SOME IDENTITIES INVOLVING CENTRAL BINOMIAL COEFFICIENTS AND CATALAN NUMBERS

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Abstract

We prove several identities involving the central binomial coefficients $B_n = \binom{2n}{n}$ and the Catalan numbers $C_n = \binom{2n}{n}/(n+1)$. Our main result states that for $n \geq 1$ we have

$$\sum_{k=0}^{n-1} (-1)^k B_k (B_{n-k} - C_{n-k}) = (-1)^n \Big(\frac{1}{2} B_{n+1} - 2^n B_{[(n+1)/2]} \Big).$$

We offer a combinatorial proof and a short algebraic proof for this identity and we present several related summation identities. Then we provide some series representations for $1/\pi$ and related constants, where the terms of the series depend on B_n and C_n. In addition, we apply an integral representation for C_n to show that the sequence $\{(-1)^{n+1} \int_0^1 P_{2n+1}(2x-1)dx\}_{n \geq 0}$ is completely monotonic. Here, P_n denotes the Legendre polynomial.

1. Introduction

The central binomial coefficients and the Catalan numbers are defined by

$$B_n = \binom{2n}{n} \quad \text{and} \quad C_n = \frac{1}{n+1} \binom{2n}{n} \quad (n = 0, 1, 2, \dots),$$

respectively. We have the integral representations

$$B_n = \frac{2^{2n+1}}{\pi} \int_0^\infty \frac{dx}{(x^2+1)^{n+1}} \quad \text{and} \quad C_n = (-1)^{n+1} 2^{2n+1} \int_0^1 P_{2n+1}(x-1)dx, \quad (1)$$

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where P_n denotes the classical Legendre polynomial. The formula for C_n was discovered by Mansour and Sun [9] in 2009. (Note that the formula given in [9, Ex. 2.2] contains a misprint.)

The generating functions for B_n and C_n are given by

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} B_n x^n \quad \text{and} \quad \frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} C_n x^n.$$

The behavior of B_n and C_n for large n is determined by the asymptotic formulae

$$B_n \sim \frac{4^n}{\sqrt{\pi n}} \quad \text{and} \quad C_n \sim \frac{4^n}{\sqrt{\pi} n^{3/2}} \quad (n \to \infty).$$

Various authors studied number theoretic properties of B_n and C_n. No central binomial coefficient with $n > 4$ is squarefree. The elegant congruence

$$B_p \equiv 2 \pmod{p^3}$$

is valid for all prime numbers $p \geq 5$. All Catalan numbers are integers. C_n is prime if and only if $n = 2$ or $n = 5$.

In the literature, we can find many series representations for classical constants which involve B_n or C_n. As examples, we mention

$$\frac{\pi}{3} = \sum_{n=0}^{\infty} \frac{B_n}{(2n+1)16^n}, \quad \frac{48}{125} \log 2 = -\frac{9}{25} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{B_n} \Big(\frac{9}{4}\Big)^n, \quad \frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(n+1)C_n^2}{16^n}.$$

The representation

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 B_n}$$

was used by Apéry to prove that $\zeta(3)$ is irrational.

The central binomial coefficients and the Catalan numbers play an important role in many combinatorial problems and they have remarkable applications in various fields, like, for instance, graph theory, analysis and statistics. Detailed information on this subject can be found, for instance, in Koshy [8], Sloane [13] and Stanley [15].

The work in this paper is inspired by some recently published papers on convolution identities involving B_n and C_n. In 2013, Witula et al. [18] published the identity

$$\sum_{k=0}^{n} \frac{1}{2k+1} B_k B_{n-k} = \frac{16^n}{(2n+1)B_n}. \tag{2}$$

They applied methods from real analysis to establish (2). The formula

$$\sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} C_{2i} C_{2j} C_{2k} = \frac{1}{2} B_{2n+1} \tag{3}$$

was proved in 2012 by Nagy [11], who offered a combinatorial proof for (3). Several related identities are presented in Lemma 2 and Lemma 3 given in the next section.

It is the aim of this paper to continue the work on this subject and to present various identities involving B_n and C_n. Our main result is given in the next section. We offer combinatorial and algebraic proofs for an identity which is related to (2). The combinatorial proof also serves as a survey on convolution identities involving central binomial coefficients and Catalan numbers. Some additional convolution identities are given in Section 3. In Section 4, we present series representations for $1/\pi$, $1/\pi^{3/2}$ and $\sqrt{\pi}$, where the terms of the series depend on B_n and C_n. We conclude the paper with a monotonicity theorem. In Section 5, we show that the representation for C_n given in (1) can be used to prove that the sequence $\{(-1)^{n+1}\int_0^1 P_{2n+1}(x-1)dx\}_{n\geq 0}$ is completely monotonic.

2. Main Result

The following identity is the main result of our paper. We provide two proofs.

