DIVISOR FUNCTIONS AND THE NUMBER OF SUM SYSTEMS

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Abstract

Divisor functions have attracted the attention of number theorists from Dirichlet to the present day. Here we consider associated divisor functions $c_j^{(r)}(n)$ which, for non-negative integers j and r, count the number of ways of representing n as an ordered product of $j+r$ factors, of which the first j must be non-trivial, and their natural extension to negative integers r. We give recurrence properties and explicit formulae for these novel arithmetic functions. Specifically, the functions $c_j^{(-j)}(n)$ count, up to a sign, the number of ordered factorizations of n into j square-free non-trivial factors. These functions are related to a modified version of the Möbius function and turn out to play a central role in counting the number of sum systems of given dimensions. Sum systems are finite collections of finite sets of non-negative integers, of prescribed cardinalities, such that their set sum generates consecutive integers without repetitions. Using a recently established bijection between sum systems and joint ordered factorizations of their component set cardinalities, we prove a formula expressing the number of different sum systems in terms of associated divisor functions.

1. Introduction

Sum systems are finite collections of finite sets of non-negative integers, of prescribed cardinalities, such that the process of taking one element from each component set and adding them up generates each number in an arithmetic progression exactly once. Thus a sum system is of the form $A_1, \dots, A_m \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$, such that

$$\sum_{k=1}^{m} A_k = \left\{ 0, 1, \dots, \prod_{k=1}^{m} |A_k| - 1 \right\},$$

where the sum on the left-hand side is the Minkowski set sum defined as

$$\sum_{k=1}^{m} A_k = \left\{ \sum_{k=1}^{m} a_k : a_k \in A_k \ (k \in \{1, \ldots, m\}) \right\}$$

and $|A_k|$ denotes the cardinality of the set A_k. Such systems play a role in the construction of principal reversible cuboids and (in the simple case of only two component sets) of other matrices with integer entries and specific symmetry properties, e.g. most-perfect pandiagonal squares, cf. [8]. A variant of sum systems with infinitely many component sets was considered, under the curious name 'British Number Systems', by de Bruijn [2].

A construction formula for all sum systems was derived in the recent work [5]. Specifically, the construction of sum systems of given cardinalities $a_1, \ldots, a_m$ for the m component sets is based on joint ordered factorizations of these cardinalities, defined as follows (cf. [5, Definition 6.6]); note that here and in the following we use the notation $\mathbb{N}_{\geq 2} = \mathbb{N} + \{1\} = \{2, 3, 4, \ldots\}$.

Definition. Let $m \in \mathbb{N}$ and $a \in \mathbb{N}^m$. Then we call

$$((j_1, f_1), (j_2, f_2), \ldots, (j_L, f_L)) \in (\{1, \ldots, m\} \times \mathbb{N}_{\geq 2})^L,$$

where $L \in \mathbb{N}$, a *joint ordered factorization* of $a = (a_1, \ldots, a_m)$ if

$$\prod_{j_l = j} f_l = a_j \qquad (j \in \{1, \ldots, m\})$$

and $j_l \neq j_{l-1}$ $(l \in \{2, \ldots, L\})$.

In other words, a joint ordered factorization of an n-tuple of natural numbers $a_1, \ldots, a_m$ arises from writing each of these numbers as a product of non-trivial factors, i.e., factors ≥ 2, and then arranging all factors in a linear chain such that no two adjacent factors arise from the factorization of the same number. It was proven in [5, Theorem 6.7] that, given a joint ordered factorization, the sets

$$A_k = \sum_{j_l = k} \left(\prod_{s=1}^{l-1} f_s \right) \{0, 1, \ldots, f_l - 1\} \qquad (j \in \{1, \ldots, m\})$$

(where we use the convention of multiplying a set of numbers M with a constant α, $\alpha M = \{\alpha x : x \in M\}$) form a sum system, and that conversely any sum system arises from some joint ordered factorization of its dimensions in this way. (In less explicit form, the structure of sum system components had been obtained in [2]). This establishes a bijection between sum systems and joint ordered factorizations.

