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ON THE THEOREM OF N. P. ROMANOFF

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Abstract

The famous theorem of Romanoff asserts that a positive proportion of the integers can be represented as a sum of a prime number and a power of an arbitrary fixed integer. In the present paper, we prove Romanoff's theorem with explicit constants and establish an asymptotic estimate for the Romanoff constant. The values of the constant are then computed using the numerical analysis program Magma.

1. Introduction and Statement of Results

An interesting source of classical problems in additive number theory is to study the proportion of integers n which can be represented as a sum of a prime number p and a power k of an arbitrary fixed integer g, where $k \geq 1$ and $g \geq 2$. In his fascinating paper [20] from 1934 Romanoff proved that this proportion is positive. This fundamental result suggests tacit knowledge about the number of representations $n = p + g^k$, albeit there are not very many integers $n \leq x$ with an extremely large number of these representations. For, the number of powers $g^k \leq x$ is $\left\lfloor \frac{\log x}{\log g} \right\rfloor$ and, by the prime number theorem [4, 15, Chap. 18, pp. 111-114; Theorem 12, p. 36], in its most basic form, $\pi(x) = \sum_{p \leq x} 1$ is asymptotically equal to $\frac{x}{\log x}$ as x tends to infinity. It is not so hard to see that $\left|\{p + g^k \leq x\}\right| \asymp x$.

The most outstanding special case of Romanoff's theorem in some respect concerns sums of prime numbers and powers of 2. Without realizing, at first, that according to Euler [10, p. 595] who corresponded with Goldbach in 1752 that the number 959 cannot be represented as $p+2^k$, in 1849 de Polignac [5] claimed that every odd number can be expressed in this particular form. Motivated by Romanoff's theorem, in 1950 Erdős [7] and van der Corput [22] independently proved that a positive proportion of all integers are not in the form $p+2^k$. In effect, Erdős forged the general concept of covering congruences and ingeniously constructed an infinite arithmetic progression of odd integers, whereof no member is of this form. (See, also, [17, Theorem 7.12, pp. 206-207].) Romanoff's theorem can be summed up as follows.

Theorem 1 (Romanoff, 1934)**.** *In the notation above, let $f(n)$ be the number of solutions of the equation $n = p + g^k$. Then one has*

$$\limsup_{x\to\infty} \frac{1}{x}\sum_{n=1}^{x} f(n)^2 < \infty.$$

Then if S is the set of all integers which can be expressed in the form $p + g^k$, there exists a positive constant c_1, called the Romanoff constant, depending upon g only such that the inequality

$$\sum_{\substack{s\in S\\ 0<s\leq x}} 1 \geq c_1 x$$

holds for sufficiently large values of x.

The purpose of this paper is threefold. First, we prove Theorem 1 with explicit constants. It is important to stress that our proof is a modification of Erdős and Turán's method [9]. In providing numerical constants for Romanoff's work, we make ancillary use of the elementary lemma below, which is an interesting result in and of itself.

Lemma 1. *Let $l_g(q)$ denote the smallest positive exponent z for which $g^z \equiv 1 \pmod{q}$. Then one has*

$$\sum_{\substack{q=1\\ (g,q)=1\\ \mu(q)^2=1}}^{\infty} \frac{1}{q l_g(q)} = O(\log\log g).$$

The proof of Theorem 1 will be given after the proof of Lemma 1. We will base our proof of Lemma 1 on Erdős's method [8], as the special case $g = 2$ has been proved by Erdős. (See, further, [17, Lemma 7.8, pp. 200-201].) To wit, we make delicate use of the approximate formulas for certain functions related to prime numbers obtained by Rosser and Schoenfeld [21], thereby making the series in Lemma 1 amenable to numerical computation.