Theorem 1. *For all integers $n \geq 1$ we have*

$$\sum_{k=0}^{n-1}(-1)^k B_k(B_{n-k}-C_{n-k}) = (-1)^n\left(\frac{1}{2}B_{n+1} - 2^n B_{[(n+1)/2]}\right), \tag{4}$$

where $[\cdot]$ denotes the greatest integer function.

A combinatorial proof. Before proving the theorem, we collect a bunch of convolution identities involving central binomial coefficients and Catalan numbers. All of them can be proved combinatorially. For completeness, we include all variations of these identities, but only a few of them will be used in the proof of Theorem 1.

In our proofs path-counting arguments are used. The terminology of paths is from [11], we quickly recall the required definitions here. A *path* is a finite sequence of 'up-steps' $(1,1)$ and 'down-steps' $(1,-1)$. The *length* of a path is the number of its steps (i.e. the number of elements in the sequence). Paths are visualized in the two-dimensional Cartesian coordinate system in a natural way: instead of a formal definition, we just refer the reader to Figures 1-2. We will also use phrases that come from this visualization, like "the path P starts from the origin" or "P never goes below the x-axis", etc. We denote the up-steps and down-steps by $\nearrow$ and $\searrow$, respectively. We will work with four types of special paths. We call a path *balanced* if, when starting from the origin, it ends on the x-axis, i.e. if it has the same number of up-steps as down-steps. We call a path *non-zero* if, when starting from the origin, it never returns to the x-axis after the first step. We call a path *Dyck path*, if it is balanced and, when starting from the origin, it never goes (strictly) below the

x-axis. As introduced in [11], we call a path *even-zeroed* if, when starting from the origin, all of its x-intercepts are divisible by 4. See Figures 1-2.

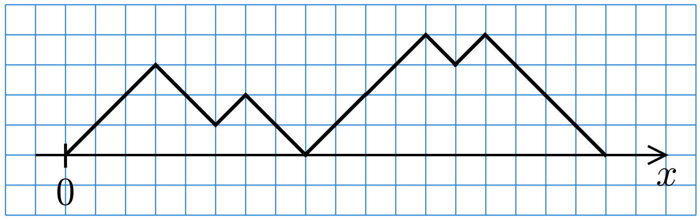

Figure 1: A Dyck path of length 18

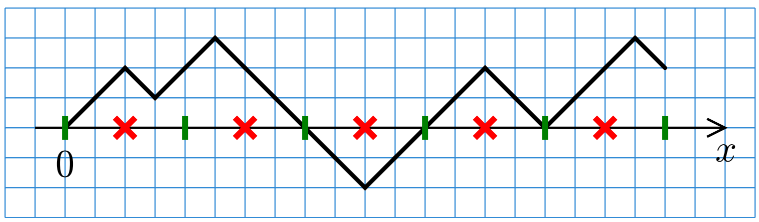

Figure 2: An even-zeroed path of length 20

We need the following combinatorial interpretations of central binomial identities and Catalan numbers.

Lemma 1. (a) *B_n counts the number of balanced paths of length $2n$.*

(b) *B_n counts the number of non-zero paths of length $2n$.*

(c) *C_n counts the number of Dyck paths of length $2n$.*

(d) *C_{2n} counts the number of* balanced *even-zeroed paths of length $4n$.*

Proof. (a) The first statement is obvious, since a balanced path consists of n up-steps and n down-steps, with no restriction on the order of steps.

(b) See [5] for combinatorial proofs.

(c) The third statement is one of the fundamental combinatorial interpretations of Catalan numbers, for example, see Theorem 1.5.1 in [15].

(d) See [12], or Lemma 3 in [11]. □

Now we prove the basic convolution identities involving central binomial coefficients and Catalan numbers. They can be easily deduced from Lemma 1.

Lemma 2. *Let n be a non-negative integer. Then:*

$$\text{(a)} \sum_{k=0}^{n} C_k C_{n-k} = C_{n+1}; \quad \text{(b)} \sum_{k=0}^{n} B_k C_{n-k} = \frac{1}{2} B_{n+1}; \textit{ and } \quad \text{(c)} \sum_{k=0}^{n} B_k B_{n-k} = 4^n.$$

Proof. (a) The first identity is the well-known recurrence relation for Catalan numbers. See Section 1.2 in [15] for more details.

(b) We will prove the multiplied form $\sum_{j=0}^{n} 2C_j B_{n-j} = B_{n+1}$ by double counting (where $j = n-k$). Both sides count the number of balanced paths of length $2n+2$. This is clearly true for the right-hand side by Lemma 1 (a).