As an illustrative example, consider the case $m = 5$, $(a_1, a_2, a_3, a_4, a_5) = (4, 6, 8, 12, 20)$. An example of a joint ordered factorization of this quintuple of dimensions is

$$((1, 2), (5, 2), (2, 2), (5, 5), (3, 4), (5, 2), (4, 4), (3, 2), (4, 3), (1, 2), (2, 3)),$$

yielding the corresponding sum system

$$\begin{aligned}
A_1 =&\{0, 1, 7680, 7681\}\\
A_2 =&\{0, 4, 15360, 15364, 30720, 30724\}\\
A_3 =&\{0, 40, 80, 120, 1280, 1320, 1360, 1400\}\\
A_4 =&\{0, 320, 640, 960, 2560, 2880, 3200, 3520, 5120, 5440, 5760, 6080\}\\
A_5 =&\{0, 2, 8, 10, 16, 18, 24, 26, 32, 34, 160, 162, 168, 170, 176, 178, 184, 186, 192, 194\}.
\end{aligned}$$

Thus, inscribing the above numbers on the faces of the five platonic polyhedra and adding the numbers obtained in each roll of these five dice, we obtain a random number generator for the integers $0, 1, \ldots, 46079$ $(= a_1a_2a_3a_4a_5 - 1)$ with uniform probability distribution.

In the present paper, we answer the question of how many different joint ordered factorizations of a given m-tuple of positive integers there are. Our main result is Theorem 4, which expresses the number of joint ordered factorizations in terms of values of certain associated divisor functions at $a_1, \ldots, a_m$ (see Equation (14)). These functions turn out to be closely linked to a modified version of the number theoretic Möbius function (cf. [1, p. 77]). In Section 2, we study some of their properties before tackling the counting problem in Sections 3 and 4.

For the reader's convenience, here is a summary of the standard multi-index notation used in this paper. For m-tuples of integers, we apply the usual componentwise addition, subtraction and scalar multiplication as well as the size function and the multi-factorial,

$$|n| = \sum_{j=1}^{m} |n_j|, \qquad n! = \prod_{j=1}^{m} n_j! \qquad (n \in \mathbb{N}_0^m),$$

respectively, and the partial ordering

$$n \leq \tilde{n} \iff n_j \leq \tilde{n}_j \ (j \in \{1, \ldots, m\}).$$

We define the special m-tuples $0_m = (0, \ldots, 0), 1_m = (1, \ldots, 1) \in \mathbb{N}_0^m$. In addition to the usual binomial coefficients, we use the multi-binomial coefficients

$$\binom{n}{\tilde{n}} = \prod_{j=1}^{m} \binom{n_j}{\tilde{n}_j} = \frac{n!}{\tilde{n}!\,(n-\tilde{n})!} \qquad (n, \tilde{n} \in \mathbb{N}_0^m, \tilde{n} \leq n)$$

and the multinomial coefficients

$$\binom{|n|}{n} = \frac{|n|!}{\prod_{j=1}^{m} n_j!} = \frac{|n|!}{n!} \qquad (n \in \mathbb{N}_0^m).$$

Note that using the same bracket notation for these different quantities does not create confusion, since the type (dimensionality) of the arguments determines which

coefficient is meant. Finally,

$$x^n = \prod_{j=1}^{m} x_j^{n_j} \qquad (n \in \mathbb{N}_0^m, x \in \mathbb{R}^m).$$

2. Non-trivial and Associated Divisor Functions

Divisor functions have been studied by many eminent number theorists, from Dirichlet to the present day (e.g. [10], [6], [7]). The non-trivial and associated divisor functions defined in [4] can be conveniently described in the framework of the commutative Dirichlet convolution algebra of arithmetic functions. The convolution of arithmetic functions $f_1, f_2, \ldots, f_j$ is given by

$$(f_1 * f_2 * \cdots * f_j)(n) = \sum_{n_1 n_2 \cdots n_j = n} f_1(n_1) f_2(n_2) \cdots f_j(n_j), \tag{1}$$

summing over all ordered factorizations of $n \in \mathbb{N}$ into j factors. We denote the jth convolution power as follows, $f^{*j} := f * f * \cdots * f$, where the right-hand side has j repetitions of f; by the usual convention, $f^{*0} = e$. The function $e(n) = \delta_{n,1}$ $(n \in \mathbb{N})$ is the neutral element of the Dirichlet convolution product, and the convolution inverse of the constant function 1 is the well-known Möbius function μ.

In analogy to the standard jth divisor function $d_j = 1^{*j}$ (cf. [11, p. 9]), which counts the ordered factorizations of its argument into j positive integer factors, we define the *jth non-trivial divisor function* $c_j = (1-e)^{*j}$, which counts the ordered factorizations of its argument into j non-trivial integer factors, i.e., into factors > 1.

