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MONOCHROMATIC SUMS EQUAL TO PRODUCTS NEAR ZERO

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Abstract

Hindman proved that whenever the set $\mathbb{N}$ of natural numbers is finitely colored, there must exist non-constant monochromatic solutions to the equation $a+b=cd$. In this paper we extend this result for dense subsemigroups of $((0,\infty),+)$ to near zero, that is, we will get the above mentioned monochromatic solutions as small as we want.

1. Introduction

In [4], P. Csikvári, A. Sárközy, and K. Gyarmati asked whether, whenever the set $\mathbb{N}$ of natural numbers is finitely colored, there must exist monochromatic a, b, c, and d with $a \neq b$ such that $a+b=cd$. In [11], Hindman answered this question affirmatively by showing, in addition, that one can demand that a, b, c, d are all distinct and the color of $a+b$ is the same as that of a, b, c, and d. In fact, he proved a considerably stronger result using the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$.

Let $(S, \cdot)$ be an infinite discrete semigroup. Now, the points of βS are taken to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, let us set $\bar{A} = \{p \in \beta S : A \in p\}$. (In this paper we have used $\bar{A}$ and clA to mean the same thing.) Then the set $\{\bar{A} : A \subseteq S\}$ will become a basis for a topology on βS. The operation $\cdot$ on S can be extended to the Stone-Čech compactification βS of S so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where

$x^{-1}A = \{y \in S : x \cdot y \in A\}$. Readers may see Sections 2.1 and 4.1 of [14] for detailed descriptions.

A nonempty subset I of a semigroup $(T, \cdot)$ is called a *left ideal* of T if $T \cdot I \subseteq I$, a *right ideal* if $I \cdot T \subseteq I$, and a *two-sided ideal* (or simply an *ideal*) if it is both a left and a right ideal. A *minimal left ideal* is a left ideal that does not contain any proper left ideal. Similarly, we can define *minimal right ideal* and the *smallest ideal*. Any compact Hausdorff right topological semigroup $(T, \cdot)$ has the unique smallest two-sided ideal

$$\begin{aligned} K(T) &= \bigcup\{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } T\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R of T, $L \cap R$ is a group and, in particular, $K(T)$ contains an idempotent. An idempotent that belongs to $K(T)$ is called a *minimal idempotent*. For more details readers may consult Section 2.2 of [14].

In [13], Hindman, Maleki, and Strauss introduced different notions of largeness for arbitrary semigroups and studied their combinatorial properties. From now on we use $\mathcal{P}_f(X)$ to denote the set of all finite nonempty subsets of a set X.

Definition 1 ([13])**.** Let $(S, \cdot)$ be a semigroup.

(a) A set $A \subseteq S$ is *syndetic* if only if there exists $G \in \mathcal{P}_f(S)$ with $S \subseteq \bigcup_{t\in G} t^{-1}A$.

(b) A set $A \subseteq S$ is *piecewise syndetic* if only if there exists $G \in \mathcal{P}_f(S)$ such that $\{y^{-1}(\bigcup_{t\in G} t^{-1}A) : y \in S\}$ has the finite intersection property.

Piecewise syndetic sets and syndetic sets play a significant role in describing the algebraic structure of βS. The following theorem from [14] tells us which ultrafilters are in $clK(\beta S)$.

Theorem 1 ([14, Theorem 4.40])**.** *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then $K(\beta S) \cap clA \neq \emptyset$ if and only if A is piecewise syndetic.*

In [8], Furstenberg introduced a number of classes of large sets originating from topological dynamics. Alternative characterizations of such sets are also available in terms of the algebraic structure of $\beta\mathbb{N}$.

We now recall the following definitions from [14].

Definition 2. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then

(a) A is called *Central* in $(S, \cdot)$ if there is some idempotent $p \in K(\beta S)$ such that $A \in p$ and

(b) A is called *Central** in $(S, \cdot)$ if $A \cap B \neq \emptyset$ for every central set B in $(S, \cdot)$.

In [2], Bergelson and Glasscock investigated the interplay between additive and multiplicative largeness. We need the following definition from [11] to state such results.

Definition 3. Let $\langle x_n\rangle_{n=1}^{\infty}$ be an infinite sequence of positive real numbers, let $m\in\mathbb{N}$ and let $\langle y_n\rangle_{n=1}^{m}$be a finite sequence of positive real numbers. Then

(a) $FS(\langle x_n\rangle_{n=1}^{\infty})=\{\sum_{n\in F}x_n : F\in\mathcal{P}_f(\mathbb{N})\}$ and
$FP(\langle x_n\rangle_{n=1}^{\infty})=\{\prod_{n\in F}x_n : F\in\mathcal{P}_f(\mathbb{N})\}$.