On the left-hand side, the term $2C_j B_{n-j}$ counts the number of those balanced paths of length $2n+2$ for which the first (non-origin) x-intercept is at $2j+2$. (Such an x-intercept must exist, as the balanced path ends on the x-axis.) This is because there are $2C_j$ ways for the segment between the points $(0,0)$ and $(2j+2,0)$, and there are B_{n-j} ways for the segment between $(2j+2,0)$ and $(2n+2,0)$: the $(0,0) \rightsquigarrow (2j+2,0)$ segment either has the form $\nearrow D \searrow$ where D is Dyck path of length $2j$, or it has the form $\searrow \overline{D} \nearrow$ where $\overline{D}$ is a Dyck path of length $2j$ reflected over the x-axis, and both D and $\overline{D}$ can be chosen in C_j ways by Lemma 1 (c). The $(2j+2,0) \rightsquigarrow (2n+2,0)$ segment is a balanced path of length $2n-2j$ without further restrictions, and hence the number of possible $(2j+2,0) \rightsquigarrow (2n+2,0)$ segments is B_{n-j} by Lemma 1 (a). See Figure 3.

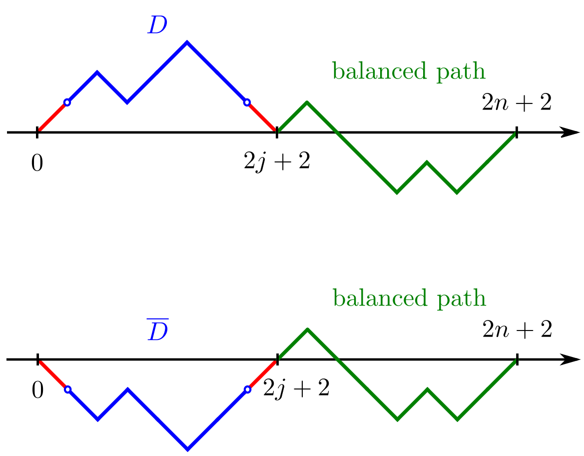

Figure 3: Illustration of the two cases in the proof of Lemma 2 (b)

(c) The third identity is widely known, but its combinatorial proof is a bit tricky. See [16] for a collection of some elegant proofs, one of them using Lemma 1 (a) and Lemma 1 (b). □

Now we consider alternating convolution identities and restricted convolution identities with condition on the parity of indices. They are notably more difficult to prove than the basic convolution identities in Lemma 2.

Remark 1. We note first that if we know the closed form of the convolution

$$\Lambda_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for some given numbers $a_0, \ldots, a_n, b_0, \ldots, b_n$, then finding a closed form for the alternating convolution

$$\Lambda_n^{\text{alt}} = \sum_{k=0}^{n} (-1)^k a_k b_{n-k}$$

is equivalent to finding a closed form for the restricted convolution

$$\Lambda_n^{\text{even}} = \sum_{\substack{k \text{ is even} \\ k \in \{0,\ldots,n\}}} a_k b_{n-k},$$

because $\Lambda_n^{\text{alt}} = 2\Lambda_n^{\text{even}} - \Lambda_n$. Finally, we note that the restricted convolution

$$\Lambda_n^{\text{odd}} = \sum_{\substack{k \text{ is odd} \\ k \in \{0,\ldots,n\}}} a_k b_{n-k}$$

can also be easily obtained from Λ_n and Λ_n^{even}, namely, $\Lambda_n^{\text{odd}} = \Lambda_n - \Lambda_n^{\text{even}}$.

Lemma 3. *Let n be a non-negative integer. Then:*

(a) $$\sum_{\substack{k \text{ is even} \\ k \in \{0,\ldots,n\}}} C_k C_{n-k} = \begin{cases} 2^n C_{n/2}, & \text{if } n \text{ is even}, \\ \frac{1}{2} C_{n+1}, & \text{if } n \text{ is odd}. \end{cases}$$

(b) $$\sum_{k=0}^{n} (-1)^k C_k C_{n-k} = \begin{cases} 2^{n+1} C_{n/2} - C_{n+1}, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

(c) $$\sum_{\substack{k \text{ is even} \\ k \in \{0,\ldots,n\}}} C_k B_{n-k} = \begin{cases} 2^n B_{n/2}, & \text{if } n \text{ is even}, \\ 2^{n-1} B_{(n+1)/2}, & \text{if } n \text{ is odd}. \end{cases}$$

(d) $$\sum_{\substack{k \text{ is even} \\ k \in \{0,\ldots,n\}}} B_k C_{n-k} = \begin{cases} 2^n B_{n/2}, & \text{if } n \text{ is even}, \\ \frac{1}{2} B_{n+1} - 2^{n-1} B_{(n+1)/2}, & \text{if } n \text{ is odd}. \end{cases}$$

(e) $$\sum_{k=0}^{n} (-1)^k B_k C_{n-k} = \begin{cases} 2^{n+1} B_{n/2} - \frac{1}{2} B_{n+1}, & \text{if } n \text{ is even}, \\ \frac{1}{2} B_{n+1} - 2^n B_{(n+1)/2}, & \text{if } n \text{ is odd}. \end{cases}$$

(f) $\sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} B_k B_{n-k} = \begin{cases} 2^{2n-1} + 2^{n-1} B_{n/2}, & \textit{if } n \textit{ is even}, \\ 2^{2n-1}, & \textit{if } n \textit{ is odd}. \end{cases}$

(g) $\sum_{k=0}^{n} (-1)^k B_k B_{n-k} = \begin{cases} 2^n B_{n/2}, & \textit{if } n \textit{ is even}, \\ 0, & \textit{if } n \textit{ is odd}. \end{cases}$

Proof. Applying the preceding Remark 1, the relationship $\Lambda_n^{\text{alt}} = 2\Lambda_n^{\text{even}} - \Lambda_n$ and Lemma 2 implies that statement (a) is equivalent to (b), statement (d) is equivalent to (e), and statement (f) is equivalent to (g).