Furthermore, for a non-negative integer r, the *associated (j,r)-divisor function* is defined as $c_j^{(r)} = (1-e)^{*j} * 1^{*r}$. In view of Equation (1), it counts the ordered factorizations of its argument into $j + r$ factors, of which the first j must be non-trivial.

As the constant function 1 has a convolution inverse, the latter definition extends naturally to negative upper indices, giving the associated $(j,-r)$-divisor function $c_j^{(-r)} = (1-e)^{*j} * \mu^{*r}$. (Note that $1-e$ does not have a convolution inverse, so there is no analogous extension to negative lower indices.) The functions $c_0^{(-r)} = \mu^{*r}$ were studied by Popovici [9]. In the associated $(j,-r)$-divisor functions, the *modified Möbius function*

$$(\mu - e)(n) = \begin{cases} (-1)^{\Omega(n)} & \text{if } n \text{ is square-free} \\ 0 & \text{otherwise (including the case } n = 1) \end{cases} \qquad (n \in \mathbb{N}),$$

where $\Omega(n)$ is the number of prime factors of n, appears naturally. Indeed, if $j \geq r$, then

$$c_j^{(-r)} = (1-e)^{*j-r} * ((1-e) * \mu)^{*r} = (-1)^r (1-e)^{*j-r} * (\mu - e)^{*r};$$

if $j < r$, then

$$c_j^{(-r)} = ((1-e) * \mu)^{*j} * \mu^{*r-j} = (-1)^j (\mu - e)^{*j} * \mu^{*r-j}.$$

Note that $c_j^{(r)}(n)$ involves factorization of n into $j + r$ factors if $r \geq 0$, into $\max\{j, -r\}$ factors if $r < 0$, of which at least j must be non-trivial, so $c_j^{(r)}(n) = 0$ if $j > \Omega(n)$. (Also, if $r < 0$, then at least $-r$ factors must be square-free.)

The special case $j = -r$,

$$c_j^{(-j)}(n) = (-1)^j \sum_{n_1 n_2 \cdots n_j = n} (\mu - e)(n_1)\,(\mu - e)(n_2) \cdots (\mu - e)(n_j) \qquad (n \in \mathbb{N}), \quad (2)$$

turns out to be of particular importance (cf. Theorem 4 below). The value of $c_j^{(-j)}(n)$ can be interpreted as $(-1)^{\Omega(n)+j}$ times the number of ordered factorizations of n into j non-trivial, square-free factors.

The following statement aids the calculation of the associated divisor functions either via a recurrent scheme similar to Pascal's triangle, or directly in terms of the prime factorization of their argument.

Theorem 1. *Let $j \in \mathbb{N}_0$, $r \in \mathbb{Z}$. Then*

(a)

$$c_j^{(r+1)} = c_{j+1}^{(r)} + c_j^{(r)}; \quad (3)$$

(b) if $n = p_1^{a_1} p_2^{a_2} \cdots p_\nu^{a_\nu}$ with distinct primes $p_1, p_2, \ldots, p_\nu$, then

$$c_j^{(r)}(n) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \prod_{l=1}^{\nu} \binom{a_l + r + j - k - 1}{a_l}. \quad (4)$$

Proof. Equation (3) follows immediately from the observation that $c_{j+1}^{(r)} = (1 - e)^{*j+1} * 1^{*r} = (1-e)^{*j} * 1 * 1^{*r} - (1-e)^{*j} * e * 1^{*r}$. For part (b), the binomial theorem gives

$$c_j^{(r)} = \sum_{k=0}^{j} (-1)^k \binom{j}{k} e^{*k} * 1^{*j-k+r}, \quad (5)$$

and Equation (4) follows from the identity (cf. [4, Lemma 1])

$$1^{*j}(n) = \prod_{k=1}^{\nu} \binom{a_k + j - 1}{a_k}, \quad (6)$$

which holds for all integers j. As 1 and $1^{*-1} = \mu$ are multiplicative arithmetic functions and the Dirichlet convolution of multiplicative functions is multiplicative, it is sufficient to verify (6) for a single prime power. For positive j, $1^{*j}(p^a)$ is, by Equation (1), equal to the number of j-part partitions of a, i.e., to $\binom{a+j-1}{a}$. Furthermore, again by Equation (1), $1^{*-j}(p^a) = \mu^{*j}(p^a)$ is equal to $(-1)^a$ times the number of ways of writing a as an ordered sum of j terms, each either 0 or 1, i.e., to $(-1)^a \binom{j}{a} = \binom{-j+a-1}{a}$. □

