# THE LARGEST $(k, \ell)$-SUM-FREE SETS IN COMPACT ABELIAN GROUPS

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*Received: 5/19/19, Revised: 1/3/20, Accepted: 7/21/20, Published: 8/14/20*

**Abstract**

A subset $A$ of a finite abelian group is called $(k, \ell)$-sum-free if $kA \cap \ell A = \emptyset$. In this paper, we extend this concept to compact abelian groups and study the question of determining the maximum size of a measurable $(k, \ell)$-sum-free set. For integers $1 \leq k < \ell$ and a compact abelian group $G$, let

$$\lambda_{k,\ell}(G) = \sup\{\mu(A) : kA \cap \ell A = \emptyset\}$$

be the maximum possible size of a $(k, \ell)$-sum-free subset of $G$. We prove that if $G = \mathbb{I} \times M$, where $\mathbb{I}$ is the identity component of $G$, then

$$\lambda_{k,\ell}(G) = \max\left\{\lambda_{k,\ell}(M), \lambda_{k,\ell}(\mathbb{I})\right\};$$

moreover, $\lambda_{k,\ell}(\mathbb{I}) = \frac{1}{k+\ell}$ if $\mathbb{I}$ is nontrivial. We also discuss how this problem motivates a new framework for studying $(k, \ell)$-sum-free sets in finite groups.

## 1. Introduction

The *Minkowski sum* of two subsets $A$ and $B$ of an additive abelian group $G$ is

$$A + B = \{a + b : a \in A, b \in B\}.$$

When $G$ is finite, a natural problem is determining the maximum size of a subset $A \subset G$ that is *sum-free*, i.e., satisfies $(A+A) \cap A = \emptyset$. In other words, $A$ is sum-free if $x + y = z$ has no solution in $A$. Early progress on this problem for cyclic groups appears in the work of Diananda and Yap [4] and Wallis, Street, and Wallis [15]. In 2005, Green and Ruzsa [6] completely resolved this problem for finte abelian groups. Let $\lambda_{1,2}(G)$ denote the maximum density of a sum-free subset of $G$.

**Theorem 1** (Green and Ruzsa [6]). *For any finite abelian group $G$ with exponent $\exp(G)$, we have*

$$\lambda_{1,2}(G) = \max_{d \mid \exp(G)} \left\{ \left\lceil \frac{d-1}{3} \right\rceil \cdot \frac{1}{d} \right\}.$$

*In particular, $\frac{2}{7} \leq \lambda_{1,2}(G) \leq \frac{1}{2}$, and both of these extremal values are achieved.*

Other statistics on sum-free sets have been the object of considerable study (see [5, 14] and the references therein).

This problem has more recently been generalized to $(k,\ell)$-sum-free sets. For a positive integer $k$, let $kA = \underbrace{A + \cdots + A}_{k}$ denote the $k$-fold Minkowski sum of $A$ with itself (not to be confused with the $k$-fold dilation of $A$). Then, for a finite abelian group $G$, let

$$\lambda_{k,\ell}(G) = \max\left\{\frac{|A|}{|G|} : kA \cap \ell A = \emptyset\right\}$$

denote the maximum density of a $(k,\ell)$-sum-free subset of $G$. Trivially, $\lambda_{k,\ell}(G) = 0$ when $k = \ell$, so by convention we take $1 \leq k < \ell$.

Most work has focused on $(k,\ell)$-sum-free sets in cyclic groups; the general abelian case remains far from understood. Important results are due to Bier and Chin [3] and Hamidoune and Plagne [7], whose approaches relied on Vosper's Theorem and Kneser's Theorem. In 2018, Bajnok and Matzke [2] found a general expression for $\lambda_{k,\ell}(\mathbb{Z}_n)$ by analyzing $(k,\ell)$-sum-free arithmetic progressions.