Statement (c) is also equivalent to (d), as for even n,

$$\sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} C_k B_{n-k} = \sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} B_k C_{n-k},$$

and for odd n,

$$\sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} C_k B_{n-k} = \sum_{\substack{k \text{ is odd} \\ k\in\{0,\ldots,n\}}} B_k C_{n-k},$$

hence the sum of the left-hand sides of (c) and (d) is

$$\sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} C_k B_{n-k} + \sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} B_k C_{n-k} = \sum_{k=0}^{n} B_k C_{n-k} = \frac{1}{2} B_{n+1},$$

by Lemma 2 (b). So it is enough to prove the statements (a), (c) and (g).

(a) This identity is trivial for odd n, because then

$$\sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} C_k C_{n-k} = \sum_{\substack{k \text{ is odd} \\ k\in\{0,\ldots,n\}}} C_k C_{n-k},$$

and hence

$$2 \cdot \sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} C_k C_{n-k} = \sum_{k=0}^{n} C_k C_{n-k} = C_{n+1}$$

by Lemma 2 (a). In case of even n, the identity in (a) becomes the hardest to prove combinatorially among the identities of this lemma. The identity is attributed to Shapiro, and based on Lemma 1 (d), combinatorial proofs were given in [11] and [7].

(c) For even n, this identity is proved in [11] by showing first that the left-hand side counts the number of even-zeroed paths of length $2n$ (this is Lemma 6.a in [11], with $n \leftarrow 2n$ substitution), and then by verifying that the number of these paths is $2^n B_{n/2}$ (this is Lemma 7 in [11]).

We can adopt the proof for odd n, too. It is still true for odd n, that the sum

$$\sum_{\substack{k \text{ is even} \\ k\in\{0,\ldots,n\}}} C_k B_{n-k}$$

counts the number of even-zeroed paths of length $2n$, since for even k, the term $C_k B_{n-k}$ counts the number of those even-zeroed paths of length $2n$ whose right-most x-intercept is at $2k$. This is because C_k is the number of possibilities for the segment $(0, 0) \rightsquigarrow (2k, 0)$ by Lemma 1 (d), and B_{n-k} is the number of possibilities for the remaining $2n - 2k$ steps of the path (which form a non-zero path) by Lemma 1 (b), see Figure 4.

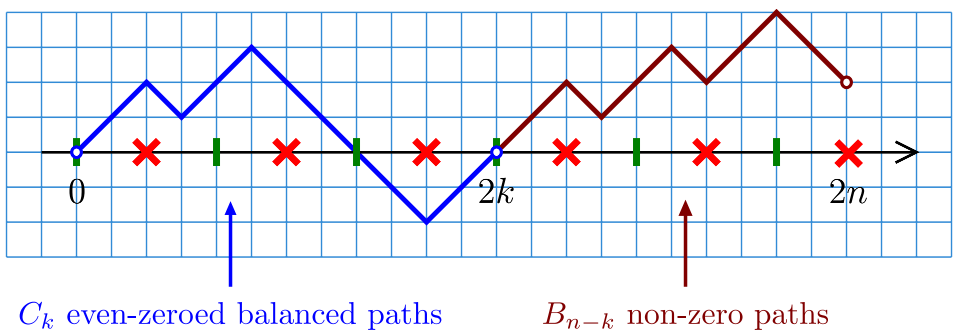

Figure 4: The number of even-zeroed paths of length $2n$

Let E_{2n} denote the number of even-zeroed paths of length $2n$ for any n. We are left to show that $E_{2n} = 2^{n-1}B_{(n+1)/2}$ for odd n. We could mimic the proof used in the case of even n, but we point out instead that the case of odd n follows from the case of even n. For odd n, every even-zeroed path of length $2n$ can be extended by two arbitrary last steps to obtain an even-zeroed path of length $2n+2$. (Note that the $(2n + 2)$-th step of the path is allowed to step onto the x-axis, as $2n + 2$ is divisible by 4; and the $(2n + 1)$-th step cannot step onto the x-axis by parity argument.) And clearly, every even-zeroed path of length $2n + 2$ is obtained in this way exactly once, hence $E_{2n+2} = 4E_{2n}$. But we know from the even case that $E_{2n+2} = 2^{n+1}B_{(n+1)/2}$, and so $E_{2n} = E_{2n+2}/4 = 2^{n-1}B_{(n+1)/2}$, as stated.

