AN ADAPTIVE UPPER BOUND ON THE RAMSEY NUMBERS $R(3,\ldots,3)$

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Abstract

Since 2002, the best known upper bound on the Ramsey numbers $R_n(3) = R(3,\ldots,3)$ is $R_n(3) \le n!(e - 1/6) + 1$ for all $n \ge 4$. It is based on the current estimate $R_4(3) \le 62$. We show here how any closing-in on $R_4(3)$ yields an improved upper bound on $R_n(3)$ for all $n \ge 4$. For instance, with our present adaptive bound, the conjectured value $R_4(3) = 51$ implies $R_n(3) \le n!(e - 5/8) + 1$ for all $n \ge 4$.

1. Introduction

For $n \ge 1$, the n-color Ramsey number $R_n(3) = R(3,\ldots,3)$ denotes the smallest N such that, for any n-coloring of the edges of the complete graph K_N, there is a monochromatic triangle. See for instance [4, 8, 11] for background on Ramsey theory. There is a well known recursive upper bound on $R_n(3)$ due to [5], namely

$$R_n(3) \le n(R_{n-1}(3) - 1) + 2 \tag{1}$$

for all $n \ge 2$. Currently, the only known exact values of $R_n(3)$ are $R_1(3) = 3$, $R_2(3) = 6$ and $R_3(3) = 17$. As for $n = 4$, the current state of knowledge is

$$51 \le R_4(3) \le 62.$$

The lower bound is due to [1] and the upper bound to [3], down from the preceding bound $R_4(3) \le 64$ in [9]. Moreover, it is conjectured in [14] that

$$R_4(3) = 51.$$

Here is a brief summary of successive upper bounds on $R_n(3)$. In [5], the authors proved that

$$R_n(3) \le n!e + 1$$

for all $n \geq 2$. Whitehead's results [13] led to

$$R_n(3) \leq n!(e - 1/24) + 1$$

for all $n \geq 2$, and Wan [12] further improved it to

$$R_n(3) \leq n!(e - e^{-1} + 3)/2 + 1.$$

The last improvement came in 2002, when it was proved in [15] that

$$R_n(3) \leq n!(e - 1/6) + 1$$

for all $n \geq 4$. That bound relies on the estimate $R_4(3) \leq 62$ by [3].

Because of the recurrence relation (1), any improved upper bound on $R_k(3)$ for some $k \geq 4$ will yield an improved upper bound on $R_n(3)$ for all $n \geq k$. Our purpose here is to make this automatic improvement explicit. For instance, combined with our adaptive upper bound, the above-mentioned conjecture $R_4(3) = 51$ implies

$$R_n(3) \leq n!(e - 5/8) + 1$$

for all $n \geq 4$. This would be a substantial improvement over the current upper bound $n!(e - 1/6) + 1$, since $e - 1/6 \approx 2.55$ while $e - 5/8 \approx 2.09$.

2. Main Results

As reported in [7], it is proved in [15] that $R_n(3) \leq n!(e-1/6)+1$ for all $n \geq 4$. But the latter paper is in Chinese and not easily accessible to English readers. In this section, we prove a somewhat more general statement. We shall need the formulas below.

2.1. Useful Formulas

In proving $R_n(3) \leq n!e + 1$, the authors of [5] used without comment the formula

$$\lfloor (n+1)!e \rfloor = (n+1)\lfloor n!e \rfloor + 1$$

for all $n \geq 1$. For convenience, we provide a proof here, as a direct consequence of the auxiliary formula below.

Proposition 1. *For all $n \geq 1$, we have $\lfloor n!e \rfloor = \sum_{i=0}^{n} n!/i!$.*

Proof. We have $e = 1/0! + 1/1! + \sum_{i=2}^{\infty} 1/i! = 2 + \sum_{i=2}^{\infty} 1/i!$. Since $e < 3$, it follows that $\sum_{i=2}^{\infty} 1/i! < 1$. Now $n!e = \sum_{i=0}^{n} n!/i! + \sum_{i=n+1}^{\infty} n!/i!$. The left-hand summand is an integer, while the right-hand one satisfies

$$\sum_{i=n+1}^{\infty} n!/i! = \sum_{j=1}^{\infty} \frac{1}{\Pi_{k=1}^{j}(n+k)} \leq \sum_{i=2}^{\infty} 1/i! < 1.$$

This concludes the proof. $\square$

Corollary 1 ([5]). *For all $n \geq 1$, we have $\lfloor (n+1)!e \rfloor = (n+1)\lfloor n!e \rfloor + 1$.*

Proof. Applying Proposition 1 for $n+1$ and then for n, we have

$$\begin{aligned}
\lfloor (n+1)!e \rfloor &= \sum_{i=0}^{n+1} (n+1)!/i! \\
&= (n+1)\sum_{i=0}^{n} n!/i! + (n+1)!/(n+1)! \\
&= (n+1)\lfloor n!e \rfloor + 1.
\end{aligned}$$

$\square$

2.2. An Optimal Model

We now exhibit an optimal model for the recursion (1).