**Theorem 2** (Bajnok and Matzke [2]). *For any integers $1 \leq k < \ell$ and $n \geq 1$, we have*

$$\lambda_{k,\ell}(\mathbb{Z}_n) = \max_{d|n}\left\{\left\lceil\frac{d-\delta_d+r_d}{k+\ell}\right\rceil \cdot \frac{1}{d}\right\},$$

*where $\delta_d = \gcd(d, \ell-k)$ and $r_d$ is the remainder of $k\left\lceil\frac{d-\delta_d}{k+\ell}\right\rceil$ modulo $\delta_d$.*

For further background, see the excellent exposition in [1].

One might wonder about the analogous problem on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the $d$-dimensional torus $\mathbb{T}^d$. In this paper, we generalize the study of $(k,\ell)$-sum-free sets to compact abelian groups. (We take all of our compact groups to be Hausdorff.) This transition from the discrete setting to the continuous setting has a number of precedents in additive combinatorics; previous fruitful generalizations include analogs of Mann's Theorem [9] and Freiman's Theorem [11, 12]. Our generalization of sum-free sets, however, is new.

For a compact abelian group $G$, let $\mu$ be the probability Haar measure (normalized so that $\mu(G) = 1$). We then define

$$\lambda_{k,\ell}(G) = \sup\{\mu(A) : kA \cap \ell A = \emptyset\},$$

where the supremum runs over all measurable subsets $A \subset G$. Note that when $G$ is finite, this definition coincides with the definition above.

We let $\mathbb{I}$ denote the identity component of $G$ (the connected component of $G$ which contains the identity element). It is well known (see, e.g., [8], Theorem 5.4) that $\mathbb{I}$ is a closed normal subgroup; the quotient $M = G/\mathbb{I}$ is the (topological) group of components of $G$. It can be shown that $G$ is isomorphic (as a topological group) to the Cartesian product $\mathbb{I} \times M$ when $\mathbb{I}$ is an open set, which is the case that we consider in this paper.[1]

The main result of this paper is the following formula for $\lambda_{k,\ell}(G)$ when $G$ can be written as $G = \mathbb{I} \times M$ (as above).

**Theorem 3.** *For any integers $1 \le k < \ell$ and any compact abelian group $G = \mathbb{I} \times M$, we have*

$$\lambda_{k,\ell}(G) = \max\left\{\lambda_{k,\ell}(M), \lambda_{k,\ell}(\mathbb{I})\right\}.$$

*If $\mathbb{I}$ is nontrivial, then we also have $\lambda_{k,\ell}(\mathbb{I}) = \frac{1}{k+\ell}$ and hence*

$$\lambda_{k,\ell}(G) = \max\left\{\lambda_{k,\ell}(M), \frac{1}{k+\ell}\right\}.$$

Note that positive-dimensional compact abelian Lie groups are included in the latter case. In particular, $\lambda_{k,\ell}(\mathbb{T}^d) = \frac{1}{k+\ell}$ answers our original question about the $d$-dimensional torus. We also remark that when $\mathbb{I}$ is trivial (consists of only a single point), we have $\lambda_{k,\ell}(\mathbb{I}) = 0$ and $\lambda_{k,\ell}(G) = \lambda_{k,\ell}(M)$, as expected.

In Section 2, we prove Theorem 3. We will make use of the following deep classical result of Kneser [10]. Here, $\mu_*$ denotes the inner Haar probability measure. (Even on $\mathbb{T}$, the Minkowski sum of two measurable sets need not be measurable.)

**Theorem 4** (Kneser [10])**.** *Let $G$ be a compact abelian group with Haar probability measure $\mu$, and let $A$ and $B$ be nonempty measurable subsets of $G$. Then*

$$\mu_*(A+B) \ge \min\{\mu(A) + \mu(B), 1\},$$

*unless the stabilizer $H = \mathrm{Stab}(A+B)$ is an open subgroup of $G$, in which case*

$$\mu_*(A+B) \ge \mu(A) + \mu(B) - \mu(H).$$

In Section 3, we discuss some consequences of our results and possible future lines of inquiry. In particular, the compact case inspires a curious new framework for investigating $(k,\ell)$-sum-free sets in the finite context.

