GENERALIZED EULER-GENOCCHI POLYNOMIALS AND LUCAS NUMBERS

Robert Frontczak[1]
Landesbank Baden-Württemberg (LBBW), Stuttgart, Germany
robert.frontczak@lbbw.de

Živorad Tomovski
University St. Cyril and Methodius Faculty of Natural Sciences and Mathematics Republic of North Macedonia
tomovski@pmf.ukim.edu.mk
and
Department of Mathematics, University of Ostrava, Ostrava, The Czech Republic

Received: 8/5/19, Revised: 4/8/20, Accepted: 6/15/20, Published: 7/24/20

Abstract

The family of Euler-Genocchi polynomials has been studied recently. We use its generating function to define an extension of this family to generalized Euler-Genocchi polynomials of order m. This family of polynomials contains the generalized Euler and generalized Genocchi polynomials as special members. We derive some combinatorial properties of these polynomials. Moreover, we prove two combinatorial identities involving generalized Euler-Genocchi polynomials and products of Lucas numbers. Some special cases are stated and compared to existing results. Finally, we define the generalized Bernoulli polynomials of order r and m and connect them combinatorially to Fibonacci numbers.

1. Introduction

For $r \in \mathbb{N} \cup \{0\}$, Belbachir et al. ([1, 2]) recently introduced and studied a new class of polynomials called Euler-Genocchi polynomials. These polynomials, denoted by the authors by $\{A_n^{(r)}(x)\}_{n\geq 0}$, are defined by means of the generating function

$$\sum_{n=0}^{\infty} A_n^{(r)}(x)\frac{z^n}{n!} = \frac{2z^r}{e^z+1}e^{xz} \qquad (|z| < \pi), \tag{1.1}$$

[1]Corresponding author; Statements and conclusions made in this article by R. F. are entirely those of the author. They do not necessarily reflect the views of LBBW.

with $A_j^{(r)}(x) = 0$ for $j < r$. The numbers $A_n^{(r)}(0) = A_n^{(r)}$ are called the Euler-Genocchi numbers of order r. Euler-Genocchi polynomials belong to the family of Appell polynomials. They contain Euler polynomials $E_n(x)$ and Genocchi polynomials $G_n(x)$ as special members. To be more precise, we have the relations

$$E_n(x) = A_n^{(0)}(x) \qquad \text{and} \qquad G_n(x) = A_n^{(1)}(x),$$

where, as usual, we have the defining equations

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z+1}e^{xz} \qquad (|z| < \pi) \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} G_n(x)\frac{z^n}{n!} = \frac{2z}{e^z+1}e^{xz} \qquad (|z| < \pi). \tag{1.3}$$

Belbachir et al. ([1, 2]) derive many interesting properties of these polynomials, such as, linear recurrences and difference equations. They also evaluate alternating power sums in terms of $A_n^{(r)}(x)$ and state an expression for sums of the Stirling numbers of the second kind in terms of these polynomials.

In this article, we move one step further and extend the definition of Euler-Genocchi polynomials to generalized Euler-Genocchi polynomials of order m, where $m \geq 0$ is an integer. The new family of polynomials contains the generalized Euler and generalized Genocchi polynomials as special members. We derive some combinatorial properties of these polynomials. Moreover, in Section 3 we focus on connections to the Lucas sequence. We prove two combinatorial identities involving generalized Euler-Genocchi polynomials and products of Lucas numbers. Some special sums are studied in detail and compared to existing results. Finally, in Section 4 we define the generalized Bernoulli polynomials of order r and m, state some basic properties, and prove connections to the Fibonacci sequence.

2. The Generalized Euler-Genocchi Polynomials

We start by generalizing the definition of Euler-Genocchi polynomials to a wider class of polynomials.

Definition 1. Let r and m be integers with $r \geq 0$ and $m \geq 0$. We define the *generalized Euler-Genocchi polynomials of order m* by the generating function

$$A(r,m;x,z) = \sum_{n=0}^{\infty} A_n^{(r,m)}(x)\frac{z^n}{n!} = \left(\frac{2z^r}{e^z+1}\right)^m e^{xz} \qquad (|z| < \pi), \tag{2.1}$$

with $A_n^{(r,0)}(x) = x^n$ for all $r \geq 0$ and $A_j^{(r,m)}(x) = 0$ for all $j < rm$. We call the numbers $A_n^{(r,m)}(0) = A_n^{(r,m)}$ *the generalized Euler-Genocchi numbers of order* m.

It is obvious that $A_n^{(r,1)}(x) = A_n^{(r)}(x)$ and $A_0^{(r,m)}(x) = 0$ for $r, m \geq 1$. Furthermore, we note that $A_n^{(0,m)}(x) = E_n^{(m)}(x)$ and $A_n^{(1,m)}(x) = G_n^{(m)}(x)$ are the generalized Euler and Genocchi polynomials, respectively. These polynomials have the generating functions (see [4] and [5] for more material on this and related topics)

$$\sum_{n=0}^{\infty} E_n^{(m)}(x)\frac{z^n}{n!} = \Big(\frac{2}{e^z+1}\Big)^m e^{xz} \qquad (|z| < \pi) \tag{2.2}$$

and

$$\sum_{n=0}^{\infty} G_n^{(m)}(x)\frac{z^n}{n!} = \Big(\frac{2z}{e^z+1}\Big)^m e^{xz} \qquad (|z| < \pi). \tag{2.3}$$

We are now going to present some properties of the family $A_n^{(r,m)}(x)$. Throughout this section we use the convention that $\binom{n}{k} = 0$ for $k \notin \{0, \ldots, n\}$. The first two properties are more or less obvious.

