 \rangle| < p-1$, $a_1+a_2+a_3 = 0$, and $b = 0$. Since $|\langle d_1, d_2, \ldots, d_6 \rangle| < p-1$ and $b = 0$, both conditions in Corollary 1 fail; hence, $\mathrm{rb}(\mathbb{Z}_p, eq) \geq 4$. Note that in this case, every $s \in \mathbb{Z}_p$ satisfies $s(a_1 + a_2 + a_3) = b$. To show that $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 4$, consider a 4-coloring of $\mathbb{Z}_p$.

`Case 3.1:` At most two color classes have size one. Combine the two smallest color classes to make a 3-coloring that has no color classes of size one. By Theorem 2, this 3-coloring contains a rainbow solution. Thus, the original 4-coloring contains a rainbow solution, which implies $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 4$. Note that if there are at least three color classes of size one, then the argument used in Case 3.1 does not hold. Essentially, combining the two smallest color classes will give a 3-coloring that has a color class with one element.

`Case 3.2`: At least three color classes have size one. Let $A = \{s_1\}$ and $B = \{s_2\}$ be two of the three color classes of size one. Let $s_3 = a_3^{-1}(-a_1s_1 - a_2s_2)$. Then (s_1, s_2, s_3) is a solution. Since $s_1 \neq s_2$, by Lemma 2, s_1, s_2, s_3 are pairwise distinct. Therefore, (s_1, s_2, s_3) is a rainbow solution. Thus, the 4-coloring contains a rainbow, which implies $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 4$. □

Note that Theorem 4 considered equations where $a_i \neq a_j$ for some $i \neq j$. For the remainder of this section it is assumed that $a_1 = a_2 = a_3$. To handle equations of this type, Theorem 1 and Lemma 3 are essential.

Lemma 3. *Suppose sets A, B, C, and D partition $\mathbb{Z}_p$. The sets $A \cup B$, $A \cup C$, $A \cup D$, B, C, and D cannot all be arithmetic progressions with common difference $d \neq 0$.*

Proof. For the sake of contradiction, suppose $A \cup B$, $A \cup C$, $A \cup D$, B, C, and D are all arithmetic progressions with common difference d. Define $B = \{\beta, \beta + d, \ldots, \beta + kd\}$. Since B and $A \cup B$ are both arithmetic progressions with the same common difference, then A contains $\beta - d$ or $\beta + (k+1)d$. Similarly, this applies to C and $A \cup C$ and applies to D and $A \cup D$. However, this implies that B, C, and D are not pairwise disjoint, a contradiction. □

It will be shown in Lemma 5 that an arithmetic progression D of $\mathbb{Z}_p$, with $2 \leq |D| \leq p-2$, of common difference d can only be viewed as an arithmetic progression of common difference $\pm d$. Consider the interval notation $[x, y]$, for $x < y$, as defined in [5] as follows. For $x, y \in \mathbb{Z}_p$, let $k = y - x$ and $[x, y] := \{x + i \in \mathbb{Z}_p \mid 0 \leq i \leq k\}$. The author in [5] defines an arithmetic progression of common difference r and length $k + 1$ as the dilated interval $r[x, y]$. The set of such arithmetic progressions is denoted, in [5], by $\mathrm{AP}(r)$.

Lemma 4. [5, Lemma 3.4] Let X be a subset of $\mathbb{Z}_p$ such that $2 \leq |X| \leq p - 2$ and $r, t \in \mathbb{Z}_p^*$. If $X, tX \in \mathrm{AP}(r)$, then $t \in \{\pm 1\}$.

This lemma can be generalized as follows.