In recent years, there have been several numerical studies on the asymptotic density of the set S, namely [3, 11, 12, 16, 19]. This is a variation of the notion of the density of a sequence of natural numbers. (See [13, 18].) The upper, and respectively lower, asymptotic density of the set S is expressed by the real number $d_U(S)$ defined by

$$d_U(S) = \limsup_{x\to\infty} \frac{1}{x} \sum_{\substack{s\in S\\ 0<s\le x}} 1,$$

and respectively by the real number $d_L(S)$ defined by

$$d_L(S) = \liminf_{x\to\infty} \frac{1}{x} \sum_{\substack{s\in S\\ 0<s\le x}} 1.$$

Second, we shall thus derive an asymptotic estimate for the Romonoff constant c_1 as g tends to infinity. Our result may be stated as follows.

Theorem 2. *The inequalities*

$$d_U(S) \le \frac{1}{\log g}$$

and

$$d_L(S) \ge \frac{1}{\log g + O((\log\log g)^2)}$$

hold.

Third, we provide a method for computing numerically the Romanoff constant c_1 in Theorem 1 for any integer $g \ge 2$. We apply the computer algebra system Magma [1] to the proofs of Theorem 1 and Lemma 1 to supply the associated numerical approximations for the cases $g = 2$ and $g = 3$.

The method in this paper gives explicit results for all general g. However, some new techniques have been developed to determine the explicit value for the special case $g = 2$. We mention, below, the detailed scientific records for this most important case.

Chen and Sun [3] proved that $d_L(S) > 0.0868$. Lü [16] showed that $d_L(S) > 0.09322$. Subsequently, Habsieger and Roblot [11] demonstrated that $d_L(S) > 0.0933$ and also that $d_U(S) \le 0.49095$. Pintz [19] proved that $d_L(S) > 0.09368$. Pintz employed Wu's result [23], whose proof is not finished, according to Habsieger and Sivak-Fischler [12]. Rectifying this would lead to $d_L(S) > 0.093626$ only. Then Habsieger and Sivak-Fischler [12] showed that $d_L(S) > 0.0936275$. Elsholtz and Schlage-Puchta [6] improved upon this last result and currently hold the best record with $d_L(S) \ge 0.107648$. The explicit result for the case $g = 2$ in this paper is much weaker than the current record.

2. Proof of Lemma 1

We follow Erdős's method directly. Let there be

$$e(h) = \sum_{\substack{l_g(q)=h \\ (g,q)=1 \\ \mu(q)^2=1}} \frac{1}{q}. \tag{1}$$

Then $e(h) < \infty$, since $l_g(q) \gg \log q$. Let further

$$s(k) = \sum_{h=1}^{k} e(h), \tag{2}$$

where $s(0) = 0$, and set

$$N(k) = \prod_{i=1}^{k} (g^k - 1).$$

We have

$$2^{\omega(N(k))} \leq N(k) \leq \prod_{i=1}^{k} g^i = g^{\frac{1}{2}k(k+1)},$$

where

$$\omega(N(k)) = \sum_{p^\nu \| N(k)} 1 = \sum_{p|N(k)} 1.$$

An integer q is counted in at most one of the sums $e(h)$. If q is counted in the sum $s(k)$, then $q \mid g^i - 1$ for some $i \leq k$. Thus $q \mid N(k)$, and thence

$$\begin{aligned} s(k) \leq \sum_{\substack{q|N(k) \\ \mu(q)^2=1}} \frac{1}{q} \leq \sum_{r=0}^{\frac{\log g}{\log 4}k(k+1)} \sum_{\substack{q|N(k) \\ \omega(q)=r \\ \mu(q)^2=1}} \frac{1}{q} \\ \leq \sum_{r=0}^{\frac{\log g}{\log 4}k(k+1)} \frac{1}{r!} \Bigg(\sum_{p|N(k)} \frac{1}{p} \Bigg)^r \\ \leq \exp\Bigg(\sum_{p|N(k)} \frac{1}{p} \Bigg), \end{aligned} \tag{3}$$

which is $O(\log\log N(k)) = O(\log k)$ at most.