## 2. Proofs

We begin by recording a few general observations.

[1]If $\mathbb{I}$ is open, then $M$ is finite (since $G$ is compact) and has the discrete topology (since $\mathbb{I}$ is open). Index the connected components of $G$ according to the corresponding elements of $M$. Write $M = \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r}$, where each $\mathbb{Z}_{a_i}$ has generator $m_i$. For each $m_i$, choose an arbitrary element $x_i$ in the corresponding connected component of $G$, and consider $a_i x_i$. Since connected compact abelian groups are divisible ([8], Theorem 24.25), there is an element $y_i \in \mathbb{I}$ such that $a_i y_i = a_i x_i$; then $z_i = x_i - y_i$ is an element with order $a_i$ in the connected component of $G$ corresponding to $m_i$. The subgroup of $G$ generated by $z_1, \ldots, z_r$ is a closed and normal subgroup isomorphic to $M$. Since this subgroup intersects $\mathbb{I}$ only at the identity element of $G$, we conclude (see, e.g., [13], page 343) that $G$ is isomorphic (as a topological group) to $\mathbb{I} \times M$.

**Lemma 1.** *Let $A$ and $B$ be (not necessarily measurable) subsets of a compact abelian group $G$. If $\mu_*(A) + \mu_*(B) > 1$, then $A + B = G$.*

*Proof.* There exist closed subsets $A_* \subseteq A$ and $B_* \subseteq B$ satisfying $\mu(A_*)+\mu(B_*) > 1$. Assume (for the sake of contradiction) that there exists some $g \in G \setminus (A+B)$. Then $g \notin A_* + B_*$, so $A_*$ and $\{g\} - B_*$ are disjoint. But $1 \geq \mu(A_*) + \mu(\{g\} - B_*) = \mu(A_*) + \mu(B_*)$ yields a contradiction. $\square$

**Lemma 2.** *Let $G = \mathbb{I} \times M$ be a compact abelian group. Then any open subgroup $H$ of $G$ is of the form $H = \mathbb{I} \times N$, where $N$ is a subgroup of $M$.*

*Proof.* The set $U = \mathbb{I} \cap H$ is open in $\mathbb{I}$ and nonempty (since it contains the identity of $G$). Then $U$ and its cosets are an open partition of $\mathbb{I}$. Since $\mathbb{I}$ is connected, $U = \mathbb{I}$. Finally, noting that $H/\mathbb{I}$ is a subgroup of $G/\mathbb{I} \cong M$ completes the proof. $\square$

**Lemma 3.** *Let $A$ be a nonempty subset of an abelian group $G$. Then for any integers $1 \leq i < j$, we have* $\mathrm{Stab}(iA) \subseteq \mathrm{Stab}(jA)$ *as an inclusion of subgroups.*

*Proof.* For any $h \in \mathrm{Stab}(iA)$, we have $\{h\} + jA = (\{h\} + iA) + (j - i)A = iA + (j - i)A = jA$, so $h \in \mathrm{Stab}(jA)$. $\square$

This is, of course, a specific instance of the general fact that $\mathrm{Stab}(A)$ is a subgroup of $\mathrm{Stab}(A + B)$ for any $A, B$. We now bound $\lambda_{k,\ell}(G)$ from above.

**Theorem 5.** *For any integers $1 \leq k < \ell$ and any compact abelian group $G = \mathbb{I} \times M$, we have*

$$\lambda_{k,\ell}(G) \leq \max\left\{\lambda_{k,\ell}(M), \frac{1}{k+\ell}\right\}.$$

*Proof.* Assume (for the sake of contradiction) that there exists a $(k, \ell)$-sum-free set $A \subseteq G$ with measure strictly greater than both $\lambda_{k,\ell}(M)$ and $\frac{1}{k+\ell}$. Let $H = \mathrm{Stab}(\ell A)$. We consider two cases depending on whether or not $H$ contains $\mathbb{I}$.