(b) $FS(\langle y_n\rangle_{n=1}^{m})=\{\sum_{n\in F}y_n : \emptyset\neq F\subseteq\{1,2,\ldots,m\}\}$ and
$FP(\langle y_n\rangle_{n=1}^{m})=\{\prod_{n\in F}y_n : \emptyset\neq F\subseteq\{1,2,\ldots,m\}\}$.

(c) The sequence $\langle x_n\rangle_{n=1}^{\infty}$ has *distinct finite sums* if and only if whenever $F,G\in\mathcal{P}_f(\mathbb{N})$ and $F\neq G$, one has $\sum_{n\in F}x_n\neq\sum_{n\in G}x_n$. The analogous definition applies to $\langle y_n\rangle_{n=1}^{m}$.

(d) The sequence $\langle x_n\rangle_{n=1}^{\infty}$ has *distinct finite products* if and only if whenever $F,G\in\mathcal{P}_f(\mathbb{N})$ and $F\neq G$, one has $\prod_{n\in F}x_n\neq\prod_{n\in G}x_n$. The analogous definition applies to $\langle y_n\rangle_{n=1}^{m}$.

(e) The sequence $\langle x_n\rangle_{n=1}^{\infty}$ is *strongly increasing* if and only if for each $n\in\mathbb{N}$, $\sum_{t=1}^{n}x_t<x_{n+1}$.

Clearly, if $\langle x_n\rangle_{n=1}^{\infty}$ is strongly increasing, then it has distinct finite sums. We also need to recall the following definition from [2].

Definition 4. Let $A\subseteq\mathbb{N}$. Then A is said to be an *IP_0-set* in $(\mathbb{N},+)$ if and only if for each $m\in\mathbb{N}$, there exists a finite sequence $\langle y_n\rangle_{n=1}^{m}$ such that $FS(\langle y_n\rangle_{n=1}^{m})\subseteq A$.

The following two theorems show that multiplicatively large sets in $\mathbb{N}$ also contain lots of large additive structure.

Theorem 2 ([1, Lemma 5.11])**.** *For all $A\subseteq\mathbb{N}$, if A is syndetic in $(\mathbb{N},\cdot)$ then A is central in $(\mathbb{N},+)$.*

Theorem 3 ([2, Theorem 6.1])**.** *For all $A\subseteq\mathbb{N}$, if A is piecewise syndetic in $(\mathbb{N},\cdot)$ then A is an IP_0-set in $(\mathbb{N},+)$.*

In [12], Hindman and Leader first introduced the semigroup consisting of ultrafilters converging to zero. If S be a dense subsemigroup of $((0,\infty),+)$, then one can define $0^+(S)=\{p\in\beta S : \text{for all } (\epsilon>0)((0,\epsilon)\cap S\in p)\}$. It is necessary to take S as a dense subsemigroup of $((0,\infty),+)$, for otherwise $0^+(S)$ will be empty.

Theorem 4 ([14, Lemma 13.29.(f)])**.** *Let S be a dense subsemigroup of $((0,\infty),+)$ such that $S\cap(0,1)$ is a subsemigroup of $((0,1),\cdot)$ and assume that for each $y\in S\cap(0,1)$ and each $x\in S$, x/y and $yx\in S$. Then $0^+(S)$ is a two-sided ideal of $\beta(S\cap(0,1),\cdot)$.*

In the following discussions, semigroups satisfying the hypothesis of the above theorem will be called *HL semigroups*.

Definition 5. Let S be a dense subsemigroup of $((0,\infty),+)$. Then S is an *HL semigroup* if and only if $S \cap (0,1)$ is a subsemigroup of $((0,1),\cdot)$ and, for each $y \in S \cap (0,1)$ and for each $x \in S$, x/y and $yx \in S$.

In an HL semigroup the requirement $y < 1$ is not essential, but makes the proof of theorems, lemmas, and corollaries related to HL semigroups simpler since, under that assumption, if $x < 1/n$, then $yx < 1/n$.

In [12], Hindman and Leader introduced the notion of a semigroup of ultrafilters near zero which is denoted as $0^+(S)$ and studied some Ramsey theoretic results near zero. In [5], De and Hindman investigated image partition regularity near zero using the algebraic structure of $0^+(S)$. In [6] and [7], De and Paul, and in [3], Biswas, De and Paul, continued their investigations on image partition regularity near zero.