We treat the last sum using the upper bound [21, Equation (3.20), p. 70]

$$\sum_{p \leq x} \frac{1}{p} < \log\log x + c_2 + \frac{1}{\log^2 x}$$

for $x > 1$, where, from [21, Equations (2.7) and (2.10), p. 65] and [15, pp. 22-23],

$$c_2 = -\int_0^\infty e^{-y} \log y \, dy + \sum_p \left\{ \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right\} = 0.26149\,72128\,47643.$$

We have

$$\sum_{p|N(k)} \frac{1}{p} \leq \sum_{p \leq f_1(N(k))} \frac{1}{p} + \sum_{\substack{p|N(k) \\ p > f_1(N(k))}} \frac{1}{p}$$
$$\leq \log\log f_1(N(k)) + c_2 + \frac{1}{(\log f_1(N(k)))^2} + \frac{\omega(N(k))}{f_1(N(k))}, \qquad (4)$$

where $f_1(N(k))$ will be specified later.

For a suitable upper bound for $\omega(N(k))$ for small values of $N(k)$ in (4), we evaluate

$$\tau(N(k)) = \sum_{d|N(k)} 1$$

at prime powers to get

$$\tau(p^k) = k + 1 \leq \sqrt[3]{p^k}$$

for $\log p > 3$ and all integers $k \geq 0$. Now, we introduce the function

$$u(k) = \frac{\tau(p^k)}{\sqrt[3]{p^k}} = (k+1)\exp\left(-\frac{k \log p}{3}\right).$$

For $p < 23$, this function attains its maximum values at the points $k = \frac{3}{\log p} - 1$. Then

$$u_{\max}(p) = \max\left(\frac{k_f + 1}{\sqrt[3]{p^{k_f}}}, \frac{k_c + 1}{\sqrt[3]{p^{k_c}}}\right),$$

where $k_f = \left\lfloor \frac{3}{\log p} - 1 \right\rfloor$ and $k_c = \left\lceil \frac{3}{\log p} - 1 \right\rceil$, implies that the maximum values over $k \geq 0$ for $p = 2, 3, 5, 7, 11, 13, 17, 19$ are, respectively, 2, $\sqrt[3]{3}$, $\frac{2}{\sqrt[3]{5}}$, $\frac{2}{\sqrt[3]{7}}, 1, 1, 1, 1$. Thence, we have $u(k) \leq u_{\max}(p)$ for $p < 23$ and all $k \geq 0$. We put $u_{\max}(p) = 1$ for $p \geq 23$, and let c_3 be the product of all the constant values $u_{\max}(p)$. Clearly, $c_3 = \frac{24}{\sqrt[3]{315}}$. Thus, if we set

$$N(k) = \prod_{i=1}^{k} p_i^{\alpha_i},$$

then

$$\tau(N(k)) = \prod_{i=1}^{k} (\alpha_i + 1) \leq \prod_{i=1}^{k} u_{\max}(p) \sqrt[3]{p_i^{\alpha_i}} \leq c_3 \prod_{i=1}^{k} \sqrt[3]{p_i^{\alpha_i}} = c_3 \sqrt[3]{N(k)}.$$

By multiplicativity, $2^{\omega(N(k))} \le \tau(N(k))$. It follows that

$$\omega(N(k)) \le \frac{1}{\log 2}\left(\log c_3 + \frac{\log N(k)}{3}\right) \tag{5}$$

for small values of $N(k)$. From (4) and (5), with $f_1(N(k)) = \log^2 N(k)$, it follows that

$$\begin{aligned}\sum_{p|N(k)} \frac{1}{p} \le \log\log\log^2 N(k) + c_2 + \frac{1}{(\log\log^2 N(k))^2} \\ + \frac{1}{\log 2}\left(\frac{\log c_3}{\log^2 N(k)} + \frac{1}{3\log N(k)}\right).\end{aligned} \tag{6}$$