Lemma 1. *Let n, m and r be three integers such that $n \geq rm$. Then it holds that*

$$A_n^{(r,m)}(x) = \sum_{k=0}^{n} \binom{n}{k} A_k^{(r,m)} x^{n-k}. \tag{2.4}$$

Proof. From Definition 1 we instantly have

$$\sum_{n=0}^{\infty} A_n^{(r,m)}(x)\frac{z^n}{n!} = A(r,m;x,z) = A(r,m;0,z)e^{xz} = \Big(\sum_{n=0}^{\infty} A_n^{(r,m)}\frac{z^n}{n!}\Big)\Big(\sum_{j=0}^{\infty} x^j\frac{z^j}{j!}\Big),$$

and the formula follows using Cauchy's rule. □

Lemma 2. *For $r, m \geq 0$ and $n \geq 1$, we have the relation*

$$\frac{d}{dx}A_n^{(r,m)}(x) = nA_{n-1}^{(r,m)}(x). \tag{2.5}$$

Proof. Simple differentiation of $A(r,m;x,z)$ with respect to x gives the result. □

The next property is a reciprocal relation that also exists for generalized Euler and Genocchi (and Bernoulli) polynomials, and that will be used in the next section.

Lemma 3. *For $m \geq 0$, the generalized Euler-Genocchi polynomials satisfy the following relation:*

$$A_n^{(r,m)}(m-x) = (-1)^{n-rm}A_n^{(r,m)}(x). \tag{2.6}$$

Especially,

$$A_n^{(r,m)}(m) = (-1)^{n-rm}A_n^{(r,m)} \quad \textit{and} \quad A_n^{(r,m)}\Big(\frac{m}{2}\Big) = (-1)^{n-rm}A_n^{(r,m)}\Big(\frac{m}{2}\Big). \tag{2.7}$$

Proof. From Definition 1 it follows that

$$\begin{aligned} A(r,m;x,-z) &= \sum_{n=0}^{\infty} A_n^{(r,m)}(x)(-1)^n \frac{z^n}{n!} \\ &= \Big(\frac{2(-1)^r z^r}{e^{-z}+1}\Big)^m e^{-xz} \\ &= (-1)^{rm}\Big(\frac{2z^r}{e^z+1}\Big)^m e^{(m-x)z} \\ &= (-1)^{rm} A(r,m;m-x,z). \end{aligned}$$

□

The next two theorems contain addition formulas for $A_n^{(r,m)}(x)$. The first one generalizes Theorem 2.1 of [2].

Theorem 1. *For $r \geq 0$ and $m \geq 1$ it holds that*

$$A_n^{(r,m)}(x+1) + A_n^{(r,m)}(x) = 2r! \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{r} A_k^{(r,m-1)} x^{n-k-r}. \tag{2.8}$$

Proof. Once more we work with the generating function. We have

$$\begin{aligned} \sum_{n=0}^{\infty}\Big(A_n^{(r,m)}(x+1) + A_n^{(r,m)}(x)\Big)\frac{z^n}{n!} &= \Big(\frac{2z^r}{e^z+1}\Big)^m e^{(x+1)z} + \Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz} \\ &= 2\Big(\frac{2z^r}{e^z+1}\Big)^{m-1}(z^r e^{xz}) \\ &= 2\Big(\sum_{n=0}^{\infty} A_n^{(r,m-1)}\frac{z^n}{n!}\Big)\Big(\sum_{j=0}^{\infty} r!\binom{j}{r} x^{j-r}\frac{z^j}{j!}\Big). \end{aligned}$$

The application of Cauchy's rule completes the proof. □

It is worthwhile to remark that, since

$$A_n^{(r,m)}(x+1) = \sum_{k=0}^{n}\binom{n}{k} A_k^{(r,m)}(x), \tag{2.9}$$

an equivalent version of equation (2.8) is

$$\sum_{k=0}^{n}\binom{n}{k} A_k^{(r,m)}(x) + A_n^{(r,m)}(x) = 2r!\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{r} A_k^{(r,m-1)} x^{n-k-r}. \tag{2.10}$$

When $m = 1$, then the above relation becomes

$$\sum_{k=0}^{n}\binom{n}{k} A_k^{(r,1)}(x) + A_n^{(r,1)}(x) = 2r!\binom{n}{r} x^{n-r}, \tag{2.11}$$

which coincides with the relation presented in [2] in a slightly different manner using the falling factorial. When $r=0$, then by (2.4), the above relation gives the identity

$$\sum_{k=0}^{n}\binom{n}{k}E_k^{(m)}(x)+E_n^{(m)}(x)=2E_n^{(m-1)}(x), \tag{2.12}$$

which has been proved by Srivastava and Pintér ([8]). When $r=1$, then we obtain the corresponding identity for generalized Genocchi polynomials as

$$\sum_{k=0}^{n}\binom{n}{k}G_k^{(m)}(x)+G_n^{(m)}(x)=2nG_{n-1}^{(m-1)}(x), \tag{2.13}$$

where we have used (2.4) and (2.5).