Remarks. 1. In the specific case of a power of a square-free number n, the product in Equation (4) turns into a power; then, using the last identity in the above proof, we can derive the formula

$$c_j^{(r-j+1-a)}(n^a) = (-1)^{a\Omega(n)+j} c_j^{(-r)}(n^a).$$

2. The discussion before Theorem 1 shows that for all $r \in \mathbb{Z}$, the factorization involved in $c_j^{(r)}$ requires j non-trivial factors. If $j = \Omega(n)$, then these must be the prime factors of n. Hence, writing $c_{|a|}^{(r+1)}(p_1^{a_1} p_2^{a_2} \cdots p_\nu^{a_\nu})$ in two different ways, and taking $j = |a|$ and changing the summation variable from k to $|a| - k$ in eq. (4), we can deduce the interesting identity

$$\binom{|a|}{a} = \sum_{k=0}^{|a|} (-1)^{|a|-k} \binom{|a|}{k} \prod_{l=1}^{\nu} \binom{a_l + k + r}{a_l}$$

for all $a \in \mathbb{N}^\nu$, $\nu \in \mathbb{N}$, and for all $r \in \mathbb{Z}$; note that the left-hand side is independent of r.

We note the following relationships between the associated divisor functions and the standard divisor functions d_j (and their inverses with respect to Dirichlet convolution).

Theorem 2. *Let $j \in \mathbb{N}_0$, $r \in \mathbb{Z}$. Then*

$$d_r = \sum_{k=0}^{\infty} \binom{k+r-1}{k} c_k^{(-k)}. \tag{7}$$

More generally, for any $u \in \mathbb{N}_0$ and $v \in \mathbb{Z}$,

$$c_{j+u}^{(r+v)} = \sum_{k=j}^{\infty} \binom{k+r-1}{k-j} c_{u+k}^{(v-k)}. \tag{8}$$

Proof. The identity (7) follows from the inverse binomial formula,

$$1^{*r} = (e + \mu - e)^{*-r} = \sum_{k=0}^{\infty} (-1)^k \binom{r+k-1}{k} (\mu - e)^{*k}.$$

The series is pointwise convergent because $\mu - e$ is pointwise nilpotent in the convolution algebra. Hence induction on j gives

$$(1 - e)^{*j} * 1^{*k} = \sum_{k=j}^{\infty} \binom{k+r-1}{k-j} (e - \mu)^{*k},$$

and Equation (8) follows by convolution with $(1 - e)^{*u} * 1^{*v}$ on both sides. □

Remark. Curiously, the binomial coefficient appearing in Equations (7) and (8) can be expressed as the associated divisor function of a kth prime power, $c_j^{(r)}(p^k) = \binom{k+r-1}{k-j}$ (cf. [4, Lemma 5] and [11, p. 62] for $j = 0$), giving an alternate form as the sum over products of the form $c_j^{(r)}(p^k)c_k^{(-k)}(n)$. For $j = 0$, this is equal to the number of weak compositions of k into r parts (cf. [12, p. 15]). We also note that Equation (5) provides a converse to Equation (7).

Taking $r = 2$ in Equation (7) gives an expression for the standard divisor function (number of divisors), $d_2 = \sum_{k=0}^{\infty} (k + 1)c_k^{(-k)}$. Taking $r = 1$ yields the identity $1 = \sum_{k=0}^{\infty} c_k^{(-k)}$. We note that the sum $\sum_{k=0}^{\infty} |c_k^{(-k)}|$, which gives the number of ordered factorizations into (any number of) square-free, non-trivial factors, generates all odd integers; indeed, $\sum_{k=0}^{\infty} |c_k^{(-k)}(p_1 p_2^m)| = 2m + 1$ $(m \in \mathbb{N}_0)$.

3. An Auxiliary Counting Problem

We now turn to the question of counting the number of joint ordered factorizations of a given m-tuple. In the present section, we first consider the following combinatorial problem. *Given a number of colored (but otherwise identical) blocks, with any number of blocks to each of several colors, in how many ways can all blocks be arranged in a linear sequence such that no two adjacent blocks have the same color?* Note that the answer may very well be 0; indeed, if there are 2 more blocks of one color than of all the other colors taken together, then there is no possible arrangement.