Corollary 2. *Let X be a subset of $\mathbb{Z}_p$ such that $2 \leq |X| \leq p-2$. If $tX, t'X \in \mathrm{AP}(r)$ for $t, t' \in \mathbb{Z}_p^*$, then $t' \in \{\pm t\}$.*

Proof. Let $tX = r[x, y]$ and $t'X = r[x', y']$. Then $X = t^{-1}r[x, y] \in \mathrm{AP}(t^{-1}r)$ and $t^{-1}t'X = t^{-1}r[x', y'] \in \mathrm{AP}(t^{-1}r)$. By Lemma 4, $t^{-1}t' \in \{\pm 1\}$, and hence $t' \in \{\pm t\}$. □

Lemma 5. *Let D be a subset of $\mathbb{Z}_p$ such that $2 \leq |D| \leq p - 2$ and $d, r \in \mathbb{Z}_p^*$. If D is an arithmetic progression with difference d and D is an arithmetic progression with difference r, then $r \in \{\pm d\}$.*

Proof. Let $D = \{x, x + d, \ldots, x + kd\}$ and $D = \{x', x' + r, \ldots, x' + kr\}$. Then $D = d[d^{-1}x, d^{-1}x + k] \in \mathrm{AP}(d)$ and $D = r[r^{-1}x', r^{-1}x' + k] \in \mathrm{AP}(r)$. Multiplying these two dilated intervals by r and d, respectively, gives $rD = rd[d^{-1}x, d^{-1}x + k] \in \mathrm{AP}(rd)$ and $dD = rd[r^{-1}x', r^{-1}x' + k] \in \mathrm{AP}(rd)$. Thus rD and dD are both in $\mathrm{AP}(rd)$. Applying Corollary 2 gives $r \in \{\pm d\}$. □

Theorem 5. *If $a \in \mathbb{Z}_p^*$, $b \in \mathbb{Z}_p$, and $p \geq 5$, then* $\mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a, a, a, b)) = 4$.

Proof. Since $p \geq 5$, then $3a \in \mathbb{Z}_p^*$. By Lemma 1, it is enough to consider $ax_1 + ax_2 + ax_3 = 0$. Furthermore, because $a \in \mathbb{Z}_p^*$, the triple (s_1, s_2, s_3) is a solution to $ax_1 + ax_2 + ax_3 = 0$ if and only if it is a solution to $x_1 + x_2 + x_3 = 0$. Thus, without loss of generality, the rest of the argument only considers $x_1 + x_2 + x_3 = 0$. The exact rainbow-free 3-coloring of $\mathbb{Z}_p$ with color classes $\{0\}, \{1, p-1\}, \{2, 3, \ldots, p-2\}$ establishes that $4 \leq \mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(1, 1, 1, 0))$. Suppose there is an exact 4-coloring of $\mathbb{Z}_p$ with color classes A, B, C, and D such that $|A| \leq |B| \leq |C| \leq |D|$. It will be shown that a rainbow solution exists in the aforementioned exact 4-coloring.

`Case 1:` At most one color class has size one. Consider the exact 3-colorings with color classes: $A \cup B$, C, D; B, $A \cup C$, D; and B, C, $A \cup D$. If each of the colorings is rainbow-free, then, by Theorem 3, each of the color classes in the three colorings are arithmetic progressions. In particular, $A \cup B$, C and D are arithmetic progressions with common difference d; $A \cup C$, B and D are arithmetic progressions with common difference d'; and $A \cup D$, B and C are arithmetic progressions with common difference d''. Since the first two partitions overlap in the set D, and $2 \leq |D| \leq p - 2$, we know that $d' = \pm d$ by Lemma 5. Similarly, $d'' = \pm d$. Without loss of generality, $d'' = d' = \pm d$. However, any arithmetic progression with common difference d is also an arithmetic progression with common difference $-d$. This gives $A \cup B$, $A \cup C$, $A \cup D$, B, C, and D are all arithmetic progressions with the same common difference d. This contradicts Lemma 3 so one of the exact 3-colorings must have a rainbow solution.

`Case 2:` Exactly two color classes have size one. Let $A = \{s\}$ and $B = \{\beta\}$. If $\beta \neq -2s$, then $\{s, \beta, -s-\beta\}$ is a rainbow solution. Thus, without loss of generality, assume $\beta = -2s$. Note this also means $s \neq 0$. Consider the exact 3-coloring with color classes $A \cup B$, C, D. If this coloring is rainbow-free, then, by Theorem 3.2, $A \cup B$, C and D must be arithmetic progressions with common difference d. Further, $d^{-1}(A \cup B)$, $d^{-1}C$ and $d^{-1}D$ are sets of consecutive integers and is a rainbow-free exact 3-coloring. Now consider the exact 3-coloring with color classes $d^{-1}A$, $d^{-1}(B \cup C)$, $d^{-1}D$. Theorem 3.1 implies that $d^{-1}(B \cup C) \backslash \{d^{-1}(-2s)\} + s2^{-1} = d^{-1}C + s2^{-1}$ and $d^{-1}D + s2^{-1}$ are symmetric. However, the color classes must also be consecutive which would imply $\beta = -s$, which is a contradiction.