Theorem 2. *For $r\geq 0$ and $m\geq 1$ it holds that*

$$A_n^{(r,m)}(x)+A_n^{(r,m)}\Big(\frac{x}{2}\Big)=2^{1-2n}\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n}\binom{n}{k}2^{2k}A_k^{(r,m)}\Big(\frac{3x}{4}\Big)x^{n-k}. \tag{2.14}$$

Proof. Using the definition of the hyperbolic cosine, $\cosh(z)=(e^z+e^{-z})/2$, we see that

$$e^{xz/2}+1=2e^{xz/4}\cosh(xz/4).$$

Hence,

$$\begin{aligned}\Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz}+\Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz/2} &= \Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz/2}\Big(e^{xz/2}+1\Big)\\ &= \Big(\frac{2z^r}{e^z+1}\Big)^m e^{3xz/4}\Big(2\cosh(xz/4)\Big),\end{aligned}$$

or

$$\sum_{n=0}^{\infty}\Big(A_n^{(r,m)}(x)+A_n^{(r,m)}\Big(\frac{x}{2}\Big)\Big)\frac{z^n}{n!}=\Big(\sum_{n=0}^{\infty}A_n^{(r,m)}\Big(\frac{3x}{4}\Big)\frac{z^n}{n!}\Big)\Big(2\cosh(xz/4)\Big).$$

The addition formula follows from using Cauchy's rule and the power series of the hyperbolic cosine,

$$\cosh(z)=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}.$$

□

Equation (2.14) also holds for $m=0$. For $(r,m)=(0,m)$ and $(r,m)=(1,m)$ we get the special results for generalized Euler and generalized Genocchi polynomials, respectively:

$$E_n^{(m)}(x)+E_n^{(m)}\Big(\frac{x}{2}\Big)=2^{1-2n}\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n}\binom{n}{k}2^{2k}E_k^{(m)}\Big(\frac{3x}{4}\Big)x^{n-k} \tag{2.15}$$

and

$$G_n^{(m)}(x)+G_n^{(m)}\Big(\frac{x}{2}\Big)=2^{1-2n}\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n}\binom{n}{k}2^{2k}G_k^{(m)}\Big(\frac{3x}{4}\Big)x^{n-k}. \tag{2.16}$$

Our next result is a Raabe-like formula for generalized Euler-Genocchi polynomials.

Theorem 3. *Let m, r and q be three integers with $m\geq 1$, $r\geq 0$ and q odd. Then we have the following identity:*

$$\sum_{k=0}^{q-1}(-1)^k A_n^{(r,m)}\Big(\frac{x+k}{q}\Big)=\sum_{j=0}^{n}\binom{n}{j}q^{r-j}A_j^{(r,1)}(x)A_{n-j}^{(r,m-1)}. \tag{2.17}$$

Proof. From Definition 1, it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{q-1}(-1)^k A_n^{(r,m)}\Big(\frac{x+k}{q}\Big)\frac{z^n}{n!} &= \sum_{k=0}^{q-1}(-1)^k\sum_{n=0}^{\infty}A_n^{(r,m)}\Big(\frac{x+k}{q}\Big)\frac{z^n}{n!}\\
&= \sum_{k=0}^{q-1}(-1)^k\Big(\frac{2z^r}{e^z+1}\Big)^m e^{(x+k)z/q}\\
&= \Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz/q}\frac{e^z+1}{e^{z/q}+1}\\
&= q^r\Big(\frac{2z^r}{e^z+1}\Big)^{m-1}\frac{2(z/q)^r}{e^{z/q}+1}e^{xz/q}\\
&= \Big(\sum_{n=0}^{\infty}A_n^{(r,m-1)}\frac{z^n}{n!}\Big)\Big(\sum_{n=0}^{\infty}q^{r-n}A_n^{(r,1)}(x)\frac{z^n}{n!}\Big).
\end{aligned}$$

Applying Cauchy's rule, the proof is completed. ☐

When $(r,m)=(r,1)$, equation (2.17) becomes

$$\sum_{k=0}^{q-1}(-1)^k A_n^{(r,1)}\Big(\frac{x+k}{q}\Big)=q^{r-n}A_n^{(r,1)}(x), \tag{2.18}$$

which appears in [2].

We conclude this section with two recurrence formulas for $A_n^{(r,m)}(x)$.

Theorem 4. *For $r\geq 0$ and $m\geq 1$ the polynomials $A_n^{(r,m)}(x)$ satisfy the following recurrence relations:*

$$\Big(1-\frac{rm}{n+1}\Big)A_{n+1}^{(r,m)}(x)=\Big(x-\frac{m}{2}\Big)A_n^{(r,m)}(x)-\frac{m}{2}\sum_{k=0}^{n-1}\binom{n}{k}A_k^{(r,m)}(x)E_{n-k}(1), \tag{2.19}$$

and

$$\Big(1-\frac{rm}{n+1}\Big)A_{n+1}^{(r,m)}(x)=(x-m)A_n^{(r,m)}(x)+\frac{m}{2}\sum_{k=0}^{n}\frac{\binom{n}{k}}{r!\binom{k+r}{r}}A_{k+r}^{(r,1)}A_{n-k}^{(r,m)}(x). \quad (2.20)$$