Proposition 2. *Given $q \in \mathbb{Q}$, let $f\colon \mathbb{N} \to \mathbb{Z}$ be defined by $f(n) = \lfloor n!(e-q) \rfloor + 1$ for $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ such that $n!q \in \mathbb{Z}$, we have*

$$f(n+1) = (n+1)(f(n)-1) + 2. \tag{2}$$

Proof. We have

$$\begin{aligned}
f(n+1) &= \lfloor (n+1)!(e-q) \rfloor + 1 \\
&= \lfloor (n+1)!e \rfloor - (n+1)!q + 1 \quad \text{[since } (n+1)!q \in \mathbb{Z}] \\
&= (n+1)\lfloor n!e \rfloor + 1 - (n+1)!q + 1 \quad \text{[by Corollary 1]} \\
&= (n+1)\lfloor n!(e-q) \rfloor + 2 \quad \text{[since } n!q \in \mathbb{Z}] \\
&= (n+1)(f(n)-1) + 2.
\end{aligned}$$

$\square$

2.3. An Adaptive Bound

Our adaptive upper bound on $R_n(3)$ is provided by the following statements.

Proposition 3. *Let $k \in \mathbb{N}$ and $q \in \mathbb{Q}$ satisfy $k \geq 2$, $R_k(3) \leq k!(e-q) + 1$ and $k!q \in \mathbb{N}$. Then $R_n(3) \leq n!(e-q) + 1$ for all $n \geq k$.*

Proof. As in Proposition 2, denote $f(n) = \lfloor n!(e-q) \rfloor + 1$ for $n \in \mathbb{N}$. By assumption, we have

$$R_k(3) \leq f(k) \tag{3}$$

and $k!q \in \mathbb{Z}$. It suffices to prove the claim for $n = k+1$, since if $k!q \in \mathbb{N}$ then $(k+1)!q \in \mathbb{N}$. By successive application of (1), (3) and (2), we have

$$\begin{aligned} R_{k+1}(3) &\leq (k+1)(R_k(3)-1)+2 \\ &\leq (k+1)(f(k)-1)+2 \\ &= f(k+1). \end{aligned}$$

Note that using (2) is allowed by Proposition 2 and the assumption $k!q \in \mathbb{N}$. □

Theorem 1. *Let $k \geq 2$ be an integer. Let $a \in \mathbb{N}$ satisfy $a \leq \lfloor k!e \rfloor - R_k(3) + 1$, and let $q = a/k!$. Then $R_n(3) \leq n!(e-q)+1$ for all $n \geq k$.*

Proof. We have $a \leq k!e - R_k(3)+1$, so $R_k(3) \leq k!e - a + 1 = k!(e-q)+1$. Moreover, we have $k!q = a \in \mathbb{N}$. The conclusion follows from Proposition 3. □

Remark 1. Theorem 1 is the best possible application of Proposition 3. Indeed, with the value $a' = \lfloor k!e \rfloor - R_k(3) + 2$ and $q' = a'/k!$, it no longer holds that $R_k(3) \leq k!(e-q')+1$.

2.4. The Case $k = 4$

We now apply the above result to the case $k = 4$. We only know $51 \leq R_4(3) \leq 62$ so far. Note that by Proposition 1, we have

$$\lfloor 4!e \rfloor = \sum_{i=0}^{4} 4!/i! = 24 + 24 + 12 + 4 + 1 = 65. \tag{4}$$

Proposition 4. *Let $a \in \mathbb{N}$ satisfy $a \leq 66 - R_4(3)$. Then setting $q = a/24$, we have $R_n(3) \leq n!(e-q)+1$ for all $n \geq 4$.*

Proof. By (4), a satisfies the hypotheses of Theorem 1. The conclusion follows. □

When the exact value of $R_4(3)$ will be known, Proposition 4 will provide an adapted upper bound on $R_n(3)$ for all $n \geq 4$. In the meantime, here are three possible outcomes.