First, suppose $H$ does not contain $\mathbb{I}$. Lemma 3 implies that $\mathbb{I}$ is not contained in any $\mathrm{Stab}(iA)$ for $2 \leq i \leq \ell$, so we conclude from Lemma 2 that these stabilizers are not open subgroups of $G$. Iterative applications of Theorem 4 then give $\mu_*(kA) \geq \min\{k\mu(A), 1\}$ and $\mu_*(\ell(-A)) = \mu_*(\ell A) \geq \min\{\ell\mu(A), 1\}$. Since

$$\mu_*(kA) + \mu_*(\ell A) > \frac{k}{k+\ell} + \frac{\ell}{k+\ell} = 1,$$

Lemma 1 tells us that $kA + \ell(-A) = G$, and, in particular, $0 \in kA + \ell(-A)$. So there exist $a_1, \ldots, a_{k+\ell} \in A$ satisfying $a_1 + \cdots + a_k = a_{k+1} + \cdots + a_{k+\ell}$, which contradicts $A$ being $(k, \ell)$-sum-free.

Second, suppose $H$ contains $\mathbb{I}$. This implies that $\ell A$ is a union of cosets of $\mathbb{I}$: if $g \in (m + \mathbb{I}) \cap (\ell A)$, then $\mathbb{I} + \{g\} = m + \mathbb{I} \subseteq H + (\ell A) = \ell A$. Let

$$P = \{p \in M : (p + \mathbb{I}) \cap A \neq \emptyset\}$$

be the set of the elements of $M$ whose corresponding components in $G$ have nonempty intersection with $A$. Since $\frac{|P|}{|M|} \geq \mu(A) > \lambda_{k,\ell}(M)$, we have $kP \cap \ell P \neq \emptyset$. Then there exist $p_1, \ldots, p_{k+\ell} \in P$ and $m \in M$ such that

$$p_1 + \cdots + p_k = p_{k+1} + \cdots + p_{k+\ell} = m.$$

So there also exist $a_1, \ldots, a_{k+\ell} \in A$ with each $a_i \in p_i + \mathbb{I}$. Then

$$a_{k+1} + \cdots + a_{k+\ell} \in (m + \mathbb{I}) \cap (\ell A),$$

whence we conclude that $m + \mathbb{I} \subseteq \ell A$. Finally,

$$a_1 + \cdots + a_k \in (m + \mathbb{I}) \cap (kA) \subseteq (\ell A)$$

contradicts $A$ being $(k, \ell)$-sum-free. This completes the proof. □

Next, we establish lower bounds by constructing large $(k, \ell)$-sum-free sets. The following lemma generalizes a common tool in the study of $(k, \ell)$-sum-free sets in finite groups.

**Lemma 4.** *Fix any positive integers $1 \leq k < \ell$ and any compact abelian group $G$. If $G$ admits a surjective measurable homomorphism $\phi$ onto the compact abelian group $H$, then $\lambda_{k,\ell}(H) \leq \lambda_{k,\ell}(G)$.*

*Proof.* Let $S \subset H$ be a $(k, \ell)$-sum-free set of density $\mu$. Then $A = \phi^{-1}(S) \subset G$ is a $(k, \ell)$-sum-free set ($\phi(kA)$ and $\phi(\ell A)$ are disjoint in $H$) with the same density (corresponding probability Haar measure). □

Lower bounds now follow from the obvious choices for $H$.

**Lemma 5.** *For any positive integers $1 \leq k < \ell$ and any compact abelian group $G = \mathbb{I} \times M$, we have*

$$\max \{\lambda_{k,\ell}(M), \lambda_{k,\ell}(\mathbb{I})\} \leq \lambda_{k,\ell}(G).$$

*Proof.* Apply Lemma 4 with $H = M$ and $H = \mathbb{I}$. □

When $\mathbb{I}$ is nontrivial, we can bound $\lambda_{k,\ell}(\mathbb{I})$ from below using Pontryagin duality.