Proof. Differentiating both sides of (2.1) yields

$$\frac{d}{dz}\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!}=\sum_{n=0}^{\infty}A_{n+1}^{(r,m)}(x)\frac{z^n}{n!},$$

and

$$\begin{aligned}\frac{d}{dz}\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!} &= \frac{rm}{z}\Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz}+x\Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz}\\ &\quad -m\Big(\frac{2z^r}{e^z+1}\Big)^m e^{(x+1)z}\frac{1}{e^z+1}.\end{aligned}$$

To get (2.19), write

$$m\Big(\frac{2z^r}{e^z+1}\Big)^m e^{(x+1)z}\frac{1}{e^z+1}=\frac{m}{2}\Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz}\frac{2e^z}{e^z+1}.$$

This gives

$$\begin{aligned}\frac{d}{dz}\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!} &= rm\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^{n-1}}{n!}+x\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!}\\ &\quad -\frac{m}{2}\Big(\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!}\Big)\Big(\sum_{n=0}^{\infty}E_n(1)\frac{z^n}{n!}\Big)\\ &= rm\sum_{n=0}^{\infty}\frac{1}{n+1}A_{n+1}^{(r,m)}(x)\frac{z^n}{n!}+x\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!}\\ &\quad -\frac{m}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}A_k^{(r,m)}(x)E_{n-k}(1)\frac{z^n}{n!},\end{aligned}$$

and (2.19) follows by comparing the coefficients of z^n and rearrangement. To prove (2.20) we need the expression

$$\begin{aligned}\frac{d}{dz}\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!} &= \frac{rm}{z}\Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz}+(x-m)\Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz}\\ &\quad +m\Big(\frac{2z^r}{e^z+1}\Big)^m e^{xz}\frac{1}{e^z+1}.\end{aligned}$$

Since,

$$\frac{1}{e^z+1}=\frac{1}{2}\sum_{n=0}^{\infty}A_n^{(r,1)}\frac{z^{n-r}}{n!},$$

we can write

$$\begin{aligned}
\frac{d}{dz}\sum_{n=0}^{\infty} A_n^{(r,m)}(x)\frac{z^n}{n!} &= rm\sum_{n=0}^{\infty}\frac{1}{n+1}A_{n+1}^{(r,m)}(x)\frac{z^n}{n!} + (x-m)\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!} \\
&\quad + \frac{m}{2}\Big(\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!}\Big)\Big(\sum_{n=0}^{\infty}A_n^{(r,1)}\frac{z^{n-r}}{n!}\Big) \\
&= rm\sum_{n=0}^{\infty}\frac{1}{n+1}A_{n+1}^{(r,m)}(x)\frac{z^n}{n!} + (x-m)\sum_{n=0}^{\infty}A_n^{(r,m)}(x)\frac{z^n}{n!} \\
&\quad + \frac{m}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\binom{n}{k}}{r!\binom{k+r}{k}}A_{k+r}^{(r,1)}A_{n-k}^{(r,m)}(x)\frac{z^n}{n!}.
\end{aligned}$$

□

Remark 1. The special case for $m=1$, which is equation (6) in Theorem 2.5 in [2], is incorrect. This becomes clear by analyzing the proof or by simply checking the case $r=0$ (Euler polynomials). The correct version for the case $(r,m)=(r,1)$ is

$$\Big(1-\frac{r}{n+1}\Big)A_{n+1}^{(r,1)}(x) = (x-1)A_n^{(r,1)}(x) + \frac{1}{2}\sum_{k=0}^{n}\frac{\binom{n}{k}}{r!\binom{k+r}{k}}A_{k+r}^{(r,1)}A_{n-k}^{(r,1)}(x). \tag{2.21}$$

In the case $(r,m)=(0,m)$, we have two recurrences for generalized Euler polynomials:

$$E_{n+1}^{(m)}(x) = (x-m)E_n^{(m)}(x) + \frac{m}{2}\sum_{k=0}^{n}\binom{n}{k}E_k^{(m)}(x)E_{n-k}(0), \tag{2.22}$$

and

$$E_{n+1}^{(m)}(x) = xE_n^{(m)}(x) - \frac{m}{2}\sum_{k=0}^{n}\binom{n}{k}E_k^{(m)}(x)E_{n-k}(1). \tag{2.23}$$

We shall indicate the equivalence of the recurrences (2.22) and (2.23). The recurrences can be transformed into each other using only the standard result $E_n(x+1)+E_n(x)=2x^n$:

$$\begin{aligned}
&(x-m)E_n^{(m)}(x) + \frac{m}{2}\sum_{k=0}^{n}\binom{n}{k}E_k^{(m)}(x)E_{n-k}(0) \\
= \;& (x-m)E_n^{(m)}(x) + \frac{m}{2}\sum_{k=0}^{n-1}\binom{n}{k}E_k^{(m)}(x)E_{n-k}(0) + \frac{m}{2}E_n^{(m)}(x)E_0(0) \\
= \;& (x-m)E_n^{(m)}(x) - \frac{m}{2}\sum_{k=0}^{n-1}\binom{n}{k}E_k^{(m)}(x)E_{n-k}(1) + \frac{m}{2}E_n^{(m)}(x)(2-E_0(1)) \\
= \;& xE_n^{(m)}(x) - \frac{m}{2}\sum_{k=0}^{n}\binom{n}{k}E_k^{(m)}(x)E_{n-k}(1).
\end{aligned}$$

3. Combinatorial Identities Associated With Generalized Euler-Genocchi Polynomials and Lucas Numbers

Identities relating Fibonacci and/or Lucas numbers to Bernoulli and/or Euler numbers (polynomials) have been derived by some researchers in the past and are still the subject of research (see [3, 6, 7, 9]). In this section we focus on the Lucas sequence and give two relations to the generalized Euler-Genocchi polynomials.