`Case 3:` At least three color classes have size one. Without loss of generality, dilate the coloring so that $A = \{1\}$, $B = \{\beta\}$, and $C = \{\gamma\}$. Note that if the exact 3-colorings with color classes A, C, $B \cup D$ and A, B, $C \cup D$ are rainbow-free, then they must be of the form described in Theorem 1 part 2.*b*. This means $B \backslash \{-2\} + 2^{-1} \in \{\emptyset, \{0\}\}$ and $C \backslash \{-2\} + 2^{-1} \in \{\emptyset, \{0\}\}$. So, without loss of generality, $\beta = -2$ and $\gamma = -(2^{-1})$. Notice that $(-2, -(2^{-1}), 2 + 2^{-1})$ is a rainbow solution because $-2 = -(2^{-1})$, $2 + 2^{-1} = -2$ or $2 + 2^{-1} = -(2^{-1})$ imply $p \in \{2, 3\}$.

In all cases, an exact 3-coloring constructed from the original exact 4-coloring has a rainbow solution. Thus the original exact 4-coloring has a rainbow solution. □

3. Rainbow Numbers of $\mathbb{Z}_n$ for $a_1x_1 + a_2x_2 + a_3x_3 = b$

In this section the rainbow number for $\mathbb{Z}_n$ will be established under certain conditions on the coefficients. Since 2 is a special case, the rainbow number for $\mathbb{Z}_{2^\alpha}$ will be considered first. Then the lower and upper bounds are established for general n.

Let A and B be sets and $m, n \in \mathbb{Z}$. (A, m, eq) is *solomorphic* to (B, n, eq') when there exists a function $\phi : A \to B$ such that $\{s_1, s_2, s_3\} \subset A$ is a solution to $eq \bmod m$ if and only if $\{\phi(s_1), \phi(s_2), \phi(s_3)\} \subset B$ is a solution to $eq' \bmod n$. Note that solomorphic sets have the same rainbow number.

Theorem 6. *If $a_1a_2a_3 \in \mathbb{Z}_2^*$, then*

$$\mathrm{rb}(\mathbb{Z}_{2^\alpha}, \mathrm{eq}(a_1, a_2, a_3, b)) = \alpha + 2.$$

Proof. The proof follows by induction on α. The base case $\alpha = 1$ holds by convention. Note that since $a_1a_2a_3 \in \mathbb{Z}_2^*$, then $a_i \equiv 1 \bmod 2$ for all i and, by Lemma 1, it can be assumed that $b = 0$. Let $0 \le \alpha \in \mathbb{Z}$ and assume the statement holds for $\alpha \ge 1$. The following cases show the statement is true for $\alpha + 1$. Let c be an exact $\alpha + 2$ coloring of $\mathbb{Z}_{2^\alpha}$ and define $R_i = \{x \in \mathbb{Z}_{2^\alpha} \mid x \equiv i \bmod 2\}$ and $P_i = \{c(x) \mid x \in R_i\}$.