If $e^{e^{\sqrt{2}}} \le N(k) \le g^{\frac{1}{2}k(k+1)}$, then from (3) and (6) we see that the sum $s(k)$ in (2) is bounded by

$$v(k) = 3.68417\,33252\,992\left(\log\frac{k(k+1)}{2} + \log\log g\right).$$

Next, using partial summation, and (1) and (2), and letting k tend to infinity, we obtain

$$\begin{aligned}\sum_{\substack{q=1\\(g,q)=1\\\mu(q)^2=1}}^{\infty} \frac{1}{q l_g(q)} &= \sum_{h=1}^{\infty} \frac{1}{h} \sum_{\substack{l_g(q)=h\\(g,q)=1\\\mu(q)^2=1}} \frac{1}{q} = \sum_{j=1}^{\infty} \frac{e(j)}{j} \\ &= \sum_{j=1}^{\infty} \frac{s(j)-s(j-1)}{j} \\ &= \sum_{j=1}^{\infty} \frac{s(j)}{j(j+1)}.\end{aligned}$$

We distinguish three cases.

(i) If $2 \le g \le 8$, then

$$\sum_{j=1}^{\infty} \frac{s(j)}{j(j+1)} \le \sum_{j=1}^{k_0} \frac{s(j)}{j(j+1)} + \sum_{j=k_0+1}^{\infty} \frac{v(j)}{j(j+1)}.$$

(ii) If $8 < g \le 62$, then

$$\sum_{j=1}^{\infty} \frac{s(j)}{j(j+1)} \le \frac{s(1)}{2} + \sum_{j=2}^{\infty} \frac{v(j)}{j(j+1)}.$$

(iii) If $g > 62$, then

$$\sum_{j=1}^{\infty} \frac{s(j)}{j(j+1)} \leq \sum_{j=1}^{\infty} \frac{v(j)}{j(j+1)}.$$

In all these cases, the series on the right-hand side can be computed numerically and exactly. These series are $O(\log\log(g+2))$ at most, thus completing the proof.

3. Proof of Theorem 1

We follow Erdős and Turán's method and apply Cauchy's inequality to

$$\sum_{n=1}^{x} f(n) = \left|\{p + g^k \leq x\}\right|.$$

We forthwith obtain

$$\begin{aligned} \left(\sum_{n=1}^{x} f(n)\right)^2 &= \left(\sum_{\substack{n=1\\ f(n)\geq 1}}^{x} f(n)\right)^2 \\ &\leq \sum_{\substack{n=1\\ f(n)\geq 1}}^{x} 1 \sum_{n=1}^{x} f(n)^2, \end{aligned} \tag{7}$$

where $f(n)^2$ is the number of quadruples (p_1, p_2, k_1, k_2) such that $n = p_1 + g^{k_1} = p_2 + g^{k_2}$, and $\sum_{n=1}^{x} f(n)^2$ is the number of these quadruples such that $(p_1, p_2, k_1, k_2) \leq x$. The latter does not exceed the number of solutions of the equation $p_2 - p_1 = g^{k_1} - g^{k_2}$ with $p_1, p_2 \leq x$ and $k_1, k_2 \leq \frac{\log x}{\log g}$. If $p \leq x(1-\epsilon)$ and $g^k \leq x\epsilon$ for $\epsilon = \frac{1}{\log x}$, then $p + g^k \leq x$. Then, by the prime number theorem,

$$\begin{aligned} \sum_{n=1}^{x} f(n) &\geq \pi(x(1-\epsilon))\frac{\log x\epsilon}{\log g} \\ &\geq (1+o(1))\frac{x}{\log g}. \end{aligned} \tag{8}$$

