CARTESIAN PRODUCTS OF SOME COMBINATORIALLY RICH SETS

Sayan Goswami
Department of Mathematics, University of Kalyani, West Bengal, India
sayan92m@gmail.com

Received: 12/3/19, Revised: 6/3/20, Accepted: 8/8/20, Published: 8/14/20

Abstract

N. Hindman and D. Strauss have shown that for discrete semigroups, the cartesian product of two central sets is a central set, the cartesian product of two J-sets is a J-set, and the cartesian product of two C-sets is a C-set. The proofs of these results used the algebraic structure of the Stone-Čech compactification of a discrete semigroup. In this paper, we will give combinatorial proofs of all three results, and the proof for J-sets will be much simpler than the previous proof.

1. Introduction

We take $\mathbb{N}$ to be the set of positive integers and $\mathbb{Z}$ to be the set of all integers. Given a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X.

In [2], H. Furstenberg defined *central* subsets of $\mathbb{N}$ in terms of notions from topological dynamics, showed that if $\mathbb{N}$ is divided into finitely many classes, then one of these classes must be central, and proved the original Central Sets Theorem.

Theorem 1.1 (Original Central Sets Theorem)**.** *Let A be a central subset of $\mathbb{N}$, let $k \in \mathbb{N}$, and for each $i \in \{1, 2, \ldots, k\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. There exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for each n, $\max H_n < \min H_{n+1}$ and

(2) for each $i \in \{1, 2, \ldots, k\}$ and each $F \in \mathcal{P}_f(\mathbb{N})$, $\sum_{n\in F}(a_n + \sum_{t\in H_n} y_{i,t}) \in A$.

Furstenberg then showed that central subsets of $\mathbb{N}$ have strong combinatorial properties.

Central subsets of a discrete semigroup S have been characterized in terms of the algebra of the Stone-Čech compactification βS of S as those subsets of S whose closure contains an idempotent in the smallest ideal $K(\beta S)$ of βS. A *quasi-central* set was then defined to be a set whose closure contains an idempotent in the topological clusure of $K(\beta S)$. Since we will not be using the algebra of βS in this paper, we will not describe that algebra here. The interested reader can consult [5, Part I].

In [1], D. De, N. Hindman, and D. Strauss came up with a stronger version of the Central Sets Theorem. We state the version for commutative semgroups now. We write ${}^{\mathbb{N}}S$ for the set of sequences in S.

Theorem 1.2 (Commutative Central Sets Theorem)**.** *Let $(S, \cdot)$ be a commutative semigroup and let A be a central subset of S. There exist functions $\alpha\colon \mathcal{P}_f({}^{\mathbb{N}}S) \to S$ and $H\colon \mathcal{P}_f({}^{\mathbb{N}}S) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) if $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has $\prod_{i=1}^{m} \big(\alpha(G_i) \cdot \prod_{t \in H(G_i)} f_i(t)\big) \in A$.

There is a more complicated version for arbitrary semigroups which we will not state because that statement is not relevant to our proofs (which do apply to arbitrary semigroups, however).

From the point of view of Ramsey Theory, the major importance of central sets is that they satisfy a long list of combinatorial properties, most of which are consequences of the Central Sets Theorem. See [5, Part III] for many examples.

A subset of a semigroup is said to be a C-set provided it satisfies the conclusion of the (general) Central Sets Theorem. An important tool for the characterization of C-sets is the notion of J-set, which we will define in the next section.

In [4], Hindman and Strauss showed that in an arbitrary semigroup, the product of central sets is central, the product of J-sets is a J-set, and the product of C-sets is a C-set. The proofs used the algebra of βS (or, in the case of J-sets, the algebra of the Stone-Čech compactification of a discrete partial semigroup). Further, an algebraic proof that the product of quasi-central sets is quasi-central is an easy consequence of the results in [4].

In this paper we will give combinatorial proofs of all four of these assertions.

2. Products of Combinatorially Rich Sets

The characterizations of quasi-central sets, central sets, and C-sets that we will use depend on some combinatorial notions that we define now. Given a subset A of a semigroup S and $x \in S$, $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

Definition 2.1. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(a) The set A is *piecewise syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$, there exists $x \in S$ such that $F \cdot x \subseteq \bigcup_{t \in G} t^{-1}A$.