If at least $\alpha + 1$ colors appear in P_0, then, by the inductive hypothesis, $\mathbb{Z}_{2^\alpha}$ contains a rainbow solution to $eq = \mathrm{eq}(a_1, a_2, a_3, 0)$ because $(\mathbb{Z}_{2^{\alpha-1}}, 2^{\alpha-1}, eq)$ is solomorphic to $(R_0, 2^\alpha, eq)$. If at most α colors appear in P_0, then there exist two colors, red and blue, that appear in P_1. Let s_1 and s_2 be two elements in R_1 such that $c(\{s_1, s_2\}) = \{red, blue\}$. Since $a_1 \equiv a_2 \equiv a_3 \equiv s_1 \equiv s_2 \equiv 1 \bmod 2$, the element $s_3 \in \mathbb{Z}_{2^\alpha}$ satisfying $a_1s_1 + a_2s_2 + a_3s_3 \equiv 0 \bmod 2^\alpha$ is such that $s_3 \in R_0$. This means $\{s_1, s_2, s_3\}$ is a rainbow solution to $\mathrm{eq}(a_1, a_2, a_3, 0)$ since $c(s_3) \notin \{red, blue\}$. Thus $\mathrm{rb}(\mathbb{Z}_{2^\alpha}, \mathrm{eq}(a_1, a_2, a_3, 0)) \le \alpha + 2$. To obtain a lower bound, color P_0 with a rainbow-free coloring of $\mathbb{Z}_{2^{\alpha-1}}$ that has α colors and color P_1 with the $(\alpha + 1)^{st}$ color. This coloring has no rainbow solutions since every solution has exactly 1 or 3 elements from R_0. □

Theorem 7 and Corollary 3 establish the lower bound for the rainbow number of $\mathbb{Z}_n$.

Theorem 7. *Let $2 \le t \in \mathbb{Z}$. If $a_1a_2a_3 \in \mathbb{Z}_p^*$ and $\mathbb{Z}_t$ has a rainbow-free, exact r_t-coloring for $eq = \mathrm{eq}(a_1, a_2, a_3, 0)$ where 0 is uniquely colored, then there exists a rainbow-free, exact $(\mathrm{rb}(\mathbb{Z}_p, eq) + r_t - 2)$-coloring of $\mathbb{Z}_{pt}$ with 0 uniquely colored.*

Proof. Since $\mathrm{rb}(\mathbb{Z}_p, eq) \le 4$, every rainbow-free coloring of $\mathbb{Z}_p$ for eq uses at most three colors. Define $r_p = \mathrm{rb}(\mathbb{Z}_p, eq) - 1$. Note there exists an exact rainbow-free

r_p-coloring of $\mathbb{Z}_p$ for eq where 0 is the only element in its color class. If $r_p = 2$ or $p = 3$, the r_p-coloring is obvious. If $r_p = 3$, $p \geq 5$, and $a_i \neq a_j$, for some $i \neq j$, such a coloring exists by Theorem 2. Lastly, if $a_1 = a_2 = a_3$ the coloring is described in the proof of Theorem 5.

Let c_p be a rainbow-free, exact r_p-coloring of $\mathbb{Z}_p$ for eq such that 0 is colored uniquely and c_t be a rainbow-free exact r_t-coloring of $\mathbb{Z}_t$ where 0 is colored uniquely. Define an exact $(r_p + r_t - 1)$-coloring of $\mathbb{Z}_{pt}$ by

$$c(x) = \begin{cases} 0 & \text{if } x = 0, \\ c_p(x \bmod p) & \text{if } x \neq 0 \bmod p, \\ (r_p - 1) + c_t\left(\frac{x}{p} \bmod t\right) & \text{if } x = 0 \bmod p \text{ and } x \neq 0. \end{cases}$$

Notice that $0 \in \mathbb{Z}_{pt}$ is the only element in its color class with respect to the coloring c. Let (s_1, s_2, s_3) be a solution in $\mathbb{Z}_{pt}$ to eq. Since $a_1a_2a_3 \in \mathbb{Z}_p^*$, p cannot divide exactly two of s_1, s_2, and s_3, so either p divides each of s_1, s_2, and s_3 or p divides at most one of s_1, s_2, and s_3.