Section 2 is devoted to studying the interplay between additive and multiplicatively large sets near zero. Our main result in this section is that for an HL semigroup S, any piecewise syndetic set in $(S \cap (0,1),\cdot)$ is both an AP-rich set near zero and an IP_0-set near zero.

Using the algebraic structure of $0^+(S)$, we establish the existence of monochromatic solutions to the equation $a + b = cd$ near zero in Section 3.

2. Additive and Multiplicative Largeness Near Zero

Hindman and Leader introduced different notions of large sets near zero in [12]. In this section we study the interplay between additive and multiplicatively large sets near zero. We now recall some definitions from [12].

Definition 6. Let S be a dense subsemigroup of $((0,\infty),+)$.

(a) A set $A \subseteq S$ is *syndetic near zero* if only if for every $\epsilon > 0$ there exist some $F \in \mathcal{P}_f((0,\epsilon) \cap S)$ and some $\delta > 0$ such that $S \cap (0,\delta) \subseteq \bigcup_{t\in F}(-t + A)$.

(b) A subset A of S is *piecewise syndetic near zero* if and only if there exist sequences $\langle F_n\rangle_{n=1}^\infty$ and $\langle \delta_n\rangle_{n=1}^\infty$ such that

   (1) for each $n \in \mathbb{N}$, $F_n \in \mathcal{P}_f((0,1/n) \cap S)$ and $\delta_n \in (0,1/n)$; and

   (2) for all $G \in \mathcal{P}_f(S)$ and all $\mu > 0$, there is some $x \in (0,\mu) \cap S$ such that for all $n \in \mathbb{N}$, $(G \cap (0,\delta_n)) + x \subseteq \bigcup_{t\in F_n}(-t + A)$.

The following theorem is a near zero analogue of Theorem 1, showing that piecewise syndetic sets near zero behave like piecewise syndetic sets existing at infinity.

Theorem 5 ([12, Theorem 3.5])**.** *Let S be a dense subsemigroup of $((0,\infty),+)$ and let $A \subseteq S$. Then $K(0^+(S)) \cap \bar{A} \neq \emptyset$ if and only if A is piecewise syndetic near zero.*

Definition 7. Let S be a dense subsemigroup of $((0,\infty),+)$. A subset A of S is said to be an *IP-set near zero* if and only if there exists a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} x_n$ converges and $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$.

The following Theorem shows that IP sets near zero are members of idempotents existing near zero.

Theorem 6 ([12, Theorem 3.1])**.** *Let S be a dense subsemigroup of $((0,\infty),+)$ and let $A \subseteq S$. Then A is an IP-set near zero if and only if there is some idempotent p in $0^+(S)$ such that $A \in p$.*

Definition 8. Let S be a dense subsemigroup of $((0,\infty),+)$. Define $\mathit{\Gamma}_0(S) = \{p \in \beta S :$ if $A \in p$ then A is an IP-set near zero$\}$.

Remark 1. Clearly $\mathit{\Gamma}_0(S) \subseteq 0^+(S)$.

Lemma 1. *Let S be an HL semigroup. Then $\mathit{\Gamma}_0(S)$ is a left ideal of $\beta(S \cap (0,1),\cdot)$.*

Proof. Note that $\mathit{\Gamma}_0(S) = cl_{0^+(S)}\{p \in 0^+(S) : p+p=p\}$, and therefore is non-empty. Let $p \in \mathit{\Gamma}_0(S)$, and $q \in \beta(S \cap (0,1),\cdot)$, and let $A \in q \cdot p$. Then $\{y \in S \cap (0,1) : y^{-1}A \in p\} \in q$, so pick $y \in S \cap (0,1)$ such that $y^{-1}A \in p$ where each $y^{-1}A$ is computed in $S \cap (0,1)$. Thus $y^{-1}A$ is an IP-set near zero. Take some sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq (y^{-1}A) \cap (0,\epsilon)$ and $\sum_{n=1}^{\infty} x_n$ converges. Let $z_n = yx_n$ for all $n \in \mathbb{N}$. Then $FS(\langle z_n\rangle_{n=1}^{\infty}) \subseteq A \cap (0,\epsilon)$ and $\sum_{n=1}^{\infty} z_n$ is a convergent series in S. Therefore $p \cdot q \in \mathit{\Gamma}_0(S)$ which completes the proof. □

We now study the multiplicative structure of subsets determined by large sets of $(S \cap (0,1),\cdot)$.