On the other hand, let $h = g^{k_1} - g^{k_2}$. If $k_1 \neq k_2$, then h is a nonzero, even integer. Let now

$$\pi(x; h) = \sum_{\substack{p_2 \leq x\\ p_2 - p_1 = h}} 1,$$

where the sum is taken over prime numbers $p_2 \leq x$ such that $p_2 - p_1 = h$, and p_1 is a previous prime numbers before p_2, but not necessarily adjacent to p_2. Then $\pi(h; x)$ is at most the number of prime numbers $p_2 \leq x$ such that $p_2 + h$ is also a prime number. Hence, by Chen's upper bound method [2, Theorem 3] on the small sieve,

$$\pi(x;h) \le c_4 \mathfrak{S}(h)\frac{x}{\log^2 x}$$

uniformly in h, where

$$c_4 = (7.8342 + o(1)),$$
$$\mathfrak{S}(h) = c_5 k(h),$$
$$c_5 = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right),$$

and

$$k(h) = \prod_{\substack{p|h\\ p>2}} \frac{p-1}{p-2}.$$

There are three cases here.

(i) If $k_2 > k_1$, then $h = -g^{k_1}(g^{k_2-k_1} - 1)$. Thus, we have

$$k(h) = \prod_{\substack{p|g^{k_1}\\ p>2}} \frac{p-1}{p-2} \prod_{\substack{p|g^{k_2-k_1}-1\\ p>2}} \frac{p-1}{p-2}$$
$$= \prod_{\substack{p|g\\ p>2}} \frac{p-1}{p-2} \prod_{\substack{p|g^{k_2-k_1}-1\\ p>2}} \frac{p-1}{p-2}.$$

(ii) If $k_1 > k_2$, then $h = g^{k_2}(g^{k_1-k_2} - 1)$. Hence, in similar fashion

$$k(h) = \prod_{\substack{p|g\\ p>2}} \frac{p-1}{p-2} \prod_{\substack{p|g^{|k_2-k_1|}-1\\ p>2}} \frac{p-1}{p-2}.$$

(iii) If $k_1 = k_2$, then the number of solutions of the equation $p_2 - p_1 = g^{k_1} - g^{k_2} = 0$ with $p_1 = p_2 \le x$ and $1 \le k_2 \le \frac{\log x}{\log g}$ is

$$\pi(x)\left[\frac{\log x}{\log g}\right] \le \pi(x)\frac{\log x}{\log g} = c_6 x$$

for $x > 1$, where

$$c_6 = (1 + o(1))\frac{1}{\log g}.$$

(The function $[x]$ gives the integer part of x.)

We now return to relation (7). Using

$$k(h) \le c_7 \prod_{\substack{p|h \\ p>2}} \left(1 + \frac{1}{p}\right),$$

where

$$c_7 = \prod_{p>2} \left(\frac{p-1}{p-2}\right)\left(1 + \frac{1}{p}\right)^{-1}$$

and

$$\sum_{\substack{q|m \\ \mu(q)^2=1}} \frac{1}{q} = \prod_{p|m} \left(1 + \frac{1}{p}\right),$$