If p divides each of s_1, s_2, and s_3, then (s_1, s_2, s_3) is not a rainbow solution under the coloring c since it is not a rainbow solution under c_t. If p divides at most one of s_1, s_2, and s_3, then (s_1, s_2, s_3) is not a rainbow solution under the coloring c since it is not a rainbow solution under c_p and 0 is the unique element in its color class under c_p. Therefore, c is a rainbow-free, exact $(\text{rb}(\mathbb{Z}_p, eq) + r_t - 2)$-coloring of $\mathbb{Z}_{pt}$. □

Corollary 3. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_\ell^{\alpha_\ell}$, p_x prime for $1 \leq x \leq \ell$, and $a_1a_2a_3 \in \mathbb{Z}_n^*$, then*

$$2 + \sum_{i=1}^{\ell} \left[\alpha_i(\text{rb}(\mathbb{Z}_{p_i}, \text{eq}(a_1, a_2, a_3, 0) - 2))\right] \leq \text{rb}(\mathbb{Z}_n, \text{eq}(a_1, a_2, a_3, 0)).$$

Proof. Define $eq = \text{eq}(a_1, a_2, a_3, 0)$. This proof is inductive on the sum of the exponents in the prime factorization of n. When n is prime, proceeding as in the first part of the proof of Theorem 7, there exists an exact rainbow-free r_n-coloring of $\mathbb{Z}_n$ where 0 is colored uniquely. Thus, the inequality holds when n is prime.

Assume that for $k = \frac{n}{p_j} = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_j^{\alpha_j - 1} \dots p_\ell^{\alpha_\ell} < n$, there is a rainbow-free, exact r_k-coloring of $\mathbb{Z}_k$ such that 0 is colored uniquely with

$$r_k = 1 + (\alpha_j - 1)(\text{rb}(\mathbb{Z}_{p_j}, eq) - 2) + \sum_{\substack{i=1 \\ i \neq j}}^{\ell} \alpha_i \left[\text{rb}(\mathbb{Z}_{p_i}, eq) - 2\right].$$

Applying Theorem 7 gives a rainbow-free, exact $(\text{rb}(\mathbb{Z}_{p_j}, eq) + r_k - 2)$-coloring of $\mathbb{Z}_{kp_j}$ where 0 is colored uniquely. Therefore,

$$\begin{aligned} \text{rb}(\mathbb{Z}_n, eq) &\geq \text{rb}(\mathbb{Z}_{p_j}, eq) + r_k - 1 \\ &\geq \text{rb}(\mathbb{Z}_{p_j}, eq) + (\alpha_j - 1)(\text{rb}(\mathbb{Z}_{p_j}, eq) - 2) + \sum_{\substack{i=1 \\ i \neq j}}^{\ell} \alpha_i \left[\text{rb}(\mathbb{Z}_{p_i}, eq) - 2\right] \\ &= 2 + \sum_{i=1}^{\ell} \alpha_i \left[\text{rb}(\mathbb{Z}_{p_i}, eq) - 2\right]. \end{aligned}$$

□

Corollary 4 generalizes Corollary 3 to $\text{eq}(a_1, a_2, a_3, b)$ using Lemma 1.

Corollary 4. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_\ell^{\alpha_\ell}$, p_k prime for $1 \leq k \leq \ell$, and $a_1 + a_2 + a_3, a_1a_2a_3 \in \mathbb{Z}_n^*$, then*

$$2 + \sum_{i=1}^{\ell} \left[\alpha_i(\text{rb}(\mathbb{Z}_{p_i}, \text{eq}(a_1, a_2, a_3, b) - 2))\right] \leq \text{rb}(\mathbb{Z}_n, \text{eq}(a_1, a_2, a_3, b)).$$

The upper bound will now be established. Suppose c is a coloring of $\mathbb{Z}_{ut}$. The remainder of this section uses residue classes $R_i = \{z \in \mathbb{Z}_{ut} \mid z \equiv i \bmod u\}$ and color palettes $P_i = \{c(z) \mid z \in R_i\}$ that were mentioned in the proof of Theorem 6.

Lemma 6. *Let $3 \leq t, u \in \mathbb{Z}$, $a_3 \in \mathbb{Z}_u^*$, and (s_1, s_2, s_3) and (s_1', s_2', s_3') be solutions in $\mathbb{Z}_{ut}$ to $\text{eq}(a_1, a_2, a_3, b)$. If $s_1' \in R_{s_1}$ and $s_2' \in R_{s_2}$, then $s_3' \in R_{s_3}$.*

Proof. Since $a_1s_1' + a_2s_2' + a_3s_3' = b \bmod ut$ implies $a_1s_1' + a_2s_2' + a_3s_3' = b \bmod u$, solving for s_3' over $\mathbb{Z}_u$ gives $s_3' = a_3^{-1}(b - (a_1s_1 + a_2s_2)) = a_3^{-1}(a_3s_3) \bmod u$. Hence, $s_3' \in R_{s_3}$. □

A similar argument to the one used in Lemma 6 can be used for $a_1, a_2 \in \mathbb{Z}_t^*$ and solving for s_1' and s_2' instead.

