AN ADAPTIVE UPPER BOUND ON THE RAMSEY NUMBERS
\( R(3,\ldots,3) \)

Shalom Eliahou

Univ. Littoral Côte d’Opale, Laboratoire de Mathématiques Pures et Appliquées
Joseph Liouville, Calais, France and CNRS, France
eliahou@univ-littoral.fr

Received: 12/10/19, Accepted: 6/22/20, Published: 7/24/20

Abstract

Since 2002, the best known upper bound on the Ramsey numbers \( R_n(3) = R(3,\ldots,3) \) is \( R_n(3) \leq n!(e - 1/6) + 1 \) for all \( n \geq 4 \). It is based on the current estimate \( R_4(3) \leq 62 \). We show here how any closing-in on \( R_4(3) \) yields an improved upper bound on \( R_n(3) \) for all \( n \geq 4 \). For instance, with our present adaptive bound, the conjectured value \( R_4(3) = 51 \) implies \( R_n(3) \leq n!(e - 5/8) + 1 \) for all \( n \geq 4 \).

1. Introduction

For \( n \geq 1 \), the \( n \)-color Ramsey number \( R_n(3) = R(3,\ldots,3) \) denotes the smallest \( N \) such that, for any \( n \)-coloring of the edges of the complete graph \( K_N \), there is a monochromatic triangle. See for instance [4, 8, 11] for background on Ramsey theory. There is a well known recursive upper bound on \( R_n(3) \) due to [5], namely

\[
R_n(3) \leq n(R_{n-1}(3) - 1) + 2
\]

for all \( n \geq 2 \). Currently, the only known exact values of \( R_n(3) \) are \( R_1(3) = 3 \), \( R_2(3) = 6 \) and \( R_3(3) = 17 \). As for \( n = 4 \), the current state of knowledge is

\[
51 \leq R_4(3) \leq 62.
\]

The lower bound is due to [1] and the upper bound to [3], down from the preceding bound \( R_4(3) \leq 64 \) in [9]. Moreover, it is conjectured in [14] that

\[
R_4(3) = 51.
\]

Here is a brief summary of successive upper bounds on \( R_n(3) \). In [5], the authors proved that

\[
R_n(3) \leq n!e + 1
\]
for all $n \geq 2$. Whitehead’s results [13] led to
\[ R_n(3) \leq n!(e - 1/24) + 1 \]
for all $n \geq 2$, and Wan [12] further improved it to
\[ R_n(3) \leq n!(e - e^{-1} + 3)/2 + 1. \]
The last improvement came in 2002, when it was proved in [15] that
\[ R_n(3) \leq n!(e - 1/6) + 1 \]
for all $n \geq 4$. That bound relies on the estimate $R_4(3) \leq 62$ by [3].

Because of the recurrence relation (1), any improved upper bound on $R_k(3)$ for some $k \geq 4$ will yield an improved upper bound on $R_n(3)$ for all $n \geq k$. Our purpose here is to make this automatic improvement explicit. For instance, combined with our adaptive upper bound, the above-mentioned conjecture $R_4(3) = 51$ implies
\[ R_n(3) \leq n!(e - 5/8) + 1 \]
for all $n \geq 4$. This would be a substantial improvement over the current upper bound $n!(e - 1/6) + 1$, since $e - 1/6 \approx 2.55$ while $e - 5/8 \approx 2.09$.

2. Main Results

As reported in [7], it is proved in [15] that $R_n(3) \leq n!(e - 1/6) + 1$ for all $n \geq 4$. But the latter paper is in Chinese and not easily accessible to English readers. In this section, we prove a somewhat more general statement. We shall need the formulas below.

2.1. Useful Formulas

In proving $R_n(3) \leq n!e + 1$, the authors of [5] used without comment the formula
\[ \lfloor (n + 1)!e \rfloor = (n + 1)\lfloor n!e \rfloor + 1 \]
for all $n \geq 1$. For convenience, we provide a proof here, as a direct consequence of the auxiliary formula below.

Proposition 1. For all $n \geq 1$, we have $\lfloor n!e \rfloor = \sum_{i=0}^{n} n!/i!$.

Proof. We have $e = 1/0! + 1/1! + \sum_{i=2}^{\infty} 1/i! = 2 + \sum_{i=2}^{\infty} 1/i!$. Since $e < 3$, it follows that $\sum_{i=2}^{\infty} 1/i! < 1$. Now $n!e = \sum_{i=0}^{n} n!/i! + \sum_{i=n+1}^{\infty} n!/i!$. The left-hand summand is an integer, while the right-hand one satisfies
\[ \sum_{i=n+1}^{\infty} n!/i! = \sum_{i=2}^{\infty} \frac{1}{\Pi_{k=1}^{i}(n + k)} \leq \sum_{i=2}^{\infty} 1/i! < 1. \]
INTEGERS: 20 (2020)

This concludes the proof.