(b) If S is commutative, then A is a *J-set* if and only if whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a \cdot \prod_{t \in H} f(t) \in A$.

(c) For arbitrary S, A is a *J-set* if and only if whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m)$ in $\mathbb{N}$ such that for each $f \in F$, $a(1) \cdot f\big(t(1)\big) \cdot a(2) \cdot f\big(t(2)\big) \cdot a(3) \cdots a(m) \cdot f\big(t(m)\big) \cdot a(m+1) \in A$.

It is shown in [5, Lemma 14.14.2] that Definitions 2.1(b) and (c) agree if S is commutative.

The next notion is unfortunately quite complicated. (There are simpler versions that are equivalent if S is countable. See for example [6, page 63].)

Definition 2.2. Let $(S, \cdot)$ be a semigroup and let $\mathcal{A} \subseteq \mathcal{P}(S)$. Then $\mathcal{A}$ is *collectionwise piecewise syndetic* if and only if there exist functions $K\colon \mathcal{P}_f(\mathcal{A}) \to \mathcal{P}_f(S)$ and $x\colon \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(S) \to S$ such that for all $F \in \mathcal{P}_f(S)$ and all $\mathcal{F}$ and $\mathcal{H}$ in $\mathcal{P}_f(\mathcal{A})$ with $\mathcal{F} \subseteq \mathcal{H}$ one has $F \cdot x(\mathcal{H}, F) \subseteq \bigcup_{t \in K(\mathcal{F})} t^{-1}(\bigcap \mathcal{F})$.

When we write that $\langle C_F \rangle_{F \in \mathcal{I}}$ is a downward directed family, we mean that $(\mathcal{I}, \geq)$ is a directed set and when $F, G \in \mathcal{I}$ with $F \geq G$, one has $C_F \subseteq C_G$.

Theorem 2.3. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.*

(a) The set A is quasi-central if and only if there is a downward directed family $\langle C_F \rangle_{F \in \mathcal{I}}$ of subsets of A such that

(i) for each $F \in \mathcal{I}$ and each $x \in C_F$, there exists $G \in \mathcal{I}$ such that $C_G \subseteq x^{-1}C_F$ and

(ii) for each $F \in \mathcal{I}$, C_F is piecewise syndetic.

(b) The set A is central if and only if there is a downward directed family $\langle C_F \rangle_{F \in \mathcal{I}}$ of subsets of A such that

(i) for each $F \in \mathcal{I}$ and each $x \in C_F$, there exists $G \in \mathcal{I}$ such that $C_G \subseteq x^{-1}C_F$ and

(ii) $\{C_F : F \in \mathcal{I}\}$ is collectionwise piecewise syndetic.

(c) The set A is a C-set if and only if there is a downward directed family $\langle C_F \rangle_{F \in \mathcal{I}}$ of subsets of A such that

(i) for each $F \in \mathcal{I}$ and each $x \in C_F$, there exists $G \in \mathcal{I}$ such that $C_G \subseteq x^{-1}C_F$ and

(ii) for each $\mathcal{F} \in \mathcal{P}_f(\mathcal{I})$, $\bigcap_{F \in \mathcal{F}} C_F$ is a J-set.

Proof. (a) is [3, Theorem 3.7]; (b) is [3, Theorem 3.8]; and (c) is [5, Theorem 14.27]. □

Theorem 2.4. *Let $(S, \cdot)$ and $(T, \cdot)$ be semigroups, let A be a quasi-central set in S, and let B be a quasi-central set in T. Then $A \times B$ is a quasi-central set in $S \times T$.*

Proof. Let $\langle C_F\rangle_{F\in\mathcal{I}}$ be as guaranteed by Theorem 2.3(a) for A and let $\langle D_G\rangle_{G\in\mathcal{J}}$ be as guaranteed by Theorem 2.3(a) for B. Direct $\mathcal{I}\times\mathcal{J}$ by agreeing that $(F,G)\geq(F',G')$ if and only if $F\geq F'$ and $G\geq G'$. We claim that $\langle C_F\times D_G\rangle_{(F,G)\in\mathcal{I}\times\mathcal{J}}$ is as required by Theorem 2.3(a) to show that $A\times B$ is quasi-central in $S\times T$.