we compute

$$\begin{aligned}
\sum_{n=1}^{x} f(n)^2 &\le c_6 x + 2c_4c_5 \frac{x}{\log^2 x} \sum_{1 \le k_1 < k_2 \le \frac{\log x}{\log g}} \prod_{\substack{p|g \\ p>2}} \frac{p-1}{p-2} \prod_{\substack{p|g^{k_2-k_1}-1 \\ p>2}} \frac{p-1}{p-2} \\
&\le c_6 x + 2c_4c_5c_7 \frac{x}{\log^2 x} \prod_{\substack{p|g \\ p>2}} \frac{p-1}{p-2} \sum_{1 \le l \le \frac{\log x}{\log g}} \prod_{\substack{p|g^{k_2-k_1}-1 \\ p>2}} \left(1 + \frac{1}{p}\right) \\
&\le c_6 x + 2c_4c_5c_7 \frac{x}{\log^2 x} \left(\frac{\log x}{\log g}\right) \prod_{\substack{p|g \\ p>2}} \frac{p-1}{p-2} \sum_{1 \le l \le \frac{\log x}{\log g}} \prod_{\substack{p|g^l-1 \\ p>2}} \left(1 + \frac{1}{p}\right) \\
&= c_6 x + 2c_4c_5c_7 \frac{1}{\log g} \left(\frac{x}{\log x}\right) \prod_{\substack{p|g \\ p>2}} \frac{p-1}{p-2} \sum_{1 \le l \le \frac{\log x}{\log g}} \sum_{\substack{q|g^l-1 \\ (g,q)=1 \\ \mu(q)^2=1}} \frac{1}{q} \\
&= c_6 x + 2c_4c_5c_7 \frac{1}{\log g} \left(\frac{x}{\log x}\right) \prod_{\substack{p|g \\ p>2}} \frac{p-1}{p-2} \sum_{\substack{q=1 \\ (g,q)=1 \\ \mu(q)^2=1}}^{\infty} \frac{1}{q} \sum_{\substack{l=1 \\ q|g^l-1}}^{\left[\frac{\log x}{\log g}\right]} 1 \\
&\le c_6 x + 2c_4c_5c_7 \frac{1}{\log g} \left(\frac{x}{\log x}\right) \prod_{\substack{p|g \\ p>2}} \frac{p-1}{p-2} \sum_{\substack{q=1 \\ (g,q)=1 \\ \mu(q)^2=1}}^{\infty} \frac{1}{q l_g(q) \log g} \log x \\
&\le c_6 x + 2c_4c_5c_7 \frac{1}{\log^2 g} x \prod_{\substack{p|g \\ p>2}} \frac{p-1}{p-2} \sum_{\substack{q=1 \\ (g,q)=1 \\ \mu(q)^2=1}}^{\infty} \frac{1}{q l_g(q)} \qquad (9)
\end{aligned}$$

for sufficiently large values of x, since

(i) $q \mid g^l - 1$ is equivalent to $g^l \equiv 1 \pmod q$ which is equivalent to $l_g(q) \mid l$,

(ii) the sum over l equals 0 if $(g,q) > 1$, and this sum is less than or equal to $\frac{\log x}{l_g(q) \log g}$ if $(g,q) = 1$, and

(iii) $\left[\frac{\log x}{\log g}\right] \leq \frac{\log x}{\log g}$.

That the series on the extreme right side of (9) converges is a consequence of Lemma 1. Then the first part of the theorem follows.

It remains to prove that the set S has positive lower asymptotic density. Combining (7), (8), and (9), we see that

$$\sum_{\substack{n=1 \\ f(n) \geq 1}}^{x} 1 \geq c_1 x$$

for sufficiently large values of x, where

$$\frac{1}{c_1} = \log g + 2c_4c_5c_7 \prod_{\substack{p \mid g \\ p>2}} \frac{p-1}{p-2} \sum_{\substack{q=1 \\ (g,q)=1 \\ \mu(q)^2=1}}^{\infty} \frac{1}{q l_g(q)}. \tag{10}$$

This proves the second part of the theorem.

4. Proof of Theorem 2

We simply note in (10) that

$$\prod_{\substack{p \mid g \\ p>2}} \frac{p-1}{p-2} \leq c_8 \prod_{\substack{p \mid g \\ p>2}} \left(1 + \frac{1}{p(p-2)}\right) \prod_{p \mid g} \left(1 - \frac{1}{p}\right)^{-1},$$

where c_8 is equal to $\frac{1}{2}$ or 1, depending on whether or not the prime number 2 is a divisor of g. From [14, Theorems 62 and 328],

$$\prod_{\substack{p \mid g \\ p>2}} \frac{p-1}{p-2} = O\left(\frac{g}{\phi(g)}\right) = O(\log\log g),$$

where $\phi(g)$ is the number of positive integers not exceeding g that are relatively prime to g. By Lemma 1, the series in (10) is $O(\log\log g)$ at most. Hence, we have

$$\prod_{\substack{p \mid g \\ p>2}} \frac{p-1}{p-2} \sum_{\substack{q=1 \\ (g,q)=1 \\ \mu(q)^2=1}}^{\infty} \frac{1}{q l_g(q)} = O((\log\log g)^2).$$

