AN IDENTITY INVOLVING BINOMIAL COEFFICIENTS EQUIVALENT TO AN IDENTITY OF LAGRANGE

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Abstract

There are many identities involving binomial coefficients such as the Chu–Vandermonde identity and Dixon's identity. In the present paper, we show that an identity involving binomial coefficients and an identity of Lagrange are equivalent. We do this by using another identity involving a recurrence relation. The identity is derived from the binomial theorem and is proved by induction.

1. Introduction and Statement of Result

There are various methods to find identities involving binomial coefficients. For example, examining the coefficient of x^c in the expansion of $(1+x)^a(1+x)^b = (1+x)^{a+b}$, we get the Chu–Vandermonde identity [2]

$$\binom{a+b}{c} = \sum_{k=0}^{c} \binom{a}{k}\binom{b}{c-k}$$

where a, b, c are non-negative integers. Also, Dixon's identity ([2]; see also [1]) below can be proved by induction:

$$\sum_{k} (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!\,b!\,c!}.$$

In this paper, we present a new identity involving binomial coefficients as follows. For natural numbers $m \geq 2$ and $i \leq [\frac{1}{2}m]$, we show that the identity

$$\sum_{k=0}^{i} (-1)^{k+1} \frac{1}{m-k}\binom{m-k}{k}\binom{m-2k}{i-k} = 0$$

and an identity of Lagrange are equivalent. The identity of Lagrange ([3]; see also

[4]) we consider is

$$(u+v)^m + \sum_{i=1}^{[\frac{1}{2}m]} \frac{m}{i}\binom{m-i-1}{i-1}(-uv)^i(u+v)^{m-2i} = u^m + v^m.$$

To prove the equivalence, we use the identity

$$(u+v)^m = \sum_{i=1}^{[\frac{1}{2}m]} N_{(m,i)} u^i v^i (u+v)^{m-2i} + u^m + v^m$$

where

$$N_{(m,i)} = \binom{m}{i} - \sum_{k=1}^{i-1} N_{(m,k)} \binom{m-2k}{i-k}.$$

This identity is derived from the binomial theorem by induction.

2. Deriving From the Binomial Theorem

We derive two identities from the binomial theorem according to the parity of its power.

2.1. An Identity of Odd Power

Let n, i, j be natural numbers with $j \leq i \leq n$ throughout the paper. In this section, let r, s be non-negative integers with $r \leq n,\ s \leq n-1$. We begin by defining several symbols.

Definition 1. The symbols $M_{(2n+1,i,j)}, N_{(2n+1,i)}, G_s, F_{(2n+1,r)}$ are defined as follows:

$$M_{(2n+1,i,j)} = \binom{2n+1}{i} - \sum_{k=1}^{j-1} M_{(2n+1,k,k)} \binom{2(n-k)+1}{i-k}, \tag{1}$$

$$N_{(2n+1,i)} = M_{(2n+1,i,i)}, \tag{2}$$

$$G_s = (u+v)^{2s+1} - \sum_{i=1}^{s} \binom{2s+1}{i} u^i v^i (u^{2(s-i)+1} + v^{2(s-i)+1}), \tag{3}$$

$$\begin{aligned} F_{(2n+1,r)} = &\sum_{i=1}^{r} \binom{2n+1}{i} u^i v^i G_{n-i} \\ &+ \sum_{i=r+1}^{n} \binom{2n+1}{i} u^i v^i (u^{2(n-i)+1} + v^{2(n-i)+1}) + u^{2n+1} + v^{2n+1}. \end{aligned}$$

From (1) and (2), we find

$$N_{(2n+1,i)} = \binom{2n+1}{i} - \sum_{k=1}^{i-1} N_{(2n+1,k)} \binom{2(n-k)+1}{i-k}. \tag{4}$$

Lemma 1. *The following equality holds:*

$$G_s = u^{2s+1} + v^{2s+1} \quad (s = 0, 1, \ldots, n-1).$$

Proof. We can transform the binomial theorem of odd power

$$(u+v)^{2n+1} = \sum_{i=0}^{2n+1} \binom{2n+1}{i} u^{2n+1-i} v^i$$

into the identity

$$(u+v)^{2n+1} = \sum_{i=1}^{n} \binom{2n+1}{i} u^i v^i (u^{2(n-i)+1} + v^{2(n-i)+1}) + u^{2n+1} + v^{2n+1}.$$

As $n = s$ in this identity, equating it with (3), we get the conclusion. $\square$

From Definition 1 and Lemma 1, we find the next proposition.