Theorem 7 ([12, Theorem 5.6])**.** *Let S be an HL semigroup. Let $r \in \mathbb{N}$ and let $S \cap (0,1) = \bigcup_{i=1}^{r} B_i$. Then there is some $i \in \{1,2,\ldots,r\}$ such that B_i is central near zero and B_i is central in $(S \cap (0,1),\cdot)$.*

As a consequence of the above theorem we have the following corollary.

Corollary 1. *Let S be an HL semigroup. Let $r \in \mathbb{N}$ and let $S \cap (0,1) = \bigcup_{i=1}^{r} B_i$. Then there is some $i \in \{1,2,\ldots,r\}$ such that B_i is an IP-set near zero and B_i is central in $(S \cap (0,1),\cdot)$.*

Proof. By Lemma 1, $\mathit{\Gamma}_0(S)$ is a left ideal of $\beta(S \cap (0,1),\cdot)$. Pick a minimal left ideal L of $\beta(S \cap (0,1),\cdot)$ with $L \subseteq \mathit{\Gamma}_0(S)$ and pick $p = p \cdot p \in L$. Note that $p \in K(\beta(S \cap (0,1),\cdot))$ with $p = p \cdot p$ so all of its members are central in $(S \cap (0,1),\cdot)$. Pick $i \in \{1,2,\ldots,r\}$ such that $B_i \in p$. Then B_i is central in $(S \cap (0,1),\cdot)$ and, since $p \in \mathit{\Gamma}_0(S)$, B_i is an IP-set near zero. □

To prove A is central near zero if A is syndetic in $(S \cap (0,1), \cdot)$, we need the following lemma.

Lemma 2 ([12, Lemma 4.8])**.** *Let S be an HL semigroup. If $A \subseteq S$ and $y^{-1}A$ is central near zero, then A is central near zero.*

Theorem 8. *Let S be an HL semigroup. If A is syndetic in $(S \cap (0,1), \cdot)$ then A is central near zero.*

Proof. Since A is syndetic in $(S \cap (0,1), \cdot)$, there exists $G \in \mathcal{P}_f(S \cap (0,1))$ such that $S \cap (0,1) = \bigcup_{t \in G} t^{-1}A$. Now take an idempotent p in $K(0^+(S))$. Choose $t \in G$ such that $t^{-1}A \in p$. Thus $t^{-1}A$ is central near zero. So by Lemma 2, A is central near zero. □

We can define IP_0-set near zero for a dense subsemigroup of $((0,\infty),+)$ as is defined on $(\mathbb{N},+)$.

Definition 9. Let S be a dense subsemigroup of $((0,\infty),+)$ and $A \subseteq S$. Then A is said to be an *IP_0-set near zero* if and only if for each $m \in \mathbb{N}$ and $\epsilon > 0$, there exists a finite sequence $\langle y_n \rangle_{n=1}^m$ of positive real numbers such that $FS(\langle y_n \rangle_{n=1}^m) \subseteq A \cap (0,\epsilon)$.

We also define AP-rich sets near zero because IP_0-sets near zero and AP-rich sets are both essential for our main result in this section.

Definition 10. Let S be a dense subsemigroup of $((0,\infty),+)$. A set $A \subseteq S$ is said to be an *AP-rich set near zero* if and only if for each $\epsilon > 0$ and $l \in \mathbb{N}$, there exist $a, d \in S$ such that $\{a, a+d, \ldots, a+(l-1)d\} \subseteq A \cap (0,\epsilon)$.

As a consequence of van der Waerden's theorem (see Theorem 10), we have the following remark.

Remark 2. Any piecewise syndetic set near zero is an AP-rich set near zero.

Definition 11. Let S be a dense subsemigroup of $((0,\infty),+)$ and $A \subseteq S$. Then

(a) (i) $I(S) = \{p \in \beta S :$ if $A \in p$ then for each $l \in \mathbb{N}$, there exist $a, d \in S$ such that $\{a, a+d, \ldots, a+(l-1)d\} \subseteq A\}$;

(ii) $I_0(S) = \{p \in \beta S :$ if $A \in p$ then A is an AP-rich set near zero$\}$;

(b) $J_0(S) = \{p \in \beta S :$ if $A \in p$ then A is an IP_0-set near zero$\}$.

Remark 3. Clearly $I_0(S) = I(S) \cap 0^+(S)$.