Recall that Lucas numbers $(L_n)_{n\geq 0}$ are defined by the recurrence

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2,$$

with initial conditions $L_0 = 2$ and $L_1 = 1$, respectively. They can be expressed compactly by the Binet form:

$$L_n = \alpha^n + \beta^n,$$

with $\alpha = (1+\sqrt{5})/2$ being the golden section and and $\beta = -1/\alpha$. Our main result of this section is the following combinatorial identity.

Theorem 5. *Let n, r and m be integers with $r \geq 0$, $m \geq 1$ and $n \geq rm$. Then, we have*

$$\sum_{j=0}^{n} \binom{n}{j} 5^{\frac{n-j}{2}} A_{n-j}^{(r,m)}(x) \sum_{k_1+k_2+\cdots+k_m=j} \binom{j}{k_1, k_2, \ldots, k_m} L_{k_1} L_{k_2} \cdots L_{k_m}$$
$$= 2^m 5^{\frac{rm}{2}} \binom{n}{rm} (rm)! \Big(m\beta + \sqrt{5}x\Big)^{n-rm}, \tag{3.1}$$

where $\beta = (1-\sqrt{5})/2$ and

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! \cdot k_2! \cdots k_m!}, \qquad k_1 + k_2 + \cdots + k_m = n,$$

$k_i \in \mathbb{N}_0, i = 1, ..., m,$ is the multinomial coefficient.

Proof. By the Binet formula, the exponential generating function for the Lucas numbers, $G(z)$, is

$$G(z) = \sum_{n=0}^{\infty} L_n \frac{z^n}{n!} = 2e^{z/2} \cosh(\sqrt{5}z/2). \tag{3.2}$$

Hence, for $m \geq 1$

$$\begin{aligned} G^m(z) &= \sum_{n=0}^{\infty} \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} L_{k_1} L_{k_2} \cdots L_{k_m} \frac{z^n}{n!} \\ &= 2^m e^{mz/2} \cosh^m(\sqrt{5}z/2). \end{aligned} \tag{3.3}$$

By (2.1) we also have

$$A(r,m;x,z) = z^{rm}\frac{e^{(x-m/2)z}}{\cosh^m(z/2)}.$$

Thus, we get

$$\begin{aligned}
G^m(z)A(r,m;x,\sqrt{5}z) &= 2^m e^{mz/2}\cosh^m(\sqrt{5}z/2)(\sqrt{5}z)^{rm}\frac{e^{(x-m/2)\sqrt{5}z}}{\cosh^m(\sqrt{5}z/2)} \\
&= 2^m 5^{\frac{rm}{2}} z^{rm} e^{(m\beta+\sqrt{5}x)z} \\
&= 2^m 5^{\frac{rm}{2}} \sum_{n=0}^{\infty}\Big(m\beta+\sqrt{5}x\Big)^n \frac{z^{n+rm}}{n!} \\
&= 2^m 5^{\frac{rm}{2}} \sum_{n=rm}^{\infty}\Big(m\beta+\sqrt{5}x\Big)^{n-rm} \frac{z^{n}}{(n-rm)!} \\
&= 2^m 5^{\frac{rm}{2}} \sum_{n=rm}^{\infty}\binom{n}{rm}(rm)!\Big(m\beta+\sqrt{5}x\Big)^{n-rm} \frac{z^{n}}{n!}.
\end{aligned}$$

On the other hand, $G^m(z)A(r,m;x,\sqrt{5}z)$ can be determined using Cauchy's product rule for power series. Comparing the coefficients of z^n gives our statement. □

When $(r,m)=(r,1)$, we get the special case

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}A_{n-j}^{(r,1)}(x)L_j = 5^{\frac{r}{2}}2\binom{n}{r}(r)!\Big(\beta+\sqrt{5}x\Big)^{n-r}. \tag{3.4}$$

When $(r,m)=(r,2)$, the equation becomes

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}A_{n-j}^{(r,2)}(x)(2^jL_j+2) = 5^{r}4\binom{n}{2r}(2r)!\Big(2\beta+\sqrt{5}x\Big)^{n-2r}, \tag{3.5}$$

where we have used the known result

$$\sum_{k=0}^{j}\binom{j}{k}L_kL_{j-k} = 2^jL_j+2.$$

The special cases involving generalized Euler and generalized Genocchi polynomials are obtained from $(r,m)=(0,m)$ and $(r,m)=(1,m)$, respectively:

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}E_{n-j}^{(m)}(x)\sum_{k_1+k_2+\cdots+k_m=j}\binom{j}{k_1,k_2,\ldots,k_m}L_{k_1}L_{k_2}\cdots L_{k_m} \\
&\qquad = 2^m\Big(m\beta+\sqrt{5}x\Big)^n,
\end{aligned} \tag{3.6}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}G_{n-j}^{(m)}(x)\sum_{k_1+k_2+\cdots+k_m=j}\binom{j}{k_1,k_2,\ldots,k_m}L_{k_1}L_{k_2}\cdots L_{k_m}$$
$$=2^m5^{\frac{m}{2}}\frac{n!}{(n-m)!}\Big(m\beta+\sqrt{5}x\Big)^{n-m}. \tag{3.7}$$

From (3.6) we also get

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}L_jE_{n-j}(x)=2\Big(\beta+\sqrt{5}x\Big)^n, \tag{3.8}$$

which has been recently stated in [6]. The alternating version of the above identity is stated next as a corollary.