Proposition 1. *The following equality holds:*

$$\begin{aligned} F_{(2n+1,r)} = &\sum_{i=1}^{r} N_{(2n+1,i)} u^i v^i (u+v)^{2(n-i)+1} \\ &+ \sum_{i=r+1}^{n} M_{(2n+1,i,r+1)} u^i v^i (u^{2(n-i)+1} + v^{2(n-i)+1}) \\ &+ u^{2n+1} + v^{2n+1}. \end{aligned} \tag{5}$$

Proof. We proceed by induction on r.

From (1), $M_{(2n+1,i,1)} = \binom{2n+1}{i}$ holds. Hence the theorem is obviously true for $r = 0$. We assume that (5) holds for r $(r = 0, 1, \ldots, n-1)$. Since $0 \le r \le n-1$ implies $0 \le n-(r+1) \le n-1$, by Lemma 1 and (3), we have

$$\begin{aligned} F_{(2n+1,r+1)} = &\sum_{i=1}^{r} N_{(2n+1,i)} u^i v^i (u+v)^{2(n-i)+1} \\ &+ M_{(2n+1,r+1,r+1)} u^{r+1} v^{r+1} \Bigg((u+v)^{2(n-(r+1))+1} \\ &- \sum_{i=1}^{n-(r+1)} \binom{2(n-(r+1))+1}{i} u^i v^i (u^{2(n-(r+1)-i)+1} + v^{2(n-(r+1)-i)+1}) \Bigg) \\ &+ \sum_{i=r+2}^{n} M_{(2n+1,i,r+1)} u^i v^i (u^{2(n-i)+1} + v^{2(n-i)+1}) + u^{2n+1} + v^{2n+1}. \end{aligned}$$

From (1) and (2),

$$\begin{aligned}F_{(2n+1,r+1)} =& \sum_{i=1}^{r} N_{(2n+1,i)} u^i v^i (u+v)^{2(n-i)+1} \\ &+ N_{(2n+1,r+1)} u^{r+1} v^{r+1} (u+v)^{2(n-(r+1))+1} \\ &\quad - M_{(2n+1,r+1,r+1)} u^{r+1} v^{r+1} \sum_{i=r+2}^{n} \binom{2(n-(r+1))+1}{i-(r+1)} \\ &\qquad\qquad u^{i-(r+1)} v^{i-(r+1)} (u^{2(n-i)+1} + v^{2(n-i)+1}) \\ &\quad + \sum_{i=r+2}^{n} \left(\binom{2n+1}{i} - \sum_{k=1}^{r} M_{(2n+1,k,k)} \binom{2(n-k)+1}{i-k} \right) \\ &\qquad\qquad u^i v^i (u^{2(n-i)+1} + v^{2(n-i)+1}) \\ &\quad + u^{2n+1} + v^{2n+1}.\end{aligned}$$

From (1) again,

$$\begin{aligned}F_{(2n+1,r+1)} =& \sum_{i=1}^{r+1} N_{(2n+1,i)} u^i v^i (u+v)^{2(n-i)+1} \\ &+ \sum_{i=r+2}^{n} M_{(2n+1,i,r+2)} u^i v^i (u^{2(n-i)+1} + v^{2(n-i)+1}) \\ &\quad + u^{2n+1} + v^{2n+1}.\end{aligned}$$

Therefore (5) holds for $r+1$. □

Proposition 2. *The following identity holds:*

$$(u+v)^{2n+1} = \sum_{i=1}^{n} N_{(2n+1,i)} u^i v^i (u+v)^{2n+1-2i} + u^{2n+1} + v^{2n+1}. \tag{6}$$

Proof. Since we have

$$G_{n-(r+1)} = u^{2(n-(r+1))+1} + v^{2(n-(r+1))+1} \quad (r = 0, 1, \ldots, n-1)$$

by Lemma 1, we find

$$F_{(2n+1,r+1)} = F_{(2n+1,r)} \quad (r = 0, 1, \ldots, n-1),$$

that is,

$$F_{(2n+1,0)} = F_{(2n+1,1)} = \cdots = F_{(2n+1,n)}. \tag{7}$$

Here,

$$F_{(2n+1,0)} = \sum_{i=0}^{2n+1} \binom{2n+1}{i} u^{2n+1-i} v^i. \tag{8}$$

On the other hand, letting $r = n$ in Proposition 1,

$$F_{(2n+1,n)} = \sum_{i=1}^{n} N_{(2n+1,i)} u^i v^i (u+v)^{2(n-i)+1} + u^{2n+1} + v^{2n+1}. \tag{9}$$

Thus, from (7), (8) and (9), we obtain (6). □

2.2. An Identity of Even Power

Let r be a non-negative integer with $r \leq n-1$ and s a natural number with $s \leq n-1$.