(g) The alternating convolution formula of the central binomial coefficients is easy to prove by applying generating functions. It has an elegant combinatorial proof using random colored permutations [14]. The identity can also be proved using a path-counting argument; this is done in [11]. □

After these preliminaries, Theorem 1 is just a corollary of the above.

$$
\begin{aligned}
\sum_{k=0}^{n-1}(-1)^k B_k(B_{n-k}-C_{n-k}) &= \sum_{k=0}^{n}(-1)^k B_k(B_{n-k}-C_{n-k})\\
&= \sum_{k=0}^{n}(-1)^k B_k B_{n-k} - \sum_{k=0}^{n}(-1)^k B_k C_{n-k}\\
&= \begin{cases} 2^n B_{n/2} - \left(2^{n+1}B_{n/2} - \frac{1}{2}B_{n+1}\right), & \text{if } n \text{ is even,}\\ -\left(\frac{1}{2}B_{n+1} - 2^n B_{(n+1)/2}\right), & \text{if } n \text{ is odd,}\end{cases}\\
&= \begin{cases} \frac{1}{2}B_{n+1} - 2^n B_{n/2}, & \text{if } n \text{ is even,}\\ 2^n B_{(n+1)/2} - \frac{1}{2}B_{n+1}, & \text{if } n \text{ is odd,}\end{cases}\\
&= (-1)^n\left(\frac{1}{2}B_{n+1} - 2^n B_{[(n+1)/2]}\right).
\end{aligned}
$$

In the first equality, the upper limit $n-1$ of the sum is replaced to n, as $B_0 - C_0 = 0$. The third equality is a direct application of Lemma 3 (g) and Lemma 3 (e). □

An algebraic proof. We define

$$S_n = \sum_{k=0}^{n}(-1)^k B_k B_{n-k} \quad \text{and} \quad T_n = \sum_{k=0}^{n-1}(-1)^k B_k(B_{n-k}-C_{n-k}). \tag{5}$$

Then,

$$S_{n+1} = \sum_{k=0}^{n-1}(-1)^k B_k B_{n+1-k} + 2\,(-1)^n B_n + (-1)^{n+1}B_{n+1}.$$

Using

$$B_{n+1-k} = 2B_{n-k} + 2(B_{n-k}-C_{n-k})$$

yields

$$
\begin{aligned}
S_{n+1} &= 2\sum_{k=0}^{n-1}(-1)^k B_k B_{n-k} + 2\sum_{k=0}^{n-1}(-1)^k B_k(B_{n-k}-C_{n-k}) + 2\,(-1)^n B_n\\
&\quad +(-1)^{n+1}B_{n+1}\\
&= 2\sum_{k=0}^{n}(-1)^k B_k B_{n-k} + 2\sum_{k=0}^{n-1}(-1)^k B_k(B_{n-k}-C_{n-k}) + (-1)^{n+1}B_{n+1}\\
&= 2S_n + 2T_n + (-1)^{n+1}B_{n+1}.
\end{aligned}
$$

It follows that

$$T_n = \frac{1}{2}S_{n+1} - S_n + \frac{(-1)^n}{2}B_{n+1}.$$

It remains to show that

$$\frac{1}{2}S_{n+1} - S_n = (-1)^{n+1}2^n B_{[(n+1)/2]}.$$

We consider two cases. If $n = 2N$, then we conclude from Lemma 3 (g) that

$$\frac{1}{2}S_{n+1} - S_n = -2^{2N} B_N = (-1)^{n+1}2^n B_{[(n+1)/2]}.$$

And, if $n = 2N - 1$, then Lemma 3 (g) reveals that

$$\frac{1}{2}S_{n+1} - S_n = 2^{2N-1} B_N = (-1)^{n+1}2^n B_{[(n+1)/2]}.$$

This completes the proof of Theorem 1. □

3. More Identities

In this section, we present some summation identities which are closely related to (4).

Theorem 2. *For all integers $n \geq 1$ we have*

$$\sum_{k=0}^{n-1} B_k(B_{n-k} - C_{n-k}) = 4^n - \frac{1}{2}B_{n+1}. \tag{6}$$

Proof. The identity in the theorem is just the difference of the identities in Lemma 2 (c) and Lemma 2 (b). □

An application of Theorems 1 and 2 leads to the following corollary.