Lemma 3. *Let S be an HL semigroup. Then both $I_0(S)$ and $J_0(S)$ are two-sided ideals of $\beta(S \cap (0,1), \cdot)$.*

Proof. First we prove the case for $I_0(S)$, and then for $J_0(S)$. By [12, Theorem 4.11], $E(K(0^+(S))) \subseteq I(S)$ (where for a semigroup S, $E(S) = \{x \in S : x$ is an idempotent$\}$). Now by Remark 3, $I_0(S) = I(S) \cap 0^+(S)$. Then clearly $I_0(S) \neq \emptyset$. By Theorem 4, $0^+(S)$ is a two-sided ideal of $\beta(S \cap (0,1), \cdot)$. To show that $I_0(S)$ is a two-sided ideal of $\beta(S \cap (0,1), \cdot)$, it is enough to prove that $I(S) \cap \beta(S \cap (0,1), \cdot)$ is a two-sided ideal of $\beta(S \cap (0,1), \cdot)$. To this end, let $p \in I(S) \cap \beta(S \cap (0,1), \cdot)$ and $q \in \beta(S \cap (0,1), \cdot)$. Suppose $A \in q \cdot p$; then $\{x \in S \cap (0,1) : x^{-1}A \in p\} \in q$. Choose $x \in S \cap (0,1)$ such that $x^{-1}A \in p$. Pick $a, d \in S$ such that $\{a, a+d, \ldots, a+(l-1)d\} \subseteq x^{-1}A$. Then $\{ax, ax+dx, \ldots, ax+(l-1)dx\} \subseteq A$. Thus $q \cdot p \in I(S) \cap \beta(S \cap (0,1), \cdot)$. Also let $A \in p \cdot q$. Then $B = \{x \in S : x^{-1}A \in q\} \in p$. Pick $a, d \in S$ such that $\{a, a+d, \ldots, a+(l-1)d\} \subseteq B$. Then $\bigcap_{i=0}^{l-1}(a+id)^{-1}A \in q$. Take $s \in S \cap (0,1)$ such that $s \in \bigcap_{i=0}^{l-1}(a+id)^{-1}A$. Then $\{as, as+ds, \ldots, as+(l-1)ds\} \subseteq A$. Thus $p \cdot q \in I(S) \cap \beta(S \cap (0,1), \cdot)$. Therefore $I(S) \cap \beta(S \cap (0,1), \cdot)$ is a two-sided ideal of $\beta(S \cap (0,1), \cdot)$. Hence $I_0(S)$ is a two-sided ideal of $\beta(S \cap (0,1), \cdot)$.

Now we prove that $J_0(S)$ is a two-sided ideal of $\beta(S \cap (0,1), \cdot)$. Notice that $\mathit{\Gamma}_0(S) \subseteq J_0(S)$ and therefore $J_0(S) \neq \emptyset$. To this end, let $p \in J_0(S)$ and $q \in \beta(S \cap (0,1), \cdot)$. Suppose $A \in q \cdot p$. Then $B = \{x \in S \cap (0,1) : x^{-1}A \in p\} \in q$. Choose $x \in S \cap (0,1)$ such that $x^{-1}A \in p$. Then for each $m \in \mathbb{N}$ and $\epsilon > 0$, there exists a finite sequence $\langle y_n \rangle_{n=1}^m$ of positive real numbers such that $FS(\langle y_n \rangle_{n=1}^m) \subseteq x^{-1}A \cap (0, \epsilon)$. Thus $FS(\langle xy_n \rangle_{n=1}^m) \subseteq A \cap (0, \epsilon)$ and therefore $q \cdot p \in J_0(S)$.

Also let $A \in p \cdot q$. Then $B = \{x \in S \cap (0,1) : x^{-1}A \in q\} \in p$. So for each $m \in \mathbb{N}$ and $\epsilon > 0$, there exists a finite sequence $\langle y_n \rangle_{n=1}^m$ of positive real numbers such that $F = FS(\langle y_n \rangle_{n=1}^m) \subseteq B \cap (0, \epsilon)$. Then $\bigcap_{y \in F} y^{-1}A \in q$. Choose $x \in \bigcap_{y \in F} y^{-1}A \cap (0,1)$. Then $FS(\langle xy_n \rangle_{n=1}^m) \subseteq A \cap (0, \epsilon)$. Therefore $p \cdot q \in J_0(S)$. Thus $J_0(S)$ is a two-sided ideal of $\beta(S \cap (0,1), \cdot)$. $\square$

The following theorem shows that multiplicatively large sets in $S \cap (0,1)$ contain additive structure near zero.