In the following, we make extensive use of the standard multi-index notation summarised at the end of the Introduction. We denote by e_n the number of different

ways $|n|$ objects, of which there are n_j of type j, $j \in \{1, \ldots, m\}$, and which are otherwise indistinguishable, can be linearly arranged such that no neighbouring objects have the same type. Then, for any $n \in \mathbb{N}^m$, the identity

$$\binom{|n|}{n} = \sum_{1_m \le k \le n} \binom{n-1_m}{k-1_m} e_k \tag{9}$$

holds. Indeed, $\binom{|n|}{n}$ is the number of linear arrangements of all objects ignoring the non-adjacency condition. Given any such arrangement, consider the associated collapsed arrangement where any group of contiguous objects of the same type is replaced with a single such object, resulting in an arrangement of size $k \le n$ satisfying the adjacency condition. There are e_k different collapsed arrangements of size k, and $\binom{n-1_m}{k-1_m}$ different arrangements giving rise to each collapsed arrangement.

Theorem 3. *Let $n \in \mathbb{N}^m$, $m \in \mathbb{N}$. Then*

$$e_n = \sum_{0_m \le k \le n-1_m} (-1)^{|k|} \binom{n-1_m}{k} \binom{|n-k|}{n-k}. \tag{10}$$

Proof. The power series for the generating function

$$\sum_{k \in \mathbb{N}_0^m} e_{1_m+k} \frac{x^k}{k!}$$

can be shown to be convergent for all $x \in \mathbb{R}^m$ by comparison with exponential series using the a priori estimate

$$e_{1_m+k} \le \binom{|k+1_m|}{k+1_m} \le m^{|k|+m},$$

where we used Equation (9) in the first and the multinomial theorem in the second inequality. Using the exponential series

$$\exp \sum_{j=1}^{m} x_j = \sum_{k \in \mathbb{N}_0^m} \frac{x^l}{k!} \qquad (x \in \mathbb{R}^m)$$

and the identity (9) between two applications of the multivariate Cauchy product

formula, we find

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} e_{1_m+k}\,\frac{x^k}{k!} &= \left(\sum_{k\in\mathbb{N}_0} e_{1_m+k}\,\frac{x^k}{k!}\right)\left(\sum_{l\in\mathbb{N}_0^m}\frac{x^l}{l!}\right)\left(\sum_{l\in\mathbb{N}_0^m}(-1)^{|l|}\frac{x^l}{l!}\right)\\ &= \left(\sum_{n\in\mathbb{N}_0}\left(\sum_{0_m\le k\le n}\binom{n}{k}e_{1_m+k}\right)\frac{x^n}{n!}\right)\left(\sum_{l\in\mathbb{N}_0^m}(-1)^{|l|}\frac{x^l}{l!}\right)\\ &= \left(\sum_{n\in\mathbb{N}_0}\binom{|n+1_m|}{n+1_m}\frac{x^n}{n!}\right)\left(\sum_{l\in\mathbb{N}_0^m}(-1)^{|l|}\frac{x^l}{l!}\right)\\ &= \sum_{k\in\mathbb{N}_0^m}\left(\sum_{0_m\le l\le k}(-1)^{|l|}\binom{k}{l}\binom{|k+1_m-l|}{k+1_m-l}\right)\frac{x^k}{k!}\qquad (x\in\mathbb{R}^m),\end{aligned}$$

from which Equation (10) can be read off. □

In the special case $m = 2$, working out the repeated binomial sums using Gould's combinatorial identities (3.48) and (3.47) [3] gives

$$e_{(n_1,n_2)} = \binom{2}{n_2-n_1+1} = \begin{cases}2 & \text{if } n_1 = n_2,\\ 1 & \text{if } |n_1-n_2| = 1,\\ 0 & \text{otherwise.}\end{cases} \tag{11}$$

This reflects the obvious fact that with only two types of objects, the non-adjacency condition enforces an alternating arrangement, for which there are two possibilities if the numbers of objects of both types are equal, one possibility if they differ by one, and no possibility otherwise. We emphasise that $m = 2$ is a rather untypical case and that for $m \ge 3$ much more complex arrangements are possible.