To verify condition (i), let $(F,G)\in\mathcal{I}\times\mathcal{J}$ and let $(x,y)\in C_F\times D_G$. Pick $H\in\mathcal{I}$ and $K\in\mathcal{J}$ such that $C_H\subseteq x^{-1}C_F$ and $D_K\subseteq y^{-1}D_G$. Then $(H,K)\in\mathcal{I}\times\mathcal{J}$ and $C_H\times D_K\subseteq(x,y)^{-1}(C_F\times D_G)$.

To verify condition (ii), let $(F,G)\in\mathcal{I}\times\mathcal{J}$. Pick $C\in\mathcal{P}_f(S)$ such that for each $H\in\mathcal{P}_f(S)$, there exists $x\in S$ such that $H\cdot x\subseteq\bigcup_{s\in C}s^{-1}A$. Pick $D\in\mathcal{P}_f(T)$ such that for each $K\in\mathcal{P}_f(T)$, there exists $y\in T$ such that $K\cdot y\subseteq\bigcup_{t\in D}t^{-1}B$. Then $C\times D\in\mathcal{P}_f(S\times T)$. Let $L\in\mathcal{P}_f(S\times T)$ be given and pick $H\in\mathcal{P}_f(S)$ and $K\in\mathcal{P}_f(T)$ such that $L\subseteq H\times K$. Pick $x\in S$ and $y\in T$ such that $H\cdot x\subseteq\bigcup_{s\in C}s^{-1}A$ and $K\cdot y\subseteq\bigcup_{t\in D}t^{-1}B$. Then $L\cdot(x,y)\subseteq(H\times K)\cdot(x,y)\subseteq\bigcup_{(s,t)\in C\times D}(s,t)^{-1}(A\times B)$. □

Theorem 2.5. *Let $(S,\cdot)$ and $(T,\cdot)$ be semigroups, let A be a central set in S, and let B be a central set in T. Then $A\times B$ is a central set in $S\times T$.*

Proof. Let $\langle C_F\rangle_{F\in\mathcal{I}}$ be as guaranteed by Theorem 2.3(b) for A and let $\langle D_G\rangle_{G\in\mathcal{J}}$ be as guaranteed by Theorem 2.3(b) for B. Direct $\mathcal{I}\times\mathcal{J}$ by agreeing that $(F,G)\geq(F',G')$ if and only if $F\geq F'$ and $G\geq G'$. We claim that $\langle C_F\times D_G\rangle_{(F,G)\in\mathcal{I}\times\mathcal{J}}$ is as required by Theorem 2.3(b) to show that $A\times B$ is central in $S\times T$.

Exactly as in the proof of Theorem 2.4 one sees that $\langle C_F\times D_G\rangle_{(F,G)\in\mathcal{I}\times\mathcal{J}}$ satisfies condition (i).

Let $\mathcal{A}=\{C_F\colon F\in\mathcal{I}\}$, let $\mathcal{B}=\{D_G\colon G\in\mathcal{J}\}$, and let $\mathcal{D}=\{C_F\times D_G\colon (F,G)\in\mathcal{I}\times\mathcal{J}\}$. We have that $\mathcal{A}$ and $\mathcal{B}$ are collectionwise piecewise syndetic so pick functions $K_1\colon\mathcal{P}_f(\mathcal{A})\to\mathcal{P}_f(S)$, $x_1\colon\mathcal{P}_f(\mathcal{A})\times\mathcal{P}_f(S)\to S$, $K_2\colon\mathcal{P}_f(\mathcal{B})\to\mathcal{P}_f(T)$, and $x_2\colon\mathcal{P}_f(\mathcal{B})\times\mathcal{P}_f(T)\to T$ as guaranteed by Definition 2.2 for A and B respectively.

Given $\mathcal{H}\in\mathcal{P}_f(\mathcal{D})$, let $\mathcal{H}_1=\{C_F\colon F\in\mathcal{I}$ and there exists $G\in\mathcal{J}$ such that $C_F\times D_G\in\mathcal{H}\}$ and let $\mathcal{H}_2=\{D_G\colon G\in\mathcal{J}$ and there exists $F\in\mathcal{I}$ such that $C_F\times D_G\in\mathcal{H}\}$. Then $\mathcal{H}_1\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{H}_2\in\mathcal{P}_f(\mathcal{B})$. Given $F\in\mathcal{P}_f(S\times T)$, let $F_1=\pi_1[F]$ and $F_2=\pi_2[F]$, where π_1 and π_2 are the projection functions.