Corollary 2 ([15]). *$R_n(3) \leq n!(e-1/6)+1$ for all $n \geq 4$.*

Proof. Since $R_4(3) \leq 62$, we may take $a = 4$ in Proposition 4. The conclusion follows from that result with $q = a/4! = 1/6$. □

Note that the above bound does not extend to $n = 3$, since $R_3(3) = 17$, whereas by Proposition 1, we have $\lfloor 3!(e-1/6) \rfloor + 1 = \lfloor 3!e \rfloor = 3! + 3! + 3 + 1 = 16$.

As mentioned earlier, it is conjectured in [14] that $R_4(3) = 51$. If true, Proposition 4 will yield the following improved upper bound.

Corollary 3. *If $R_4(3) = 51$, then $R_n(3) \leq n!(e - 5/8) + 1$ for all $n \geq 4$.*

Proof. By Proposition 4, with $a = 66 - 51 = 15$ and $q = 15/4! = 5/8$. □

As noted in the Introduction, this would be a substantial improvement over the current upper bound $n!(e - 1/6) + 1$, since $e - 1/6 \approx 2.55$ whereas $e - 5/8 \approx 2.09$.

An intermediate step would be, for instance, to show $R_4(3) \leq 54$ if at all true. This would yield the following weaker improvement.

Corollary 4. *If $R_4(3) \leq 54$, then $R_n(3) \leq n!(e - 1/2) + 1$ for all $n \geq 4$.*

Proof. By Proposition 4, with $a = 66 - 54 = 12$ and $q = a/4! = 1/2$. □

Remark 2. *The above three corollaries are best possible applications of Proposition 4, as in each case we took the largest admissible value for $a \in \mathbb{N}$.*

2.5. The Case $k = 5$

Let us also briefly consider the case $k = 5$. At the time of writing, we only know $162 \leq R_5(3) \leq 307$. See [7].

Proposition 5. *Let $a \in \mathbb{N}$ satisfy $a \leq 327 - R_5(3)$. Then setting $q = a/120$, we have $R_n(3) \leq n!(e - q) + 1$ for all $n \geq 5$.*

Proof. By Theorem 1 and the value $\lfloor 5!e \rfloor = 326$ given by Proposition 1. □

Here again are three possible outcomes. Knowing only $R_5(3) \leq 307$ does not allow us to improve the current estimate $R_n(3) \leq n!(e - 1/6) + 1$. At the other extreme, if $R_5(3) = 162$ holds true, it would yield $R_n(3) \leq n!(e - 11/8) + 1$ for all $n \geq 5$. As an intermediate estimate, if $R_5(3) \leq 227$ holds true, it would imply $R_n(3) \leq n!(e - 5/6) + 1$ for all $n \geq 5$.

3. Concluding Remarks

3.1. On $\lim_{n\to\infty} R_n(3)^{1/n}$

The adaptive upper bound on $R_n(3)$ given by Theorem 1 may still be quite far from reality, as the asymptotic behavior of $R_n(3)$ remains poorly understood. For instance, is there a constant c such that $R_{n+1}(3) \leq cR_n(3)$ for all n? Or, maybe, such that $R_n(3) \geq cn!$ for all n? The former would imply that $\lim_{n\to\infty} R_n(3)^{1/n}$, known by [2] to exist, is finite, whereas the latter would imply $\lim_{n\to\infty} R_n(3)^{1/n} = \infty$. At the time of writing, it is not known whether that limit is finite or infinite. See for instance [6], where this question is discussed together with related problems.

3.2. Link With the Schur Numbers

The Schur number $S(n)$ is defined as the largest integer N such that for any n-coloring of the integers $\{1, 2, \ldots, N\}$, there is a monochromatic triple of integers $1 \leq x, y, z \leq N$ such that $x + y = z$. The existence of $S(n)$ was established by Schur in [10], an early manifestation of Ramsey theory. Still in [10], Schur proved the upper bound

$$S(n) \leq n!e - 1 \tag{5}$$

for all $n \geq 2$. The similarity with the upper bound $R_n(3) \leq n!e + 1$ proved 40 years later in [5] is striking. In fact, there is a well-known relationship between these numbers, namely

$$S(n) \leq R_n(3) - 2. \tag{6}$$

Thus, via (6), our adaptive upper bound on $R_n(3)$ given by Theorem 1 also yields an upper bound on $S(n)$.

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