Corollary 1. *For $r\ge 0$, $m\ge 1$ and $n\ge rm$, it holds that*

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}(-1)^{n-j}A_{n-j}^{(r,m)}(x)\sum_{k_1+k_2+\cdots+k_m=j}\binom{j}{k_1,k_2,\ldots,k_m}L_{k_1}L_{k_2}\cdots L_{k_m}$$
$$=(-1)^{rm}2^m5^{\frac{rm}{2}}\binom{n}{rm}(rm)!\Big(m\alpha-\sqrt{5}x\Big)^{n-rm}. \tag{3.9}$$

Proof. Replace x by $m-x$ in (3.1) and use the reciprocal relation from (2.6). □

When $(r,m)=(r,1)$ or $(r,m)=(r,2)$, we get the identities

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}(-1)^{n-j}A_{n-j}^{(r,1)}(x)L_j=2(-1)^r5^{\frac{r}{2}}\binom{n}{r}(r)!\Big(\alpha-\sqrt{5}x\Big)^{n-r}, \tag{3.10}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}(-1)^{n-j}A_{n-j}^{(r,2)}(x)(2^jL_j+2)=5^r4\binom{n}{2r}(2r)!\Big(2\alpha-\sqrt{5}x\Big)^{n-2r}. \tag{3.11}$$

Moreover, from $(r,m)=(0,m)$ and $(r,m)=(1,m)$ the following relations follow:

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}(-1)^{n-j}E_{n-j}^{(m)}(x)\sum_{k_1+k_2+\cdots+k_m=j}\binom{j}{k_1,k_2,\ldots,k_m}L_{k_1}L_{k_2}\cdots L_{k_m}$$
$$=2^m\Big(m\alpha-\sqrt{5}x\Big)^n, \tag{3.12}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}(-1)^{n-j}G_{n-j}^{(m)}(x)\sum_{k_1+k_2+\cdots+k_m=j}\binom{j}{k_1,k_2,\ldots,k_m}L_{k_1}L_{k_2}\cdots L_{k_m}$$
$$=(-1)^m2^m5^{\frac{m}{2}}\frac{n!}{(n-m)!}\Big(m\alpha-\sqrt{5}x\Big)^{n-m}. \tag{3.13}$$

The case $(r,m)=(0,1)$ in (3.12) becomes

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}(-1)^{n-j}L_jE_{n-j}(x)=2\Big(\alpha-\sqrt{5}x\Big)^n. \tag{3.14}$$

It has also been proved recently in [6].

4. Further Remarks

This section contains some further results involving generalized Bernoulli polynomials of order r and m, which will be defined below. This polynomial family has not been the subject of this paper so far. We think, however, that these results complement our findings and are worth stating.

Definition 2. For $r,m\geq 1$ and $n\geq m(r-1)$ we define the *generalized Bernoulli polynomials of order r and m* by the equation

$$B(r,m;x,z)=\sum_{n=0}^{\infty}B_n^{(r,m)}(x)\frac{z^n}{n!}=\left(\frac{z^r}{e^z-1}\right)^m e^{xz} \qquad (|z|<2\pi). \tag{4.1}$$

The numbers $B_n^{(r,m)}(0)=B_n^{(r,m)}$ are called *generalized Bernoulli numbers of order r and m.*

It is obvious that $B_n^{(1,m)}(x)=B_n^{(m)}(x)$ are the generalized Bernoulli polynomials (see [4]) and $B_n^{(1,1)}=B_n$ are the famous Bernoulli numbers. Also, we set $B_j^{(r,m)}(x)=0$ for $j<m(r-1)$.

We point out that all the results from Section 2 have an analogue, which involves $B_n^{(r,m)}(x)$ instead $A_n^{(r,m)}(x)$. As the proofs are similar in structure, we do not intend to repeat all the calculations and state most of the properties without proofs (we can provide proofs upon request). In addition, to give the reader an idea of how far the "symmetries" reach out, we present combinatorial connections between $B_n^{(r,m)}(x)$ and Fibonacci numbers $(F_n)_{n\geq 0}$.

Lemma 4. *Let $r,m\geq 1$ be integers and $n\geq m(r-1)$. Then, the generalized Bernoulli polynomials of order r and m have the following basic properties:*
a)

$$B_n^{(r,m)}(x)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(r,m)}x^{n-k}, \tag{4.2}$$

b)

$$\frac{d}{dx}B_n^{(r,m)}(x)=nB_{n-1}^{(r,m)}(x), \tag{4.3}$$

c)

$$B_n^{(r,m)}(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(r,m)}(x), \tag{4.4}$$

d)

$$B_n^{(r,m)}(m-x) = (-1)^{n-(r-1)m} B_n^{(r,m)}(x). \tag{4.5}$$

The addition formulas for $A_n^{(r,m)}(x)$ turn into difference equations for $B_n^{(r,m)}(x)$.