Theorem 9. *Let S be an HL semigroup. If A is piecewise syndetic in $(S \cap (0,1), \cdot)$, then A is both an AP-rich set near zero and an IP_0-set near zero.*

Proof. Since A is piecewise syndetic in $(S \cap (0,1), \cdot)$, $K(\beta(S \cap (0,1), \cdot)) \cap clA \neq \emptyset$. Now by Lemma 3, we have $K(\beta(S \cap (0,1), \cdot)) \subseteq I_0(S)$ and $K(\beta(S \cap (0,1), \cdot)) \subseteq J_0(S)$. Therefore, $I_0(S) \cap clA \neq \emptyset$ and $J_0(S) \cap clA \neq \emptyset$. Hence A is both an AP-rich set near zero and an IP_0-set near zero. $\square$

There is a strengthened version of Theorem 9. An elegant and combinatorial proof is provided by the referee and is given in Theorem 12. To prove the theorem we need two famous theorems. The following is the well-known, equivalent, finite version of van der Waerden's theorem.

Theorem 10 ((van der Waerden) [16])**.** *For all k and l there exists n such that for every partition of $\{1, 2, \ldots, n\}$ into k parts, one of the parts contains an arithmetic progression of length l.*

There is another well-known Ramsey theoretic result, which is known as Folkman's theorem. We now use the phrase "finite sums set on l generators" which means $FS(\langle x_k \rangle_{k=1}^{l}) = \{\sum_{k \in F} x_k : \emptyset \neq F \subseteq \{1, 2, \ldots, l\}\}$ for some finite sequence $\langle x_k \rangle_{k=1}^{l}$ of natural numbers.

Theorem 11 ((Folkman) [9])**.** *For all k and l there exists n such that for every partition of $\{1, 2, \ldots, n\}$ into k parts, one of the parts contains a finite sums set on l generators.*

Theorem 11 actually follows from much earlier work of Rado [15].

The following theorem states that a set containing multiplicative structure in $(S \cap (0, 1), \cdot)$ also contains additive structure near zero.

Theorem 12. *Let S be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$. If A is piecewise syndetic in $(S \cap (0, 1), \cdot)$, then A is IP_0 near zero and AP-rich near zero.*

Proof. Since A is piecewise syndetic in $(S \cap (0, 1), \cdot)$, there exist $y_1, y_2, \ldots, y_k \in S \cap (0, 1)$ such that $\{y^{-1}B : y \in S \cap (0, 1)\}$ has the finite intersection property, where $B = y_1^{-1}A \cup \cdots \cup y_k^{-1}A$ (each $y_i^{-1}A$ is computed in $S \cap (0, 1), i = 1, 2, \ldots, k$). Let $\epsilon > 0$ and $l \in \mathbb{N}$. Now by Theorem 10, there exists n_1 such that, for every partition of $\{1, 2, \ldots, n_1\}$ into k parts, one of the parts contains an arithmetic progression of length l, and by Theorem 11, there exists n_2 such that, for every partition of $\{1, 2, \ldots, n_2\}$ into k parts, one of the parts contains a finite sums set on l generators. Let $n = max\{n_1, n_2\}$. Now choose $\delta \in (0, \epsilon/(3n))$ and let $D = \delta\{1, 2, \ldots, n\}$. Note that $D \subseteq S \cap (0, 1)$. Clearly $\bigcap_{y \in D} y^{-1}B \neq \emptyset$. Now choose $s \in \bigcap_{y \in D} y^{-1}B$ and hence $sD \subseteq B$. The partition $y_1^{-1}A \cup \cdots \cup y_k^{-1}A$ of B yields a partition $sD = (sD \cap y_1^{-1}A) \cup \cdots \cup (sD \cap y_k^{-1}A)$ of sD. By the assumption above, there exist $i, j \in \{1, 2, \ldots, k\}$ such that $\{a, a + d, \ldots, a + (l - 1)d\} \subseteq sD \cap y_i^{-1}A$ for some $a, d \in S$, and $FS(\langle x_k \rangle_{k=1}^{n}) \subseteq sD \cap y_j^{-1}A$ for some $\langle x_k \rangle_{k=1}^{n}$. Thus $\{ay_i, ay_i + dy_i, \ldots, ay_i + (l - 1)dy_i\} \subseteq A \cap (0, \epsilon)$ and $FS(\langle y_j x_k \rangle_{k=1}^{n}) \subseteq A \cap (0, \epsilon)$. Since l is arbitrary, A is both IP_0 near zero and AP-rich near zero. □

3. Monochromatic Solutions To $\sum_{t=1}^{n} x_t = \prod_{t=1}^{n} y_t$ Near Zero

In [11], Hindman generalized the affirmative answer to the question raised by P. Csikvári, A. Sárközy, and K. Gyarmati regarding monochromatic solutions to $a+b = cd$, whenever the set of natural numbers $\mathbb{N}$ is finitely colored.