Corollary 1. *For all integers $n \geq 1$ we have*

$$\sum_{k=0}^{n-1} B_{2k}(B_{2(n-k)} - C_{2(n-k)}) = 2^{2n-1}(4^n - B_n), \tag{7}$$

$$\sum_{k=0}^{n-1} B_{2k}(B_{2(n-k)-1} - C_{2(n-k)-1}) = 2^{4n-3} + 4^{n-1}B_n - \frac{1}{2}B_{2n}, \tag{8}$$

$$\sum_{k=0}^{n-1} B_{2k+1}(B_{2(n-k)} - C_{2(n-k)}) = 4^n(2^{2n+1} - B_{n+1}), \tag{9}$$

$$\sum_{k=0}^{n-1} B_{2k+1}(B_{2(n-k)-1} - C_{2(n-k)-1}) = 2^{4n-1} + 2^{2n-1}B_n - \frac{1}{2}B_{2n+1}. \tag{10}$$

Proof. In view of Remark 1, these identities are easy consequences of (4) and (6). Using the notations of Remark 1, the identities (7)-(10) are the closed forms of $\Lambda_{2n}^{\text{even}}$, $\Lambda_{2n-1}^{\text{even}}$, $\Lambda_{2n+1}^{\text{odd}}$ and $\Lambda_{2n}^{\text{odd}}$, respectively, with the numbers $a_i = B_i$ and $b_i = B_i - C_i$. Theorem 2 provides a closed form of Λ_N, and Theorem 1 provides a closed form of Λ_N^{alt} for any N. These results lead to the required closed forms Λ_N^{even} and Λ_N^{odd}, using the relationships $\Lambda_N^{\text{alt}} = 2\Lambda_N^{\text{even}} - \Lambda_N$ and $\Lambda_N^{\text{odd}} = \Lambda_N - \Lambda_N^{\text{even}}$. □

The next two theorems offer relatives of (4) and (6).

Theorem 3. *For all integers $n \geq 1$ we have*

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{n-k+2} B_k(B_{n-k} - C_{n-k}) = (-1)^n \frac{7n+12}{6(n+2)} B_{n+1} + \frac{2^n}{3}\theta_n B_{[(n+1)/2]}, \quad (11)$$

where

$$\theta_n = \begin{cases} -2(5n+6)/(n+2), & \text{if } n \text{ is even,} \\ 5, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let

$$U_n = \sum_{k=0}^{n-1} \frac{(-1)^k}{n-k+2} B_k(B_{n-k} - C_{n-k})$$

and let S_n and T_n be the sums defined in (5). Using

$$B_{n+1-k} - C_{n+1-k} = B_{n-k} + 3(B_{n-k} - C_{n-k}) - \frac{3}{n-k+2}(B_{n-k} - C_{n-k})$$

yields

$$\begin{aligned} T_{n+1} &= \sum_{k=0}^{n} (-1)^k B_k (B_{n+1-k} - C_{n+1-k}) \\ &= \sum_{k=0}^{n} (-1)^k B_k B_{n-k} + 3\sum_{k=0}^{n-1} (-1)^k B_k (B_{n-k} - C_{n-k}) \\ &\quad -3\sum_{k=0}^{n-1} \frac{(-1)^k}{n-k+2} B_k (B_{n-k} - C_{n-k}) \\ &= S_n + 3T_n - 3U_n. \end{aligned}$$

Thus,

$$U_n = \frac{1}{3}S_n + T_n - \frac{1}{3}T_{n+1}.$$

We consider two cases.

Case 1. n is even. Let $n = 2N$. Applying Lemma 3 (g), identity (4) and $B_{k+1} = 2(2k+1)B_k/(k+1)$ gives

$$\begin{aligned}\frac{1}{3}S_n + T_n - \frac{1}{3}T_{n+1} &= \frac{1}{3}S_{2N} + T_{2N} - \frac{1}{3}T_{2N+1}\\ &= \frac{2^{2N}}{3}B_N + \frac{1}{2}B_{2N+1} - 2^{2N}B_N\\ &\quad + \frac{1}{3}\Big(\frac{1}{2}B_{2(N+1)} - 2^{2N+1}B_{N+1}\Big)\\ &= \frac{7n+12}{6(n+2)}B_{n+1} - \frac{5n+6}{3(n+2)}2^{n+1}B_{[(n+1)/2]}.\end{aligned}$$

Case 2. n is odd. Let $n = 2N-1$. Then,

$$\begin{aligned}\frac{1}{3}S_n + T_n - \frac{1}{3}T_{n+1} &= \frac{1}{3}S_{2N-1} + T_{2N-1} - \frac{1}{3}T_{2N}\\ &= -\frac{1}{2}B_{2N} + 2^{2N-1}B_N - \frac{1}{3}\Big(\frac{1}{2}B_{2N+1} - 2^{2N}B_N\Big)\\ &= -\frac{7n+12}{6(n+2)}B_{n+1} + \frac{5\cdot 2^n}{3}B_{[(n+1)/2]}.\end{aligned}$$

The proof of Theorem 3 is complete. □

The following companion of (11) is related to (2).