Theorem 6. *For $r, m \geq 1$ it holds that*

$$B_n^{(r,m)}(x+1) - B_n^{(r,m)}(x) = r! \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{r} B_k^{(r,m-1)} x^{n-k-r}. \tag{4.6}$$

Especially, for $r = 1$ we rediscover the known difference equation for generalized Bernoulli polynomials

$$B_n^{(m)}(x+1) - B_n^{(m)}(x) = n B_{n-1}^{(m-1)}(x).$$

Theorem 7. *For $r, m \geq 1$ it holds that*

$$B_n^{(r,m)}(x) - B_n^{(r,m)}\left(\frac{x}{2}\right) = 2^{1-2n} \sum_{\substack{k=0 \\ n-k \equiv 1 \pmod 2}}^{n} \binom{n}{k} 2^{2k} B_k^{(r,m)}\left(\frac{3x}{4}\right) x^{n-k}. \tag{4.7}$$

A Raabe-like formula for $B_n^{(r,m)}(x)$ is given below.

Theorem 8. *Let m, r and q be three integers with r, m and $q \geq 1$. Then, for $n \geq m(r-1)$ we have the following identity:*

$$\sum_{k=0}^{q-1} B_n^{(r,m)}\left(\frac{x+k}{q}\right) = \sum_{j=0}^{n} \binom{n}{j} q^{r-j} B_j^{(r,1)}(x) B_{n-j}^{(r,m-1)}. \tag{4.8}$$

The recurrence formulas for $B_n^{(r,m)}(x)$ are stated in the next theorem. The proof of the theorem is similar to the proof of Theorem 4 and we leave it to the reader of the paper.

Theorem 9. *The polynomials $B_n^{(r,m)}(x)$ satisfy the following recurrence relations:*

$$\left(1 - \frac{m(r-1)}{n+1}\right) B_{n+1}^{(r,m)}(x) = x B_n^{(r,m)}(x) - \frac{m}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k^{(r,m)}(x) B_{n+1-k}(1), \tag{4.9}$$

and

$$\left(1 - \frac{rm}{n+1}\left(1 - \frac{1}{r!}\right)\right) B_{n+1}^{(r,m)}(x) = (x-m) B_n^{(r,m)}(x) - m \sum_{k=0}^{n} \frac{\binom{n}{k}}{r! \binom{n-k+r}{r}} B_k^{(r,m)}(x) B_{n-k+r}^{(r,1)}. \tag{4.10}$$

Especially for $(r,m)=(1,m)$, we obtain two recurrences for generalized Bernoulli polynomials:

$$B_{n+1}^{(m)}(x)=xB_n^{(m)}(x)-\frac{m}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k^{(m)}(x)B_{n+1-k}(1), \tag{4.11}$$

and

$$B_{n+1}^{(m)}(x)=(x-m)B_n^{(m)}(x)-m\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n-k+1}B_k^{(m)}(x)B_{n+1-k}(0). \tag{4.12}$$

Remark 2. A direct transformation of (4.11) into (4.12) and vice versa uses $B_n(x+1)-B_n(x)=nx^{n-1}$ and $B_1(x)=x-1/2$, and can by carried out as follows:

$$\begin{aligned}
B_{n+1}^{(m)}(x) &= xB_n^{(m)}(x)-\frac{m}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k^{(m)}(x)B_{n+1-k}(1)\\
&= xB_n^{(m)}(x)-\frac{m}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k^{(m)}(x)B_{n+1-k}(1)-mB_n^{(m)}(x)B_1(1)\\
&= (x-\frac{m}{2})B_n^{(m)}(x)-\frac{m}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k^{(m)}(x)B_{n+1-k}(0)\\
&= (x-\frac{m}{2})B_n^{(m)}(x)-\frac{m}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k^{(m)}(x)B_{n+1-k}(0)+mB_n^{(m)}(x)B_1(0)\\
&= (x-m)B_n^{(m)}(x)-\frac{m}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k^{(m)}(x)B_{n+1-k}(0)\\
&= (x-m)B_n^{(m)}(x)-m\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n-k+1}B_k^{(m)}(x)B_{n+1-k}(0).
\end{aligned}$$

We conclude this paper by providing combinatorial connections between $B_n^{(r,m)}(x)$ and Fibonacci numbers. The identities that we will state must be treated as counterparts of Theorem 5 and Corollary 1 of the last section.

Theorem 10. *Let n,r and m be integers with $r,m\geq 1$ and $n\geq rm$. Then,*

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}B_{n-j}^{(r,m)}(x)\sum_{k_1+k_2+\cdots+k_m=j}\binom{j}{k_1,k_2,\ldots,k_m}F_{k_1}F_{k_2}\cdots F_{k_m}\\
&\qquad = 5^{(r-1)m/2}\binom{n}{rm}(rm)!\left(m\beta+\sqrt{5}x\right)^{n-rm},
\end{aligned} \tag{4.13}$$

where F_n is the nth Fibonacci number given by the recurrence $F_0=0$, $F_1=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n\geq 2$.