Define $K_3\colon\mathcal{P}_f(\mathcal{D})\to\mathcal{P}_f(S\times T)$ by $K_3(\mathcal{H})=K_1(\mathcal{H}_1)\times K_2(\mathcal{H}_2)$ and define $x_3\colon\mathcal{P}_f(\mathcal{D})\times\mathcal{P}_f(S\times T)\to S\times T$ by, for $\mathcal{H}\in\mathcal{P}_f(\mathcal{D})$ and $F\in\mathcal{P}_f(S\times T)$, $x(\mathcal{H},F)=\big(x_1(\mathcal{H}_1,F_1),x_2(\mathcal{H}_2,F_2)\big)$.

Now let $\mathcal{F}$ and $\mathcal{H}$ in $\mathcal{P}_f(\mathcal{D})$ such that $\mathcal{F}\subseteq\mathcal{H}$ and let $F\in\mathcal{P}_f(S\times T)$. Then $\mathcal{F}_1\subseteq\mathcal{H}_1$ and $\mathcal{F}_2\subseteq\mathcal{H}_2$ so $F_1\cdot x(\mathcal{H}_1,F_1)\subseteq\bigcup_{s\in K_1(\mathcal{F}_1)}s^{-1}(\bigcap\mathcal{F}_1)$ and $F_2\cdot x(\mathcal{H}_2,F_2)\subseteq\bigcup_{t\in K_2(\mathcal{F}_2)}t^{-1}(\bigcap\mathcal{F}_2)$. Consequently, $F\cdot x_3(\mathcal{H},F)\subseteq\bigcup_{(s,t)\in K_3(\mathcal{F})}(s,t)^{-1}(\bigcap\mathcal{F})$ as required. □

We now set out to show that the product of J-sets is a J-set. The proof for general semigroups is essentially the same as for commutative semigroups, but the commutative version is simpler, so we begin with that.

Lemma 2.6. *Let $(S,\cdot)$ be a commutative semigroup, let A be a J-set in S, and let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. Let $\Theta = \{L \in \mathcal{P}_f(\mathbb{N})\colon (\exists a \in S)(\forall f \in F)(a \cdot \prod_{t\in L} f(t) \in A)\}$. Let $\langle H_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$. There exists $K \in \mathcal{P}_f(\mathbb{N})$ such that $\bigcup_{n\in K} H_n \in \Theta$.*

Proof. For each $f \in F$, define $g_f \in {}^{\mathbb{N}}S$ by, for $n \in \mathbb{N}$, $g_f(n) = \prod_{t\in H_n} f(t)$. Since A is a J-set, pick $a \in S$ and $K \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a \cdot \prod_{n\in K} g_f(n) \in A$. Then $\bigcup_{n\in K} H_n \in \Theta$. □

Recall that, given a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$, $FU(\langle H_n\rangle_{n=1}^{\infty}) = \{\bigcup_{n\in F} H_n : F \in \mathcal{P}_f(\mathbb{N})\}$, and a sequence $\langle G_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$, is a *union subsystem* of $\langle H_n\rangle_{n=1}^{\infty}$ if and only if there exists a sequence $\langle K_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max K_n < \min K_{n+1}$ for each $n \in \mathbb{N}$ and $G_n = \bigcup_{t\in K_n} H_t$ for each n.

Lemma 2.7. *Let $(S,\cdot)$ be a commutative semigroup, let A be a J-set in S, and let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. Let $\Theta = \{L \in \mathcal{P}_f(\mathbb{N})\colon (\exists a \in S)(\forall f \in F)(a \cdot \prod_{t\in L} f(t) \in A)\}$. Let $\langle H_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$. There is a union subsystem $\langle G_n\rangle_{n=1}^{\infty}$ of $\langle H_n\rangle_{n=1}^{\infty}$ such that $FU(\langle G_n\rangle_{n=1}^{\infty}) \subseteq \Theta$.*