Theorem 4. *For all integers $n \geq 1$ we have*

$$\sum_{k=0}^{n-1}\frac{1}{n-k+2}B_k(B_{n-k} - C_{n-k}) = \frac{n}{6(n+2)}B_{n+1}. \tag{12}$$

Proof. The proof is very similar to the proof of (11). Instead of S_n and T_n we consider the sums

$$S_n^* = \sum_{k=0}^{n} B_k B_{n-k} \quad \text{and} \quad T_n^* = \sum_{k=0}^{n-1} B_k(B_{n-k} - C_{n-k}).$$

Moreover, we apply Lemma 2 (c) and identity (6) instead of Lemma 3 (g) and identity (4). Then we obtain (12). □

We conclude this section with a counterpart of Corollary 1.

Corollary 2. *For all integers $n \geq 1$ we have*

$$\sum_{k=0}^{n-1}\frac{1}{n-k+1}B_{2k}(B_{2(n-k)} - C_{2(n-k)}) = -\frac{5n+3}{3(n+1)}2^{2n+1}B_n + \frac{4n+3}{3(n+1)}B_{2n+1},$$

$$\sum_{k=0}^{n-1} \frac{1}{2(n-k)+1} B_{2k}(B_{2(n-k)-1} - C_{2(n-k)-1}) = \frac{5}{3} 4^{n-1} B_n - \frac{1}{2} B_{2n},$$

$$\sum_{k=1}^{n-1} \frac{1}{n-k+1} B_{2k-1}(B_{2(n-k)} - C_{2(n-k)}) = -\frac{5}{3} 2^{2n-1} B_n + \frac{2(4n+1)}{3(2n+1)} B_{2n},$$

$$\sum_{k=0}^{n-1} \frac{1}{2(n-k)+1} B_{2k+1}(B_{2(n-k)-1} - C_{2(n-k)-1}) = \frac{5n+3}{3(n+1)} 4^n B_n - \frac{1}{2} B_{2n+1}.$$

Proof. Analogously to the proof of Corollary 1, these identities are implied by Theorems 3 and 4, using Remark 1. We omit the details. □

4. Series Representations

In 1992, Ewell [6] published an interesting series representation which relates the Catalan numbers and $1/\pi$,

$$\frac{1}{\pi} = \frac{3}{16} + \frac{9}{4} \sum_{k=1}^{\infty} \frac{4k^2-1}{16^k (k+1)^2} C_{k-1}^2.$$

We modify Ewell's method of proof and obtain series representations for $1/\pi$, $1/\pi^{3/2}$ and $\sqrt{\pi}$, where the terms of the series involve B_k and C_k.

Theorem 5. *We have*

$$\frac{1}{\pi} = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{B_k C_{k-1}}{16^k}, \tag{13}$$

$$\frac{1}{\pi} = \frac{3}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \sum_{\nu=0}^{k} \binom{1/4}{\nu} \binom{3/4}{k-\nu} B_\nu B_{k-\nu}, \tag{14}$$

$$\frac{1}{\pi^{3/2}} = \frac{1}{2\sqrt{2}\Gamma^2(3/4)} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \binom{1/4}{k} B_k, \tag{15}$$

$$\sqrt{\pi} = \frac{3\Gamma^2(3/4)}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \binom{3/4}{k} B_k. \tag{16}$$

Proof. Using

$$(1-z)^\beta = \sum_{k=0}^{\infty} \binom{\beta}{k} (-z)^k \quad (\beta > 0;\ |z| \le 1)$$

and

$$\int_0^{\pi/2} \sin^{2n}(t) dt = \frac{\pi}{2^{2n+1}} B_n \quad (n = 0, 1, 2, \dots)$$

we obtain

$$\begin{aligned}\frac{\pi}{2}-\int_0^{\pi/2}(1-\sin^2(t))^\beta dt &= \int_0^{\pi/2}(1-(1-\sin^2(t))^\beta)dt\\ &= -\int_0^{\pi/2}\sum_{\nu=0}^{\infty}\binom{\beta}{\nu+1}(-\sin^2(t))^{\nu+1}dt\\ &= -\sum_{\nu=0}^{\infty}(-1)^{\nu+1}\binom{\beta}{\nu+1}\int_0^{\pi/2}\sin^{2(\nu+1)}(t)dt\\ &= -\sum_{k=1}^{\infty}(-1)^k\binom{\beta}{k}\int_0^{\pi/2}\sin^{2k}(t)dt\\ &= \frac{\pi}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\binom{\beta}{k}\frac{1}{2^{2k}}B_k.\end{aligned}$$

On the other hand, we have

$$\frac{\pi}{2}-\int_0^{\pi/2}(1-\sin^2(t))^\beta dt=\frac{\pi}{2}-\frac{\sqrt{\pi}\Gamma(\beta+1/2)}{2\Gamma(\beta+1)}\quad(\beta>0).$$

Thus,

$$\frac{\pi}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\binom{\beta}{k}\frac{1}{2^{2k}}B_k=\frac{\pi}{2}-\frac{\sqrt{\pi}\Gamma(\beta+1/2)}{2\Gamma(\beta+1)}.$$