**Lemma 6.** *For any positive integers $1 \leq k < \ell$ and any nontrivial compact connected abelian group $G$, we have*

$$\lambda_{k,\ell}(G) \geq \frac{1}{k + \ell}.$$

*Proof.* Consider the Pontryagin dual $\hat{G}$ (the group of characters of $G$). It is known (see, e.g., [8], Theorem 24.25) that a compact abelian group is connected if and only if its Pontryagin dual is torsion-free. Thus, $\hat{G}$ is torsion-free, and it is nontrivial since $G$ is nontrivial. Let $\chi \in \hat{G}$ be an element of infinite order. Since $\chi(G)$ is closed and dense in $\mathbb{T}$, we conclude that $\chi(G) = \mathbb{T}$. By Lemma 4, we have $\lambda_{k,\ell}(\mathbb{T}) \leq \lambda_{k,\ell}(G)$. Finally, note that the set

$$S = \left( \frac{k}{\ell^2 - k^2}, \frac{\ell}{\ell^2 - k^2} \right) \subset \mathbb{T}$$

is a $(k, \ell)$-sum-free set with measure $\frac{1}{k+\ell}$: the intervals $kS = \left( \frac{k^2}{\ell^2 - k^2}, \frac{k\ell}{\ell^2 - k^2} \right)$ and $\ell S = \left( \frac{k\ell}{\ell^2 - k^2}, \frac{\ell^2}{\ell^2 - k^2} \right)$ are disjoint in $\mathbb{T}$. $\square$

At last, we show how these results imply Theorem 3.

*Proof of Theorem 3.* We condition on whether or not $\mathbb{I}$ is trivial. If $\mathbb{I}$ is trivial, then $\lambda_{k,\ell}(\mathbb{I}) = 0$ and $G$ is isomorphic to $M$. In this case, the first statement of the theorem holds trivially. If $\mathbb{I}$ is nontrivial, then it suffices to observe that the upper bound of Theorem 5 coincides with the lower bound of Lemma 5 (by Lemma 6). $\square$

## 3. Discussion

Theorem 3 completely determines $\lambda_{k,\ell}(G)$ in terms of the largest possible $(k, \ell)$-sum-free sets of its "connected" and "discrete" parts when the identity component of $G$ is open. In the finite case (cf. Theorem 2), one must take into consideration the largest $(k, \ell)$-sum-free sets in all subgroups; our Theorem 3 shows that for compact $G = \mathbb{I} \times M$, it suffices to look for $(k, \ell)$-sum-free sets in only $\mathbb{I}$ and $M$.

There remain many interesting questions in the case where the identity component of $G$ is not open. Profinite groups (totally disconnected compact groups) are a particularly natural avenue for further inquiry. Consider, for instance, the case where $G$ is the direct product of countably many finite cyclic groups:

$$G = (\mathbb{Z}_2)^{e_2} \times (\mathbb{Z}_3)^{e_3} \times (\mathbb{Z}_4)^{e_4} \times \cdots$$

(with the $e_i$'s either finite or $\infty$). Roughly speaking, the measurable subsets of $M$ can be approximated by subsets of the form $S \times (G/H)$, where $H$ is a finite subgroup of $G$ and $S \subseteq H$. Then we expect $\lambda_{k,\ell}(G) = \sup\{\lambda_{k,\ell}(H)\}$, where $H$ ranges over the finite subgroups of $G$. As a starting point, Theorem 2 provides lower bounds. When $k = 1$ and $\ell = 2$, we can also apply Theorem 1: for example, $\lambda_{1,2}((\mathbb{Z}_p)^\infty) = \lceil \frac{p-1}{3} \rceil \cdot \frac{1}{p}$.