Lemma 7. *If $k, n \in \mathbb{Z}$ are such that $3 \leq n$, then*

$$\text{rb}(\mathbb{Z}_n, \text{eq}(a_1, a_2, a_3, b)) = \text{rb}(\mathbb{Z}_n, \text{eq}(a_1, a_2, a_3, b + (a_1 + a_2 + a_3)k)).$$

Proof. Let $eq = \text{eq}(a_1, a_2, a_3, b)$, $a = a_1 + a_2 + a_3$, $eq' = \text{eq}(a_1, a_2, a_3, b + ak)$ and c be an exact r-coloring of $\mathbb{Z}_n$ for eq. If (s_1, s_2, s_3) is a solution in $\mathbb{Z}_n$ to eq, then $a_1s_1 + a_2s_2 + a_3s_3 + (a_1 + a_2 + a_3)k = b + ak$ and $(s_1 + k, s_2 + k, s_3 + k)$ is a solution in $\mathbb{Z}_n$ to eq'. Define $c_k : \mathbb{Z}_n \to [r]$ by $c_k(x) = c(x + k \bmod n)$. Thus, (s_1, s_2, s_3) is a rainbow solution to eq with respect to c if and only if $(s_1 + k, s_2 + k, s_3 + k)$ is a rainbow solution to eq' with respect to c_k. Since c_k is a translation of the coloring c, $\text{rb}(\mathbb{Z}_n, eq) = \text{rb}(\mathbb{Z}_n, eq')$. □

Lemma 8. *Let* $2 \leq t \in \mathbb{Z}$ *and* c *be a rainbow-free coloring of* $\mathbb{Z}_{ut}$ *for* $\mathrm{eq}(a_1, a_2, a_3, b)$ *and* $a_1a_2a_3 \in \mathbb{Z}^*_{ut}$ *that does not use color yellow. If there exists* $j \in \mathbb{Z}_t$ *such that for all* $i \in \mathbb{Z}_t$, $|P_i \backslash P_j| \leq 1$, *then the coloring of* $\mathbb{Z}_t$ *given by*

$$\hat{c}(i) = \begin{cases} \text{yellow} & P_i \subseteq P_j, \\ P_i \setminus P_j & \text{otherwise}, \end{cases}$$

is well-defined and rainbow-free.

Proof. Since $|P_i \backslash P_j| \leq 1$, $\hat{c}$ is well-defined. Let $eq = \mathrm{eq}(a_1, a_2, a_3, b)$ and assume that (s_1, s_2, s_3) is a rainbow solution of eq in $\mathbb{Z}_t$ with respect to $\hat{c}$. Since (s_1, s_2, s_3) is a rainbow solution, without loss of generality, $\hat{c}(s_1) = red$ and $\hat{c}(s_2) = blue$. Thus, there exist $\alpha \in R_{s_1}$, $\delta \in R_{s_2}$ such that $c(\alpha) = red$ and $c(\delta) = blue$. Therefore, (α, δ, γ) is a solution to eq in $\mathbb{Z}_{ut}$ for some $\gamma \in R_{s_3}$. Note that $\hat{c}(s_3)$ is not *red* or *blue*. However, $P_{s_3} \setminus \{\hat{c}(s_3)\} \subseteq P_j$. Therefore $c(\gamma)$ is not *red* or *blue* so (α, δ, γ) is a rainbow solution to eq in $\mathbb{Z}_{ut}$ with respect to c, a contradiction. □

Lemma 9. *If* c *is a rainbow-free coloring of* $\mathbb{Z}_{ut}$ *for* $\mathrm{eq}(a_1, a_2, a_3, b)$, $a_1a_2a_3 \in \mathbb{Z}^*_{ut}$ *and* $|P_0| \geq |P_i|$ *for* $0 \leq i \leq u-1$, *then* $|P_i \backslash P_0| \leq 1$.