Definition 2. The symbols $M_{(2n,i,j)}, N_{(2n,i)}, H_s, F_{(2n,r)}$ are defined as follows:

$$M_{(2n,i,j)} = \binom{2n}{i} - \sum_{k=1}^{j-1} M_{(2n,k,k)} \binom{2(n-k)}{i-k}, \tag{10}$$

$$N_{(2n,i)} = M_{(2n,i,i)}, \tag{11}$$

$$H_s = (u+v)^{2s} - \sum_{i=1}^{s-1} \binom{2s}{i} u^i v^i (u^{2(s-i)} + v^{2(s-i)}) - \binom{2s}{s} u^s v^s,$$

$$\begin{aligned} F_{(2n,r)} =& \sum_{i=1}^{r} \binom{2n}{i} u^i v^i H_{n-i} \\ &+ \sum_{i=r+1}^{n-1} \binom{2n}{i} u^i v^i (u^{2(n-i)} + v^{2(n-i)}) + \binom{2n}{n} u^n v^n + u^{2n} + v^{2n}. \end{aligned}$$

From (10) and (11), we find

$$N_{(2n,i)} = \binom{2n}{i} - \sum_{k=1}^{i-1} N_{(2n,k)} \binom{2(n-k)}{i-k}. \tag{12}$$

The following three statements are proved similarly to the previous subsection.

Lemma 2. *The following equality holds:*

$$H_s = u^{2s} + v^{2s} \quad (s = 1, \ldots, n-1).$$

Proposition 3. *The following equality holds:*

$$\begin{aligned} F_{(2n,r)} =& \sum_{i=1}^{r} N_{(2n,i)} u^i v^i (u+v)^{2(n-i)} \\ &+ \sum_{i=r+1}^{n-1} M_{(2n,i,r+1)} u^i v^i (u^{2(n-i)} + v^{2(n-i)}) \\ &+ M_{(2n,n,r+1)} u^n v^n + u^{2n} + v^{2n}. \end{aligned}$$

Proposition 4. *The following identity holds:*

$$(u+v)^{2n} = \sum_{i=1}^{n} N_{(2n,i)} u^i v^i (u+v)^{2n-2i} + u^{2n} + v^{2n}. \tag{13}$$

2.3. An Identity of General Power

By the unification of (6) and (13), we deduce

$$(u+v)^m = \sum_{i=1}^{[\frac{1}{2}m]} N_{(m,i)} u^i v^i (u+v)^{m-2i} + u^m + v^m \quad (m \geq 2). \tag{14}$$

From (4) and (12), we get

$$N_{(m,i)} = \binom{m}{i} - \sum_{k=1}^{i-1} N_{(m,k)} \binom{m-2k}{i-k}. \tag{15}$$

3. The Main Theorem

By using (14) and (15), we have the main theorem.

Theorem 1. *The identity*

$$\sum_{k=0}^{i} (-1)^{k+1} \frac{1}{m-k} \binom{m-k}{k} \binom{m-2k}{i-k} = 0 \tag{16}$$

is equivalent to the identity of Lagrange

$$(u+v)^m + \sum_{i=1}^{[\frac{1}{2}m]} \frac{m}{i} \binom{m-i-1}{i-1} (-uv)^i (u+v)^{m-2i} = u^m + v^m$$

where m and i are natural numbers with $m \geq 2$ and $i \leq [\frac{1}{2}m]$.

Proof. Equating the identity of Lagrange with (14), we find

$$N_{(m,i)} = (-1)^{i+1} \frac{m}{i} \binom{m-i-1}{i-1}. \tag{17}$$

From (15) and (17), we obtain (16). Conversely, equating (15) with (16), we have (17). Substituting (17) into (14), we get the identity of Lagrange. □

In Theorem 1, one might feel uncomfortable with the fact that (16) is an expression depending only on m, whereas the identity of Lagrange is an expression depending on m, u, and v. The sense of incongruity can be resolved by considering the following proposition.

Proposition 5. *If* (14) *holds, then* (16) *is equivalent to the identity of Lagrange where m and i are natural numbers with $m \geq 2$ and $i \leq [\frac{1}{2}m]$.*

References

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