Remark. Equation (10) can be given a direct combinatorial interpretation (and a somewhat more convoluted proof) in the following manner. We call any arrangement of the $|n|$ objects ignoring the non-adjacency condition with t objects that are each followed by an object of the same type marked with a tick an *annotated arrangement with t ticks.* (Clearly there is no annotated version with t ticks of any arrangement which has fewer than t objects followed by an object of the same type.) For each $t \in \mathbb{N}_0$, let A_t be the set of all annotated arrangements with t ticks. Its cardinality is

$$|A_t| = \sum_{k\in\mathbb{N}_0^m,|k|=t}\binom{n-1_m}{k}\binom{|n-k|}{n-k}. \tag{12}$$

Indeed, given any element of A_t, we find k_j ticked objects of type $j \in \{1,\dots,m\}$, so $|k| = t$. Considering the n_j objects of type j in the arrangement (ignoring the

other types for the moment), the ticks can occur in $n_j - 1$ places, so there are $\binom{n_j - 1}{k_j}$ possibilities. Taking ticked objects together with their following object and single unticked objects as groups, there will be $n_j - k_j$ such groups. Among all types, the groups can be arranged in $\binom{|n-k|}{n-k}$ ways, hence we obtain Equation (12). Now to verify Equation (10), consider an arrangement of the objects. Let $l \in \{0, \ldots, |n| - m\}$ be the number of objects in this arrangement followed by an object of the same type. Annotated versions of this arrangement will appear in the sets $A_0, \ldots, A_l$. In the set A_t, it will have t ticks which can be placed in l places, so there are $\binom{l}{t}$ annotated versions of this (fixed) arrangement in this set. We now count the total of its appearances (as different annotated arrangements) in the sets $A_0, \ldots, A_l$, counting its appearances in odd-indexed sets negative, those in even-indexed sets positive. Thus in total we count this arrangement

$$\sum_{t=0}^{l} (-1)^t \binom{l}{t} = (1-1)^l = \delta_{l,0}$$

times. Hence the only arrangements counted in the alternating total are those with $l = 0$ objects followed by an object of the same color. This gives

$$e_n = \sum_{t=0}^{|n|-m} (-1)^t |A_t|$$

and hence, by Equation (12), formula (10).

4. The Number of Joint Ordered Factorizations

Given an m-tuple of integers $a = (a_1, a_2, \ldots, a_m) \in \mathbb{N}^m$, with $a_j \geq 2$ ($j \in \{1, \ldots, m\}$), we can use Theorem 3 to count the joint ordered factorizations of a where a_j is split into a prescribed number n_j of non-trivial factors. Indeed, we can think of taking n_j placeholders marked as type j (and otherwise indistinguishable), for $j \in \{1, \ldots, m\}$, and arranging all these placeholders according to the rules of the auxiliary counting problem of Section 3, and then putting the factors for each a_j into the blocks of type j in their given order. As there are e_n admissible arrangements of the placeholders and $c_{n_j}(a_j)$ different non-trivial ordered factorizations of a_j, we obtain the number of joint ordered factorizations from Equation (10) as

$$e_n \prod_{j=1}^{m} c_{n_j}(a_j) = \sum_{0_m \leq k \leq n-1_m} (-1)^{|k|} \binom{n-1_m}{k} \binom{|n-k|}{n-k} \prod_{j=1}^{m} c_{n_j}(a_j). \tag{13}$$

The sum over all $n \in \mathbb{N}^m$ (which is a finite sum since $c_{n_j}(a_j) = 0$ if n_j exceeds $\Omega(a_j)$, the number of prime factors of a_j counting multiplicities) then gives the total number of joint ordered factorizations of a, which can be expressed as follows.

Theorem 4. *Let $m \in \mathbb{N}$ and $a \in \mathbb{N}^m_{\geq 2}$. Then the number of different joint ordered factorizations of a is*

$$N_a = \sum_{l \in \mathbb{N}^m} \binom{|l|}{l} \prod_{j=1}^{m} c_{l_j}^{(-l_j)}(a_j). \tag{14}$$

Proof. Summing the expression (13) over $n \in \mathbb{N}^m$, changing the summation variable by setting $n = k + l$ and interchanging the order of the sums, we find

$$\begin{aligned} N_a &= \sum_{k \in \mathbb{N}_0^m} \sum_{l \in \mathbb{N}^m} \binom{k + l - 1_m}{k} \binom{|l|}{l} \prod_{j=1}^{m} c_{k_j + l_j}(a_j) \\ &= \sum_{l \in \mathbb{N}^m} \binom{|l|}{l} \prod_{j=1}^{m} \left(\sum_{k=0}^{\infty} \binom{k + l_j - 1}{k} c_{k+l_j}(a_j) \right). \end{aligned}$$