Proof. Assume $|P_i \backslash P_0| \geq 2$ for some $1 \leq i \leq u-1$ and let $red, blue \in P_i \backslash P_0$. Let $j \in \mathbb{Z}_{ut}$ be such that $a_1 i + a_2 0 + a_3 j = b$. Suppose there is an $\alpha \in R_j$ such that $c(\alpha) \notin P_0$. Choose $\beta \in R_i$ such that $c(\beta) \in \{red, blue\} \backslash \{c(\alpha)\}$. Now there exists $\gamma \in R_0$ such that $\{\beta, \gamma, \alpha\}$ is a rainbow solution to $\mathrm{eq}(a_1, a_2, a_3, b)$, a contradiction. Therefore, $P_j \subseteq P_0$. A similar argument gives that $P_0 \subseteq P_j$, so $P_0 = P_j$.

Since $|P_0|$ is maximum there must exist two colors, both in P_0 and P_j, that are not in P_i. Let $yellow, green \in P_0 \backslash P_i$. Choosing a *yellow* element in R_0 and a *green* element in R_j and solving for the appropriate element in R_i will give a rainbow solution, which is a contradiction. Therefore, $|P_i \backslash P_0| \leq 1$ for all $0 \leq i \leq u-1$. □

Using an inductive argument with the following Lemma 10, similar to the argument made in Corollary 3, and Theorem 6 gives Corollary 7.

Lemma 10. *If* $2 \leq t \in \mathbb{Z}$, $3 \leq p$ *prime,* $a_1a_2a_3 \in \mathbb{Z}^*_{pt}$, *then*

$$\mathrm{rb}(\mathbb{Z}_{pt}, \mathrm{eq}(a_1, a_2, a_3, b)) \leq \mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a_1, a_2, a_3, b_2)) + \mathrm{rb}(\mathbb{Z}_t, \mathrm{eq}(a_1, a_2, a_3, b_1)) - 2,$$

for some $b_1, b_2 \in \mathbb{Z}$.

Proof. Let $eq = \mathrm{eq}(a_1, a_2, a_3, b)$ and c be a rainbow-free exact $(\mathrm{rb}(\mathbb{Z}_{pt}, eq) - 1)$-coloring of $\mathbb{Z}_{pt}$. Create the coloring c_k from Lemma 7 to get $|P_0| \geq |P_i|$ for $1 \leq i \leq p-1$, where P_i are defined with respect to coloring c_k. This implies that $(\mathbb{Z}_{pt}, pt, eq)$ is solomorphic to $(\mathbb{Z}_{pt}, pt, eq_1)$ with $eq_1 = \mathrm{eq}(a_1, a_2, a_3, b_1)$ for some $b_1 \in \mathbb{Z}_{pt}$.

Since c_k is rainbow-free and $|P_0| \geq |P_i|$ for all i, Lemma 8 and Lemma 9 give a well-defined coloring $\hat{c}$ using P_0. If $\hat{c}$ has a rainbow solution, then c_k has a rainbow solution, so $\hat{c}$ must be rainbow-free. However, since $\hat{c}$ is coloring $\mathbb{Z}_t$, $\hat{c}$ uses at most $\text{rb}(\mathbb{Z}_t, eq_1) - 1$ colors which contributes at most $\text{rb}(\mathbb{Z}_t, eq_1) - 2$ colors to c because, without loss of generality, *yellow* is not a color from c. Furthermore, (R_0, pt, eq_1) is solomorphic to $(\mathbb{Z}_p, p, eq_2)$ so $|P_0| \leq \text{rb}(\mathbb{Z}_p, eq_2) - 1$, where $eq_2 = \text{eq}(a_1, a_2, a_3, b_2)$ for some $b_2 \in \mathbb{Z}_p$. In order for $\hat{c}$ to be rainbow-free, c_k must use at most $\text{rb}(\mathbb{Z}_p, eq_2) + \text{rb}(\mathbb{Z}_t, eq_1) - 3$ colors. This implies $\text{rb}(\mathbb{Z}_{pt}, \text{eq}(a_1, a_2, a_3, b)) - 1 \leq \text{rb}(\mathbb{Z}_p, eq_2) + \text{rb}(\mathbb{Z}_t, eq_1) - 3$. □