**Corollary 1 ([5]).** For all \( n \geq 1 \), we have \([(n+1)!e] = (n+1)[n!e] + 1\).

**Proof.** Applying Proposition 1 for \( n+1 \) and then for \( n \), we have

\[
[(n+1)!e] = \sum_{i=0}^{n+1} (n+1)!/i! \\
= (n+1) \sum_{i=0}^{n} n!/i! + (n+1)!/(n+1)! \\
= (n+1)[n!e] + 1.
\]

\[\square\]

2.2. An Optimal Model

We now exhibit an optimal model for the recursion (1).

**Proposition 2.** Given \( q \in \mathbb{Q} \), let \( f: \mathbb{N} \to \mathbb{Z} \) be defined by \( f(n) = [n!(e-q)] + 1 \) for \( n \in \mathbb{N} \). Then, for all \( n \in \mathbb{N} \) such that \( n!q \in \mathbb{Z} \), we have

\[
f(n+1) = (n+1)(f(n) - 1) + 2.
\]  

**Proof.** We have

\[
f(n+1) = [(n+1)!(e-q)] + 1 \\
= [(n+1)!e] - (n+1)!q + 1 \quad \text{[since \( (n+1)!q \in \mathbb{Z} \)]} \\
= (n+1)[n!e] + 1 - (n+1)!q + 1 \quad \text{[by Corollary 1]} \\
= (n+1)[n!(e-q)] + 2 \quad \text{[since \( n!q \in \mathbb{Z} \)]} \\
= (n+1)(f(n) - 1) + 2.
\]

\[\square\]

2.3. An Adaptive Bound

Our adaptive upper bound on \( R_n(3) \) is provided by the following statements.

**Proposition 3.** Let \( k \in \mathbb{N} \) and \( q \in \mathbb{Q} \) satisfy \( k \geq 2 \), \( R_k(3) \leq k!(e-q) + 1 \) and \( k!q \in \mathbb{N} \). Then \( R_n(3) \leq n!(e-q) + 1 \) for all \( n \geq k \).

**Proof.** As in Proposition 2, denote \( f(n) = [n!(e-q)] + 1 \) for \( n \in \mathbb{N} \). By assumption, we have

\[
R_k(3) \leq f(k)
\]  

(3)
and $k!q \in \mathbb{Z}$. It suffices to prove the claim for $n = k + 1$, since if $k!q \in \mathbb{N}$ then $(k + 1)!q \in \mathbb{N}$. By successive application of (1), (3) and (2), we have
\[
R_{k+1}(3) \leq (k + 1)(R_k(3) - 1) + 2 \\
\leq (k + 1)(f(k) - 1) + 2 \\
= f(k + 1).
\]

Note that using (2) is allowed by Proposition 2 and the assumption $k!q \in \mathbb{N}$. □

**Theorem 1.** Let $k \geq 2$ be an integer. Let $a \in \mathbb{N}$ satisfy $a \leq \lfloor k!e \rfloor - R_k(3) + 1$, and let $q = a/k!$. Then $R_n(3) \leq n!(e - q) + 1$ for all $n \geq k$.

**Proof.** We have $a \leq k!e - R_k(3) + 1$, so $R_k(3) \leq k!e - a + 1 = k!(e - q) + 1$. Moreover, we have $k!q = a \in \mathbb{N}$. The conclusion follows from Proposition 3. □

**Remark 1.** Theorem 1 is the best possible application of Proposition 3. Indeed, with the value $a' = \lfloor k!e \rfloor - R_k(3) + 2$ and $q' = a'/k!$, it no longer holds that $R_k(3) \leq k!(e - q') + 1$.

**2.4. The Case $k = 4$**

We now apply the above result to the case $k = 4$. We only know $51 \leq R_4(3) \leq 62$ so far. Note that by Proposition 1, we have
\[
\lfloor 4!e \rfloor = \sum_{i=0}^{4} 4!/i! = 24 + 24 + 12 + 4 + 1 = 65. \tag{4}
\]

**Proposition 4.** Let $a \in \mathbb{N}$ satisfy $a \leq 66 - R_4(3)$. Then setting $q = a/24$, we have $R_n(3) \leq n!(e - q) + 1$ for all $n \geq 4$.

**Proof.** By (4), $a$ satisfies the hypotheses of Theorem 1. The conclusion follows. □

When the exact value of $R_4(3)$ will be known, Proposition 4 will provide an adapted upper bound on $R_n(3)$ for all $n \geq 4$. In the meantime, here are three possible outcomes.