Proof. Let $\mathcal{A}_1 = \Theta \cap FU(\langle H_n\rangle_{n=1}^{\infty})$ and let $\mathcal{A}_2 = FU(\langle H_n\rangle_{n=1}^{\infty}) \setminus \Theta$. By [5, Corollary 5.17] pick $i \in \{1,2\}$ and a union subsystem $\langle G_n\rangle_{n=1}^{\infty}$ of $\langle H_n\rangle_{n=1}^{\infty}$ such that $FU(\langle G_n\rangle_{n=1}^{\infty}) \subseteq \mathcal{A}_i$. By Lemma 2.6, pick $K \in \mathcal{P}_f(\mathbb{N})$ such that $\bigcup_{n\in K} G_n \in \Theta$. Since $\bigcup_{n\in K} G_n \in FU(\langle G_n\rangle_{n=1}^{\infty})$ we must have that $i = 1$. □

Theorem 2.8. *Let $(S,\cdot)$ and $(T,\cdot)$ be commutative semigroups, let A be a J-set in S, and let B be a J-set in T. Then $A \times B$ is a J-set in $S \times T$.*

Proof. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}(S\times T))$, let $F_1 = \{\pi_1 \circ f\colon f \in F\}$ and let $F_2 = \{\pi_2 \circ f\colon f \in F\}$. Then $F_1 \in \mathcal{P}_f({}^{\mathbb{N}}S)$ so by Lemma 2.7 pick a union subsystem $\langle H_n\rangle_{n=1}^{\infty}$ of $\langle\{n\}\rangle_{n=1}^{\infty}$ such that $FU(\langle H_n\rangle_{n=1}^{\infty}) \subseteq \{L \in \mathcal{P}_f(\mathbb{N})\colon (\exists a \in S)(\forall f \in F)(a \cdot \prod_{t\in L} \pi_1\big(f(t)\big) \in A)\}$. Since $F_2 \in \mathcal{P}_f({}^{\mathbb{N}}T)$, using Lemma 2.6 pick $K \in \mathcal{P}_f(\mathbb{N})$ such that

$$\bigcup_{n\in K} H_n \in \{L \in \mathcal{P}_f(\mathbb{N})\colon (\exists b \in T)(\forall f \in F)(b \cdot \prod_{t\in L} \pi_2\big(f(t)\big) \in B)\}\,.$$

Let $G = \bigcup_{n\in K} H_n$. Pick $a \in S$ and $b \in T$ such that for each $f \in F$, $a \cdot \prod_{t\in G} \pi_1\big(f(t)\big) \in A$ and $b \cdot \prod_{t\in G} \pi_2\big(f(t)\big) \in B$. Then for each $f \in F$, $(a,b) \cdot \prod_{t\in G} f(t) \in A \times B$. □

Given $L \in \mathcal{P}_f(\mathbb{N})$, we write $L = \{t(1), t(2), \ldots, t(m)\}_<$ to indicate that $|L| = m$, $t(1) < t(2) < \ldots < t(m)$, and $L = \{t(1), t(2), \ldots, t(m)\}$.

Lemma 2.9. *Let $(S,\cdot)$ be an arbitrary semigroup, let A be a J-set in S, and let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. Let $\Theta = \{L \in \mathcal{P}_f(\mathbb{N})\colon L = \{t(1), t(2), \ldots, t(m)\}_<$ and $(\exists a \in S^{m+1})$ $(\forall f \in F)(a(1) \cdot f\big(t(1)\big) \cdot a(2) \cdots a(m) \cdot f\big(t(m)\big) \cdot a(m+1) \in A)\}$. Let $\langle H_n\rangle_{n=1}^{\infty}$ be* *a sequence in* $\mathcal{P}_f(\mathbb{N})$ *such that* $\max H_n < \min H_{n+1}$ *for each* $n \in \mathbb{N}$. *There exists* $K \in \mathcal{P}_f(\mathbb{N})$ *such that* $\bigcup_{n\in K} H_n \in \Theta$.

Proof. For each $n \in \mathbb{N}$, let $\alpha_n = |H_n|$ and write

$$H_n = \{b(n,1), b(n,2), \ldots, b(n,\alpha_n)\}_< .$$

Pick $d \in S$ and for $f \in F$, define $g_f \in {}^{\mathbb{N}}S$ by, for $n \in \mathbb{N}$,

$$g_f(n) = f\big(b(n,1)\big) \cdot d \cdot f\big(b(n,2)\big) \cdot d \cdot \cdots d \cdot f\big(b(n,\alpha_n)\big) .$$

Now $\{g_f\colon f \in F\} \in \mathcal{P}_f({}^{\mathbb{N}}S)$ so pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m)$ in $\mathbb{N}$ such that $a(1) \cdot g_f\big(t(1)\big) \cdot a(2) \cdots a(m) \cdot g_f\big(t(m)\big) \cdot a(m+1) \in A$. Let $K = \big\{t(1), t(2), \ldots, t(m)\big\}$. Then K is as required.