The problem of finding $(k,\ell)$-sum-free subsets when $\mathbb{I}$ is a $d$-dimensional torus and $M$ is finite motivates a set of related questions for finite abelian groups. Consider the maps $\phi_n$ taking subsets of $\mathbb{T}$ to subsets of $\mathbb{Z}_n$ via

$$A \mapsto \left\{ i \in \mathbb{Z}_n : \left( \frac{i}{n}, \frac{i+1}{n} \right) \subseteq A \right\}.$$

The fact $\left(\frac{i}{n}, \frac{i+1}{n}\right) + \left(\frac{j}{n}, \frac{j+1}{n}\right) = \left(\frac{i+j}{n}, \frac{i+j+2}{n}\right)$ implies that for any sets $A, B \subseteq \mathbb{T}$,

$$\{0,1\} + \phi_n(A) + \phi_n(B) \subseteq \phi_n(A+B),$$

with equality when (but not only when) $A$ and $B$ are the unions of open intervals of the form $\left(\frac{i}{n}, \frac{i+1}{n}\right)$. We remark that any open sets can be arbitrarily well approximated from the inside in this manner for large enough $n$. This observation motivates the following set of definitions.

Let $A$, $B$, and $C$ be subsets of a finite abelian group $G$. For lack of better notation, let $A *_C B = C + A + B$ be a "modified" sumset of $A$ and $B$, and let

$$k *_C A = \underbrace{A *_C \cdots *_C A}_{k} = kA + (k-1)C$$

be the $k$-fold iteration of this sumset. We can investigate $(k,\ell)$-sum-free sets under this operation (i.e., with respect to fixed $C$) by defining

$$\lambda^C_{k,\ell}(G) = \max \left\{ \frac{|A|}{|G|} : (k *_C A) \cap (\ell *_C A) = \emptyset \right\}.$$

Of course, $\lambda^{\{0\}}_{k,\ell}(G) = \lambda_{k,\ell}(G)$ recovers the ordinary definition of the maximum size of a $(k,\ell)$-sum-free set.

When $G = \mathbb{Z}_n$ and $C = \{0,1\}$, we see that $\lambda^{\{0,1\}}_{k,\ell}(\mathbb{Z}_n)$ reflects the problem of finding open $(k,\ell)$-sum-free subsets of $\mathbb{T}$. Theorem 3 for $G = \mathbb{T}$ gives

$$\lambda^{\{0,1\}}_{k,\ell}(\mathbb{Z}_n) \leq \frac{1}{k+\ell}.$$

(This bound also follows from Kneser's Theorem for finite groups.) Note that equality is achieved at least whenever $n$ is a multiple of $\ell^2 - k^2$ (Lemma 6). This group invariant seems an interesting object of study.

**Question 1.** *What can we say about $\lambda^{\{0,1\}}_{k,\ell}(\mathbb{Z}_n)$? Which values of $n$ satisfy $\lambda^{\{0,1\}}_{k,\ell}(\mathbb{Z}_n) = \frac{1}{k+\ell}$? For fixed $1 \leq k < \ell$, which $n$ minimizes $\lambda^{\{0,1\}}_{k,\ell}(\mathbb{Z}_n)$?*

Other compact abelian groups of the form $\mathbb{T}^d \times M$ (with $M$ finite) analogously give rise to the more general problem of computing $\lambda^C_{k,\ell}(G)$ with $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d} \times M$ and $C = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d, 0) : \varepsilon_i \in \{0,1\}\}$. Finally, we propose that other choices of

$C$ could lead to questions of future interest. (Zhang [16] has recently investigated some of these questions.)

**Acknowledgment.** The author wishes to thank Milo Brandt and Stefan Steinerberger for reading drafts of this paper. The author is also grateful to Sean Eberhard for suggesting the use of Pontryagin duality for the case of nontrivial connected $\mathbb{I}$. This paper benefitted from the suggestions of an anonymous referee.

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