Corollary 5. *If $n = p_1 p_2 \cdots p_\ell$, $3 \leq p_k$ prime for $1 \leq k \leq \ell$, $a_1 a_2 a_3 \in \mathbb{Z}_n^*$, and $eq = \text{eq}(a_1, a_2, a_3, b)$, then*

$$\text{rb}(\mathbb{Z}_n, eq) \leq 2 + \sum_{k=1}^{\ell} \left[\text{rb}(\mathbb{Z}_{p_k}, eq_k) - 2\right],$$

where $eq_k = \text{eq}(a_1, a_2, a_3, b_k)$ for some $b_k \in \mathbb{Z}$.

Corollary 6. *Let $n = p_1 p_2 \cdots p_\ell$, p_k prime for $1 \leq k \leq \ell$, $a_1 a_2 a_3 \in \mathbb{Z}_n^*$, where $a_1 + a_2 + a_3 \in \mathbb{Z}_3^*$ if $3 \mid n$. Let $eq = \text{eq}(a_1, a_2, a_3, 0)$, then*

$$\text{rb}(\mathbb{Z}_n, eq) \leq 2 + \sum_{k=1}^{\ell} \left[\text{rb}(\mathbb{Z}_{p_k}, eq) - 2\right].$$

Proof. By Theorems 4, 5, and 6, if $p \neq 3$, then $\text{rb}(\mathbb{Z}_p, \text{eq}(a_1, a_2, a_3, b)) \leq \text{rb}(\mathbb{Z}_p, eq)$ for all b. If $a_1 + a_2 + a_3 \in \mathbb{Z}_3^*$, Proposition 1 gives $\text{rb}(\mathbb{Z}_3, \text{eq}(a_1, a_2, a_3, b)) \leq \text{rb}(\mathbb{Z}_3, eq)$ for all b. The result follows by Corollary 5. □

Note that the assumption $a_1 + a_2 + a_3 \in \mathbb{Z}_3^*$ is necessary when $3 \mid n$. For example, $\text{rb}(\mathbb{Z}_3, \text{eq}(1, 1, 1, 0)) = 3$ and $\text{rb}(\mathbb{Z}_9, \text{eq}(1, 1, 1, 0)) = 5$. In particular, $\mathbb{Z}_9$ has the rainbow-free coloring $c : \mathbb{Z}_9 \to [4]$ given by $c(2) = 2$, $c(5) = 3$, $c(8) = 4$, and $c(x) = 1$ else.

Corollary 6 and Lemma 1 combine to give Corollary 7.

Corollary 7. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$, p_k prime for $1 \leq k \leq \ell$, $a_1 + a_2 + a_3, a_1 a_2 a_3 \in \mathbb{Z}_n^*$ and $eq = \text{eq}(a_1, a_2, a_3, b)$, then*

$$\text{rb}(\mathbb{Z}_n, eq) \leq 2 + \sum_{k=1}^{\ell} \left[\alpha_k (\text{rb}(\mathbb{Z}_{p_k}, eq) - 2)\right].$$

Finally, Corollaries 3, 4 and 7 combine to give Theorem 8.

Theorem 8. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$, with p_k prime for $1 \leq k \leq \ell$, and $a_1 a_2 a_3 \in \mathbb{Z}_n^*$. If one of the following holds:*

1) $b \neq 0$ and $a_1 + a_2 + a_3 \in \mathbb{Z}_n^*$, *2) $b = 0$ and $3 \nmid n$, or*

3) $b = 0$, $3 \mid n$, and $a_1 + a_2 + a_3 \in \mathbb{Z}_3^*$,

then

$$\mathrm{rb}(\mathbb{Z}_n, \mathrm{eq}(a_1, a_2, a_3, b)) = 2 + \sum_{k=1}^{\ell} [\alpha_k(\mathrm{rb}(\mathbb{Z}_{p_k}, \mathrm{eq}(a_1, a_2, a_3, b) - 2)] .$$

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