Our main aim is to show that there exists a monochromatic solution to the equation $a+b=cd$ near zero for dense subsemigroups of $((0,\infty),+)$, which means we have to show that, for any finite partition of $S\cap(0,1)$, at least one cell contains an abundance of finite sums that equal finite products.

Theorem 13 ([11, Theorem 5]). *Let $r\in\mathbb{N}$ and let $\mathbb{N}=\bigcup_{i=1}^{r}A_i$. There exists $i\in\{1,2,\ldots,r\}$ such that for each $m\in\mathbb{N}$,*

(1) there exists an increasing sequence $\langle y_n\rangle_{n=1}^{\infty}$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^{\infty})\subseteq A_i$ and whenever $F\in\mathcal{P}_f(\mathbb{N})$, there exists a strongly increasing sequence $\langle x_n\rangle_{n=1}^{m}$ such that $FS(\langle x_n\rangle_{n=1}^{m})\subseteq A_i$ and $\sum_{n=1}^{m}x_n=\prod_{n\in F}y_n$ and

(2) there exists a strongly increasing sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty})\subseteq A_i$ and whenever $F\in\mathcal{P}_f(\mathbb{N})$, there exists an increasing sequence $\langle y_n\rangle_{n=1}^{m}$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^{m})\subseteq A_i$ and $\prod_{n=1}^{m}y_n=\sum_{n\in F}x_n$.

In this section we extend this result for dense subsemigroups of $((0,\infty),+)$ to near zero. Here we follow closely the arguments used by Hindman in [11]. To establish the main result we need the following lemma.

Lemma 4. *Let S be an HL semigroup. Let $\langle w_t\rangle_{t=1}^{\infty}$ be a sequence in $S\cap(0,1)$ such that $\sum_{t=1}^{\infty}w_t$ converges. Then there exist sequences $\langle x_t\rangle_{t=1}^{\infty}$ and $\langle y_t\rangle_{t=1}^{\infty}$ such that $\langle x_t\rangle_{t=1}^{\infty}$ is strictly decreasing (and therefore all the elements of the sequence $\langle x_t\rangle_{t=1}^{\infty}$ are distinct), $\langle y_t\rangle_{t=1}^{\infty}$ is decreasing and has distinct finite products, $FS(\langle x_t\rangle_{t=1}^{\infty})\subseteq FS(\langle w_t\rangle_{t=1}^{\infty})$ and $FP(\langle y_t\rangle_{t=1}^{\infty})\subseteq FP(\langle w_t\rangle_{t=1}^{\infty})$.*

Proof. We construct the sequence $\langle x_t\rangle_{t=1}^{\infty}$ inductively. Since $\sum_{t=1}^{\infty}w_t$ converges, $\langle w_t\rangle_{t=1}^{\infty}$ converges to 0. Hence we can construct a strictly decreasing subsequence of $\langle w_t\rangle_{t=1}^{\infty}$ in the following way. Let $x_1=w_1$. Then clearly, $(0,w_1/2)$ contains infinitely many elements of the sequence $\langle w_t\rangle_{t=2}^{\infty}$. Let us take $x_2=w_l$ such that $w_l\in(0,w_1/2)$. Then clearly $x_1>x_2$. Inductively, let $\langle x_t\rangle_{t=1}^{k}$ be a strictly decreasing sequence such that for each t, $x_t=w_{p_t}$ and $x_t\in(0,x_{t-1}/2)$, for some $p_t\in\mathbb{N}$ and $x_t>x_{t+1}$ for $t\in\{1,2,\ldots,k-1\}$. Clearly $FS(\langle x_t\rangle_{t=1}^{k})\subseteq FS(\langle w_t\rangle_{t=1}^{\infty})$. Now $(0,x_k/2)$ contains infinitely many elements of the sequence $\langle w_t\rangle_{t=1}^{\infty}$ hence we can choose min $\{t\in\mathbb{N}:w_t\in(0,x_k/2)\}=q$. Let $x_{k+1}=w_q$. Then $x_k>x_{k+1}$ and $FS(\langle x_t\rangle_{t=1}^{k+1})\subseteq FS(\langle w_t\rangle_{t=1}^{\infty})$.