We multiply both sides by $2/\pi$ and simplify. This gives

$$\frac{1}{\sqrt{\pi}}\frac{\Gamma(\beta+1/2)}{\Gamma(\beta+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}\binom{\beta}{k}B_k. \tag{17}$$

Let

$$F(\beta)=\frac{\Gamma(\beta+1/2)}{\Gamma(\beta+1)}.$$

We set in (17) $\beta=1/2$, divide both sides by 2 and apply $F(1/2)=2/\sqrt{\pi}$ and

$$\binom{1/2}{k}=\frac{(-1)^{k+1}}{2^{2k-1}}C_{k-1}\quad(k=1,2,\dots).$$

This leads to (13). Next, we apply (17) with $\beta=1/4$ and $\beta=3/4$. We make use of

$$F(1/4)=\frac{2\sqrt{2}\Gamma^2(3/4)}{\pi}\quad\text{and}\quad F(3/4)=\frac{\sqrt{2}\pi}{3\Gamma^2(3/4)}.$$

Then, we obtain (15) and (16). From (15) and (16) we get by multiplication the series representation for $1/\pi$ given in (14). □

5. A Monotonicity Theorem

In the recent past, numerous authors proved that certain functions which are defined in terms of classical functions, like, for instance, Euler's gamma function and its relatives, are completely monotonic. We recall that a function f is said to be completely monotonic on $[a,\infty)$, if f is continuous on $[a,\infty)$ and satisfies

$$(-1)^k f^{(k)}(x) \geq 0 \quad (x > a; k = 0, 1, 2, \dots).$$

These functions have interesting applications in different fields, like, for instance, potential theory, probability theory and numerical analysis. For detailed information on this subject we refer to the monograph Widder [17, chapter IV] and the research papers Alzer and Berg [2, 3].

The following lemma is a very helpful tool for proving the complete monotonicity of a function. It can be proved by induction and by using the Leibniz rule for differentiation. An extension is given in Bochner [4, p. 83].

Lemma 4. *If the function $(-\log f)'$ is completely monotonic, then f is completely monotonic.*

A sequence $(\mu_n)_{n\geq 0}$ is said to be completely monotonic, if

$$(-1)^k \Delta^k \mu_n \geq 0 \quad (k, n = 0, 1, 2, \dots),$$

where

$$\Delta^0 \mu_n = \mu_n \quad \text{and} \quad \Delta^k \mu_n = \Delta^{k-1}\mu_{n+1} - \Delta^{k-1}\mu_n \quad (k = 1, 2, \dots; n = 0, 1, 2, \dots).$$

In particular, a completely monotonic sequence is decreasing and convex. The next lemma reveals a connection between completely monotonic functions and completely monotonic sequences; see Widder [17, p. 158].

Lemma 5. *If the function f is completely monotonic on $[0,\infty)$, then the sequence $(f(n))_{n\geq 0}$ is completely monotonic.*

In this section, we show that the integral representation for C_n given in (1) can be applied to prove a monotonicity property of the Legendre polynomials, defined by

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} B_{n-k} x^{n-2k}.$$

A collection of the main facts about these functions can be found, for example, in Milovanović et al. [10, chapter 1.2].

Theorem 6. *The sequence*

$$\sigma_n = (-1)^{n+1} \int_0^1 P_{2n+1}(x-1)dx \quad (n = 0, 1, 2, \dots)$$

is completely monotonic.

Proof. We define for $x \geq 0$,

$$h(x) = \frac{1}{(x+1)2^{2x+1}} \frac{\Gamma(2x+1)}{\Gamma^2(x+1)}.$$

Let $\psi = \Gamma'/\Gamma$ be the logarithmic derivative of the gamma function. Using

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \quad 2\psi(2x) = \psi(x) + \psi(x+1/2) + 2\log 2$$

and

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt,$$

where γ denotes Euler's constant (see Abramowitz and Stegun [1, chapter 6]), we obtain by differentiation

$$(-\log h(x))' = \psi(x+2) - \psi(x+1/2) = \int_0^\infty e^{-(x+1)t} \frac{e^{3t/2} - 1}{e^t - 1} dt.$$

Thus,

$$(-1)^k (-\log h(x))^{(k+1)} = \int_0^\infty e^{-(x+1)t} t^k \frac{e^{3t/2} - 1}{e^t - 1} dt > 0 \quad (k = 0, 1, 2, \dots).$$

This means that $(-\log h)'$ is completely monotonic on $[0, \infty)$. An application of Lemma 4 and Lemma 5 reveals that the sequence $(h(n))_{n \geq 0}$ is completely monotonic. From (1) we obtain

$$\sigma_n = \frac{C_n}{2^{2n+1}} = h(n) \quad (n = 0, 1, 2, \dots).$$

It follows that $(\sigma_n)_{n \geq 0}$ is completely monotonic. □

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