Proof. Let $F(z)$ be the exponential generating function for the Fibonacci numbers. Then, by the Binet formula for F_n we immediately get

$$F(z) = \sum_{n=0}^{\infty} F_n \frac{z^n}{n!} = \frac{2}{\sqrt{5}} e^{z/2} \sinh(\sqrt{5}z/2), \tag{4.14}$$

and for $m \geq 1$

$$F^m(z) = \Big(\frac{2}{\sqrt{5}}\Big)^m e^{mz/2} \sinh^m(\sqrt{5}z/2). \tag{4.15}$$

By (4.1) we also have

$$B(r, m; x, z) = 2^{-m} z^{rm} \frac{e^{(x-m/2)z}}{\sinh^m(z/2)}.$$

Hence,

$$\begin{aligned}
F^m(z)B(r, m; x, \sqrt{5}z) &= \Big(\frac{2}{\sqrt{5}}\Big)^m e^{mz/2} \sinh^m(\sqrt{5}z/2) 2^{-m} 5^{rm/2} z^{rm} \frac{e^{(x-m/2)\sqrt{5}z}}{\sinh^m(\sqrt{5}z/2)} \\
&= 5^{(r-1)m/2} z^{rm} e^{(m\beta+\sqrt{5}x)z} \\
&= 5^{(r-1)m/2} \sum_{n=0}^{\infty} \Big(m\beta + \sqrt{5}x\Big)^n \frac{z^{n+rm}}{n!} \\
&= 5^{(r-1)m/2} \sum_{n=rm}^{\infty} \Big(m\beta + \sqrt{5}x\Big)^{n-rm} \frac{z^n}{(n-rm)!} \\
&= 5^{(r-1)m/2} \sum_{n=rm}^{\infty} \binom{n}{rm} (rm)! \Big(m\beta + \sqrt{5}x\Big)^{n-rm} \frac{z^n}{n!}.
\end{aligned}$$

Using Cauchy's rule for the power series of $F^m(z)B(r, m; x, \sqrt{5}z)$ and comparing the coefficients of z^n gives our statement. □

The alternating variant is obtained easily from our previous findings.

Corollary 2. *Let r and m be two integers with $r, m \geq 1$ and $n \geq rm$. Then, it holds that*

$$\begin{aligned}
\sum_{j=0}^{n} \binom{n}{j} 5^{\frac{n-j}{2}} (-1)^{n-j} B_{n-j}^{(r,m)}(x) \sum_{\substack{k_1+k_2+\cdots+k_m=j \\ k_1,k_2,\ldots,k_m \geq 0}} \binom{j}{k_1, k_2, \ldots, k_m} F_{k_1} F_{k_2} \cdots F_{k_m} \\
= (-1)^{(r-1)m} 5^{(r-1)m/2} \binom{n}{rm} (rm)! \Big(m\alpha - \sqrt{5}x\Big)^{n-rm}. \qquad (4.16)
\end{aligned}$$

Proof. Replace x by $m - x$ in (4.13) and use the reciprocal relation for generalized Bernoulli polynomials from equation (4.5). □

When $(r,m)=(1,m)$, then (4.13) and (4.16) reduce, respectively, to

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}F_jB_{n-j}^{(m)}(x)=\binom{n}{m}(m)!\Big(m\beta+\sqrt{5}x\Big)^{n-m}, \tag{4.17}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}5^{\frac{n-j}{2}}(-1)^{n-j}F_jB_{n-j}^{(m)}(x)=\binom{n}{rm}(rm)!\Big(m\alpha-\sqrt{5}x\Big)^{n-rm}. \tag{4.18}$$

Both cases with $(r,m)=(1,1)$ appear in [6].

Acknowledgments. The authors thank the anonymous referee for her/his interest in this work and some valuable suggestions on the submitted manuscript. Further, the authors are grateful to Professor Bruce Landman for his constant support during the publication process.

References

[1] H. Belbachir, S. Hadj-Brahim and M. Rachidi, Another determinental approach for a family of Appell polynomials, *Filomat* **12** (2018), 4155–4164.

[2] H. Belbachir and S. Hadj-Brahim, Some explicit formulas for Euler-Genocchi polynomials, *Integers* **19** (2019), #A28, 14 pp.

[3] P. F. Byrd, Relations between Euler and Lucas numbers, *Fibonacci Quart.* **13** (1975), 111–114.

[4] R. Dere, Y. Simsek and H. M. Srivastava, A unified presentation of three families of generalized Apostol-type polynomials based upon the theory of the umbral calculus and the umbral algebra, *J. Number Theory* **133** (2013), 3245–3263.

[5] K. Dilcher and I. Sh. Slavutskii, A Bibliography of Bernoulli Numbers, https://www.mscs.dal.ca/ dilcher/bernoulli.html.

[6] R. Frontczak, Relating Fibonacci numbers to Bernoulli numbers via balancing polynomials, *J. Integer Sequences* **22** (2019), Article 19.5.3.

[7] G. Ozdemir and Y. Simsek, Identities and relations associated with Lucas and some special sequences, *AIP Conference Proceedings 1863 300003* (2017).

[8] H. M. Srivastava and Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, *Appl. Math. Lett.* **17** (2004), 375–380.

[9] T. Zhang and Y. Ma, On generalized Fibonacci polynomials and Bernoulli numbers, *J. Integer Sequences* **8** (2005), Article 05.5.3.