Combining this estimate and (10), we see that

$$d_L(S) \geq \frac{1}{\log g + O((\log\log g)^2)}.$$

To finish the proof of the theorem, we compute

$$d_U(S) = \limsup_{x\to\infty} \frac{1}{x} \sum_{\substack{s\in S\\ 0<s\leq x}} 1 \leq \limsup_{x\to\infty} \frac{1}{x}\left(\pi(x)\frac{\log x}{\log g}\right) \leq \frac{1}{\log g}.$$

The prime number theorem is invoked in the penultimate step.

5. Numerical Computation

We shall use Case (i) in the proof of Lemma 1 to produce the numerical approximations for the Romanoff constant c_1 in (10) for small values of g. We readily find that

$$c_5 \leq \prod_{2<p\leq 1\,089\,469} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\,18566\,793.$$

The constant c_7 is given by

$$\begin{aligned} c_7 &= \frac{3^2}{2^4}\zeta(2)^2 \prod_{p>2} \frac{p(p-1)}{(p-2)(p+1)} \prod_{p>2} \frac{(p-1)^2(p+1)^2}{p^4} \\ &= \frac{21^2}{2^{10}}\zeta(2)^2\zeta(3)^2 \prod_{p>2}\left(1 + \frac{3p^6 - 3p^4 + 2p - 1}{p^9(p-2)}\right). \end{aligned}$$

It satisfies the inequality

$$P \leq c_7 \leq PR,$$

where

$$P = \prod_{2<p\leq 1\,089\,469} \frac{p(p-1)}{(p-2)(p+1)} = 1.86878\,48462\,41371\,38130$$

and

$$R \leq \exp\left(\int_{1\,089\,469}^{\infty} \frac{3t^6 - 3t^4 + 2t - 1}{t^9(t-2)}\, dt\right) = 1.00000\,00000\,00000\,00077.$$

Thus, we conclude that

$$c_7 \leq 1.86878\,48462\,41371\,382.$$

Let now

$$S_1(g, k_0) = \sum_{j=1}^{k_0} \frac{s(j)}{j(j+1)}$$

and

$$S_2(g,k_0)=\sum_{j=k_0+1}^{\infty}\frac{v(j)}{j(j+1)}.$$

We proceed to sharpen the upper bound (5) for $\omega(N(k))$ for large values of $N(k)$. To that end, we consider the decomposition

$$\tau(N(k))=\prod_{p_i<f_2(N(k))}(\alpha_i+1)\prod_{p_i\geq f_2(N(k))}(\alpha_i+1), \tag{11}$$

where $f_2(N(k))$ will be specified later. We write

$$\prod_{p_i<f_2(N(k))}(\alpha_i+1)=\exp\Bigg(\sum_{p_i<f_2(N(k))}\log(\alpha_i+1)\Bigg). \tag{12}$$

On noting that $N(k)\geq p_i^{\alpha_i}\geq 2^{\alpha_i}$, we have

$$\alpha_i\leq\frac{\log N(k)}{\log 2},$$

and thus

$$\alpha_i+1\leq\frac{\log N(k)}{\log 2}+1<\log^2 N(k)$$

if $N(k)\geq N_0(k)$ for some $N_0(k)$. This being so, we have

$$\log(\alpha_i+1)<2\log\log N(k). \tag{13}$$

Thus, we obtain by (12) and (13)

$$\begin{aligned}\prod_{p_i<f_2(N(k))}(\alpha_i+1)&<\exp\Bigg(2\log\log N(k)\sum_{p_i<f_2(N(k))}1\Bigg)\\&\leq\exp\{(2\log\log N(k))\pi(f_2(N(k)))\}.\end{aligned}$$