As that last assertion is a bit complicated, we will illustrate assuming $m = 2$, $\alpha_{t(1)} = 3$, and $\alpha_{t(2)} = 2$. Then

$$\textstyle\bigcup_{n\in K} H_n = \{b(t(1),1), b(t(1),2), b(t(1),3), b(t(2),1), b(t(2),2)\}$$

and $a(1) \cdot g_f\big(t(1)\big) \cdot a(2) \cdots a(m) \cdot g_f\big(t(m)\big) \cdot a(m+1) = a(1) \cdot f\big(b(t(1),1)\big) \cdot d \cdot f\big(b(t(1),2)\big) \cdot d \cdot f\big(b(t(1),3)\big) \cdot a(2) \cdot f\big(b(t(2),1)\big) \cdot d \cdot f\big(b(t(2),2)\big) \cdot a(3)$. If $G = \bigcup_{n\in K} H_n$, then $|G| = 5$. If $c(1) = a(1)$, $c(2) = c(3) = d$, $c(4) = a(2)$, $c(5) = d$, and $c(6) = a(3)$, then $c \in S^6$ and c is as required to show that $G \in \Theta$. □

Lemma 2.10. *Let* $(S,\cdot)$ *be an arbitrary semigroup, let* A *be a J-set in* S, *and let* $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. *Let* $\Theta = \{L \in \mathcal{P}_f(\mathbb{N})\colon L = \{t(1), t(2), \ldots, t(m)\}_<$ *and* $(\exists a \in S^{m+1})$ $(\forall f \in F)(a(1) \cdot f\big(t(1)\big) \cdot a(2) \cdots a(m) \cdot f\big(t(m)\big) \cdot a(m+1) \in A)\}$. *Let* $\langle H_n\rangle_{n=1}^\infty$ *be a sequence in* $\mathcal{P}_f(\mathbb{N})$ *such that* $\max H_n < \min H_{n+1}$ *for each* $n \in \mathbb{N}$. *There is a union subsystem* $\langle G_n\rangle_{n=1}^\infty$ *of* $\langle H_n\rangle_{n=1}^\infty$ *such that* $FU(\langle G_n\rangle_{n=1}^\infty) \subseteq \Theta$.

Proof. This may be taken verbatim from the proof of Lemma 2.7, substituting an appeal to Lemma 2.9 for the appeal to Lemma 2.6. □

Theorem 2.11. *Let* $(S,\cdot)$ *and* $(T,\cdot)$ *be a arbitrary semigroups, let* A *be a J-set in* S, *and let* B *be a J-set in* T. *Then* $A \times B$ *is a J-set in* $S \times T$.

Proof. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}(S \times T))$, let $F_1 = \{\pi_1 \circ f\colon f \in F\}$ and let $F_2 = \{\pi_2 \circ f\colon f \in F\}$. Then $F_1 \in \mathcal{P}_f({}^{\mathbb{N}}S)$ so by Lemma 2.10 pick a union subsystem $\langle H_n\rangle_{n=1}^\infty$ of $\langle\{n\}\rangle_{n=1}^\infty$ such that $FU(\langle H_n\rangle_{n=1}^\infty) \subseteq \{L \in \mathcal{P}_f(\mathbb{N})\colon L = \{t(1), t(2), \ldots, t(m)\}_<$ and $(\exists a \in S^{m+1})(\forall f \in F)(a(1) \cdot \pi_1\big(f\big(t(1)\big)\big) \cdot a(2) \cdots a(m) \cdot \pi_1\big(f\big(t(m)\big)\big) \cdot a(m+1) \in A)\}$. Using Lemma 2.9, pick $K \in \mathcal{P}_f(\mathbb{N})$ such that $\bigcup_{n\in K} H_n \in \{L \in \mathcal{P}_f(\mathbb{N}) : L = \{t(1), t(2), \ldots, t(m)\}_<$ and $(\exists b \in T^{m+1})(\forall f \in F)(b(1) \cdot \pi_2\big(f\big(t(1)\big)\big) \cdot b(2) \cdots b(m) \cdot \pi_2\big(f\big(t(m)\big)\big) \cdot b(m+1) \in B)\}$. Let $L = \bigcup_{n\in K} H_n$, write $L = \{t(1), t(2), \ldots, t(m)\}_<$, pick $a \in S^{m+1}$ and $b \in T^{m+1}$ such that for all $f \in F$, $(a(1) \cdot \pi_1\big(f\big(t(1)\big)\big) \cdot$