**Corollary 2 ([15]).** $R_n(3) \leq n!(e - 1/6) + 1$ for all $n \geq 4$.

**Proof.** Since $R_4(3) \leq 62$, we may take $a = 4$ in Proposition 4. The conclusion follows from that result with $q = a/4! = 1/6$. □

Note that the above bound does not extend to $n = 3$, since $R_3(3) = 17$, whereas by Proposition 1, we have $\lfloor 3!(e - 1/6) \rfloor + 1 = \lfloor 3!e \rfloor = 3! + 3! + 3 + 1 = 16$.

As mentioned earlier, it is conjectured in [14] that $R_4(3) = 51$. If true, Proposition 4 will yield the following improved upper bound.
Corollary 3. If $R_4(3) = 51$, then $R_n(3) \leq n!(e - 5/8) + 1$ for all $n \geq 4$.

Proof. By Proposition 4, with $a = 66 - 51 = 15$ and $q = 15/4! = 5/8$. \hfill \Box

As noted in the Introduction, this would be a substantial improvement over the current upper bound $n!(e - 1/6) + 1$, since $e - 1/6 \approx 2.55$ whereas $e - 5/8 \approx 2.09$.

An intermediate step would be, for instance, to show $R_4(3) \leq 54$ if at all true. This would yield the following weaker improvement.

Corollary 4. If $R_4(3) \leq 54$, then $R_n(3) \leq n!(e - 1/2) + 1$ for all $n \geq 4$.

Proof. By Proposition 4, with $a = 66 - 54 = 12$ and $q = a/4! = 1/2$. \hfill \Box

Remark 2. The above three corollaries are best possible applications of Proposition 4, as in each case we took the largest admissible value for $a \in \mathbb{N}$.

2.5. The Case $k = 5$

Let us also briefly consider the case $k = 5$. At the time of writing, we only know $162 \leq R_5(3) \leq 307$. See [7].

Proposition 5. Let $a \in \mathbb{N}$ satisfy $a \leq 327 - R_5(3)$. Then setting $q = a/120$, we have $R_n(3) \leq n!(e - q) + 1$ for all $n \geq 5$.

Proof. By Theorem 1 and the value $[5!e] = 326$ given by Proposition 1. \hfill \Box

Here again are three possible outcomes. Knowing only $R_5(3) \leq 307$ does not allow us to improve the current estimate $R_n(3) \leq n!(e - 1/6) + 1$. At the other extreme, if $R_5(3) = 162$ holds true, it would yield $R_n(3) \leq n!(e - 11/8) + 1$ for all $n \geq 5$. As an intermediate estimate, if $R_5(3) \leq 227$ holds true, it would imply $R_n(3) \leq n!(e - 5/6) + 1$ for all $n \geq 5$.

3. Concluding Remarks

3.1. On $\lim_{n \to \infty} R_n(3)^{1/n}$

The adaptive upper bound on $R_n(3)$ given by Theorem 1 may still be quite far from reality, as the asymptotic behavior of $R_n(3)$ remains poorly understood. For instance, is there a constant $c$ such that $R_{n+1}(3) \leq cR_n(3)$ for all $n$? Or, maybe, such that $R_n(3) \geq cn!$ for all $n$? The former would imply that $\lim_{n \to \infty} R_n(3)^{1/n}$, known by [2] to exist, is finite, whereas the latter would imply $\lim_{n \to \infty} R_n(3)^{1/n} = \infty$. At the time of writing, it is not known whether that limit is finite or infinite. See for instance [6], where this question is discussed together with related problems.
3.2. Link With the Schur Numbers

The Schur number $S(n)$ is defined as the largest integer $N$ such that for any $n$-coloring of the integers $\{1, 2, \ldots, N\}$, there is a monochromatic triple of integers $1 \leq x, y, z \leq N$ such that $x + y = z$. The existence of $S(n)$ was established by Schur in [10], an early manifestation of Ramsey theory. Still in [10], Schur proved the upper bound

$$S(n) \leq n!e - 1$$

(5)

for all $n \geq 2$. The similarity with the upper bound $R_n(3) \leq n!e + 1$ proved 40 years later in [5] is striking. In fact, there is a well-known relationship between these numbers, namely

$$S(n) \leq R_n(3) - 2.$$  

(6)

Thus, via (6), our adaptive upper bound on $R_n(3)$ given by Theorem 1 also yields an upper bound on $S(n)$.

Acknowledgements. The author wishes to thank L. Boza and S.P. Radziszowski for informal discussions during the preparation of this note and for their useful comments on a preliminary version of it.

References


[10] I. Schur, Uber die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$, *Jahresber. Dtsch. Math.-Ver.* 25 (1916), 114-117.


[12] H. Wan, Upper bounds for Ramsey numbers $R(3, 3, \ldots, 3)$ and Schur numbers, *J. Graph Theory* 26 (1997), 119-122.