Now consider the functions which appear in the right-hand side product. By the binomial formula for negative powers in the Dirichlet convolution algebra of arithmetic functions,

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k \binom{k + l_j - 1}{k} c_{l_j + k} &= (1 - e)^{*l_j} * \sum_{k=0}^{\infty} \binom{k + l_j - 1}{k} (e - 1)^{*k} \\ &= (1 - e)^{*l_j} * (e - (e - 1))^{*-l_j} = (1 - e)^{*l_j} * \mu^{*l_j} = c_{l_j}^{(-l_j)}, \end{aligned}$$

and hence the result. □

Remarks. 1. In view of the interpretation given to Equation (2) in Section 2, Equation (14) can be read as

$$N_a = (-1)^{\sum_{j=1}^{m} \Omega(a_j)} \sum_{n=m}^{\infty} (-1)^n \sum_{l \in \mathbb{N}^m, |l| = n} \binom{|l|}{l} \prod_{j=1}^{m} F_{l_j}(a_j),$$

where we denote by $F_{l_j}(a_j)$ the number of ordered factorizations of a_j into l_j non-trivial, square-free factors. Thus N_a can be construed as an alternating sum over n of the number of ways the integers $a_1, a_2, \ldots, a_m$ can be split into a total of n non-trivial, square-free factors, and these factors can be linearly arranged without further constraints.

2. In the two-dimensional case $m = 2$, Equation (11) gives a simple explicit form for $e_{(n_1,n_2)}$, and we obtain directly from the left-hand side of Equation (13) that

$$N_{(a_1,a_2)} = \sum_{n=1}^{\infty} 2\, c_n(a_1)\, c_n(a_2) + \sum_{n=1}^{\infty} c_n(a_1)\, c_{n+1}(a_2) + \sum_{n=1}^{\infty} c_{n+1}(a_1)\, c_n(a_2).$$

In the symmetric case $a_1 = a_2 = a$ considered in [8] and [4], this gives, via $c_n + c_{n+1} = (1-e)^{*n} * (e+1-e) = c_n^{(1)}$, the expression $N_{(a,a)} = 2 \sum\limits_{n=1}^{\infty} c_n(a)\, c_n^{(1)}(a)$, which involves different divisor functions compared to Equation (14) and reproduces Theorem 4 of [4]. Note that the count given in [4] is $N_{(a,a)}/2$, using the permutation symmetry of the two equally-sized component sets of the sum system. More generally, when we have m equally sized component sets in the sum system, then by the same permutation symmetry, $N_{(a,a,\ldots,a)}/m! \in \mathbb{N}_0$. Clearly, this property extends to more general m-tuples $a \in \mathbb{N}^m$ provided that all numbers a_j have the same factorization structure, i.e., the multisets of exponents in the prime factorization coincide. The integer sequences $(N_{(a,a)}/2!)_{a\in\mathbb{N}}$ and $(N_{(a,a,a)}/3!)_{a\in\mathbb{N}}$ are equal to sequences A0273013 and A0131514 in the OEIS (http://oeis.org), respectively, but it seems that no such OEIS record exists for $m \geq 4$.

We conclude with the observation that the number of m-part sum systems is at least $m!$, and this value is attained if and only if all parts have prime cardinality.

Theorem 5. *Let $m \in \mathbb{N}$ and $a \in \mathbb{N}^m_{\geq 2}$. Then $N_a \geq m!$ and equality holds if and only if all a_j are prime numbers.*

Proof. Starting from the left-hand side of Equation (13) and considering that $e_{1_m} = m!$ by Equation (10) and $c_1(a_j) = 1$, we find

$$N_a = m! + \sum_{n\in\mathbb{N}\setminus\{1_m\}} e_n \prod_{j=1}^{m} c_{n_j}(a_j) \geq m!. \tag{15}$$

Suppose one of the dimensions, w.l.o.g. a_1, is not a prime, and consider $n = (2, 1, \ldots, 1)$. Then $\prod\limits_{j=1}^{m} c_{n_j}(a_j) = c_2(a_1) \geq 1$ and $e_{(2,1,\ldots,1)} = \frac{(m+1)!}{2!}$ by Equation (10), so the sum in (15) is strictly greater than 0. It is obvious that $N_a = m!$ if all a_j, $j \in \{1, \ldots, m\}$, are prime. □

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