$a(2)\cdots a(m)\cdot \pi_1\big(f\big(t(m)\big)\big)\cdot a(m+1)\in A$ and $\big(b(1)\cdot \pi_2\big(f\big(t(1)\big)\big)\cdot b(2)\cdots b(m)\cdot \pi_2\big(f\big(t(m)\big)\big)\cdot b(m+1)\in B\big)$. For $i\in\{1,2,\ldots,m\}$ let $c(i)=\big(a(i),b(i)\big)$. Then $c\in(S\times T)^{m+1}$ and for all $f\in F$, $\big(c(1)\cdot f\big(t(1)\big)\cdot c(2)\cdots c(m)\cdot f\big(t(m)\big)\cdot c(m+1)\in A\times B$. □

Theorem 2.12. *Let $(S,\cdot)$ and $(T,\cdot)$ be semigroups, let A be a C-set in S, and let B be a C-set set in T. Then $A\times B$ is a C-set in $S\times T$.*

Proof. Let $\langle C_F\rangle_{F\in\mathcal{I}}$ be as guaranteed by Theorem 2.3(c) for A and let $\langle D_G\rangle_{G\in\mathcal{J}}$ be as guaranteed by Theorem 2.3(c) for B. Direct $\mathcal{I}\times\mathcal{J}$ by agreeing that $(F,G)\geq(F',G')$ if and only if $F\geq F'$ and $G\geq G'$. We claim that $\langle C_F\times D_G\rangle_{(F,G)\in\mathcal{I}\times\mathcal{J}}$ is as required by Theorem 2.3(c) to show that $A\times B$ is a C-set in $S\times T$.

Exactly as in the proof of Theorem 2.4 one sees that $\langle C_F\times D_G\rangle_{(F,G)\in\mathcal{I}\times\mathcal{J}}$ satisfies condition (i).

To see that condition (ii) holds, let $\mathcal{F}\in\mathcal{P}_f(\mathcal{I}\times\mathcal{J})$ be given. Let $\mathcal{F}_1=\{C_F\colon F\in\mathcal{I}$ and there exists $G\in\mathcal{J}$ such that $C_F\times D_G\in\mathcal{F}\}$ and let $\mathcal{F}_2=\{D_G\colon G\in\mathcal{J}$ and there exists $F\in\mathcal{I}$ such that $C_F\times D_G\in\mathcal{F}\}$.

Then $\bigcap\mathcal{F}_1$ is a J-set in S and $\bigcap\mathcal{F}_2$ is a J-set in T so by Theorem 2.11, $\big(\bigcap\mathcal{F}_1\big)\times\big(\bigcap\mathcal{F}_2\big)$ is a J-set in $S\times T$ and $\big(\bigcap\mathcal{F}_1\big)\times\big(\bigcap\mathcal{F}_2\big)\subseteq\bigcap\mathcal{F}$. □

Acknowledgment. The author acknowledges the grant UGC-NET SRF fellowship with id no. 421333 of CSIR-UGC NET 2016. We would like to thank the anonymous referee for his enormous help to make the paper in its present form.

References

[1] D. De, N. Hindman, and D. Strauss, A new and stronger Central Sets Theorem, *Fund. Math.* **199** (2008), 155-175.

[2] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorical Number Theory*, Princeton University Press, Princeton, 1981.

[3] N. Hindman, A. Maleki, and D. Strauss, Central sets and their combinatorial characterization, *J. Combin. Theory Ser. A* **74** (1996), 188-208.

[4] N. Hindman and D. Strauss, Cartesian products of sets satisfying the Central Sets Theorem, *Topology Proc.* **35** (2010), 203-223.

[5] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification: Theory and Applications, 2nd edition*, Walter de Gruyter & Co., Berlin, 2012.

[6] R. McCutcheon, *Elemental Methods in Ergodic Ramsey Theory*, Springer-Verlag, Berlin, 1999.