The inequality

$$\pi(x)<\frac{x}{\log x-\frac{3}{2}}$$

for $x>e^{\frac{3}{2}}$ [21, Equation (3.4), p. 69] then gives us

$$\begin{aligned}\prod_{p_i<f_2(N(k))}(\alpha_i+1)&<\exp\Bigg(2\log\log N(k)\frac{f_2(N(k))}{\log f_2(N(k))-\frac{3}{2}}\Bigg)\\&<2^{\frac{2}{\log 2}\left(\frac{f_2(N(k))\log\log N(k)}{\log f_2(N(k))-\frac{3}{2}}\right)}\end{aligned} \tag{14}$$

for $N(k)\geq 8$ and $f_2(N(k))\geq 5$.

Next, using the inequalities $\alpha_i + 1 \le 2^{\alpha_i}$ and $p_i \ge f_2(N(k))$, where $i = 1, \ldots, k$, we obtain

$$\prod_{p_i \ge f_2(N(k))} (\alpha_i + 1) \le 2^{S(N(k),k)},$$

where

$$S(N(k), k) = \sum_{\substack{i=1 \\ p_i \ge f_2(N(k))}}^{k} \alpha_i.$$

Then we have

$$\begin{aligned} N(k) = \prod_{i=1}^{k} p_i^{\alpha_i} &\ge \prod_{p_i \ge f_2(N(k))} p_i^{\alpha_i} \\ &\ge \prod_{p_i \ge f_2(N(k))} f_2(N(k))^{\alpha_i} \\ &= f_2(N(k))^{S(N(k),k)}. \end{aligned}$$

We have further

$$S(N(k), k) \le \frac{\log N(k)}{\log f_2(N(k))},$$

whence

$$\prod_{p_i \ge f_2(N(k))} (\alpha_i + 1) \le 2^{\frac{\log N(k)}{\log f_2(N(k))}}. \tag{15}$$

Assembling (11), (14), and (15), we obtain

$$\tau(N(k)) < 2^{\frac{2}{\log 2}\left(\frac{f_2(N(k)) \log\log N(k)}{\log f_2(N(k)) - \frac{3}{2}}\right) + \frac{\log N(k)}{\log f_2(N(k))}}$$

for $N(k) \ge 8$ and $f_2(N(k)) \ge 5$. Now, we choose $f_2(N(k)) = \log^{\beta} N(k)$ with $\beta \ge 1$. Thus, we have

$$\omega(N(k)) < \frac{2}{\log 2}\left(\frac{(\log^{\beta} N(k))(\log\log N(k))}{\log\log^{\beta} N(k) - \frac{3}{2}}\right) + \frac{\log N(k)}{\log\log^{\beta} N(k)}$$

for $N(k) > e^5$.

The Case (i) in the proof of Lemma 1 and (10) are used to produce the numerical approximations for the sums $S_1(g, k_0)$ and $S_2(g, k_0)$ and the Romanoff constant c_1 for the case $g = 2$ with $k_0 = 239$ and $\beta = 1.1$, and for the case $g = 3$ with $k_0 = 151$ and $\beta = 1.2$. The computer results are indicated in Table 1. These numerical approximations are slightly lowered when we take larger values of k_0. All of our computations make use of the Magma algebra system running on a Sun Microsystem Ultra Enterprise 450 Server and require one month of Central Processing Unit time to complete their execution.

g	$S_1(g, k_0)$	$S_2(g, k_0)$	c_1
2	1.36224 46268 651	0.07880 99357 695	0.03502 76266 383
3	1.71790 58458 532	0.12363 21221 815	0.01383 26200 012

Table 1: Numerical approximations for the sums $S_1(g, k_0)$ and $S_2(g, k_0)$ and the Romanoff constant c_1 for the cases $g = 2$ and $g = 3$.

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