By similar arguments, we construct the sequence $\langle y_t\rangle_{t=1}^{\infty}$ inductively. Let $\langle y_k\rangle_{k=1}^{n}$ be a strictly decreasing subsequence of $\langle w_t\rangle_{t=1}^{\infty}$ such that $y_k=w_{t_k}$ with $t_1<t_2\ldots<t_n$. Let $E=FP(\langle y_k\rangle_{k=1}^{n})$, and let $\mu=\min E\cup\{u^{-1}v:u,v\in E\}$. Now $(0,\mu)$ contains infinitely many elements of the sequence $\langle w_t\rangle_{t=t_n+1}^{\infty}$. Let $\min\{t\in\mathbb{N}:t\geq$

$t_n+1, w_t \in (0,\mu)\} = r$. Let $y_{n+1} = w_r$. Therefore, $FP\langle y_t\rangle_{t=1}^{n+1} \subseteq FP(\langle w_t\rangle_{t=1}^{\infty})$. This completes the proof. □

Theorem 14. *Let S be an HL semigroup. Let $r \in \mathbb{N}$, $\epsilon > 0$ and let $S \cap (0,1) = \bigcup_{i=1}^{r} A_i$. There exists $i \in \{1,2,\ldots,r\}$ such that for each $m \in \mathbb{N}$,*

(1) there exists a decreasing sequence $\langle y_n\rangle_{n=1}^{\infty}$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i \cap (0,\epsilon)$, and whenever $F \in \mathcal{P}_f(\mathbb{N})$, there exists a strictly decreasing sequence $\langle x_n\rangle_{n=1}^{m}$ such that $FS(\langle x_n\rangle_{n=1}^{m}) \subseteq A_i \cap (0,\epsilon)$ and $\sum_{n=1}^{m} x_n = \prod_{n\in F} y_n$
and

(2) there exists a strictly decreasing sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_i \cap (0,\epsilon)$, and whenever $F \in \mathcal{P}_f(\mathbb{N})$, there exists a decreasing sequence $\langle y_n\rangle_{n=1}^{m}$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^{m}) \subseteq A_i \cap (0,\epsilon)$ and $\prod_{n=1}^{m} y_n = \sum_{n\in F} x_n$.

Proof. Pick $p \in \beta(S\cap(0,1),\cdot)$ such that, for every $A \in p$ there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A \cap (0,\epsilon)$ and $FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A \cap (0,\epsilon)$ (It follows from the proof of Corollary 1 that there always exists such a p). Pick $i \in \{1,2,\ldots,r\}$ such that $A_i \cap (0,\epsilon) \in p$.

Let $m \in \mathbb{N}$ be given. Let $B_0 = \{z \in A_i \cap (0,\epsilon)$: there exists a strictly decreasing sequence $\langle x_n\rangle_{n=1}^{m}$ such that $FS(\langle x_n\rangle_{n=1}^{m}) \subseteq A_i \cap (0,\epsilon)$ and $z = \sum_{n=1}^{m} x_n\}$. Let $C_0 = \{z \in A_i \cap (0,\epsilon)$: there exists a decreasing sequence $\langle y_n\rangle_{n=1}^{m}$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^{m}) \subseteq A_i \cap (0,\epsilon)$ and $z = \prod_{n=1}^{m} y_n\}$.

We claim that $B_0 \in p$. If possible, let $B_0 \notin p$, in which case $(A_i \cap (0,\epsilon))\backslash B_0 \in p$. Pick a sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq (A_i\cap(0,\epsilon))\backslash B_0$. By Lemma 4 we may assume that $\langle x_n\rangle_{n=1}^{\infty}$ is strictly decreasing. But then $\sum_{n=1}^{m} x_n \in B_0$, a contradiction. Similarly $C_0 \in p$.

For conclusion (1) pick a decreasing sequence $\langle y_n\rangle_{n=1}^{\infty}$ with distinct finite products such that $FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq B_0$. For conclusion (2) pick a strictly decreasing sequence $\langle x_n\rangle_{n=1}^{\infty}$ with distinct finite products such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq C_0$. □

Corollary 2. *Let S be an HL semigroup. Let $r \in \mathbb{N}$, $\epsilon > 0$ and let $S \cap (0,1) = \bigcup_{i=1}^{r} A_i$. There exists $i \in \{1,2,\ldots,r\}$ and a, b, c, and d in S such that $\{a,b,c,d\} \subseteq A_i \cap (0,\epsilon)$ with $a+b = cd$ and $a \neq b$.*

Proof. This is the special case of the above theorem when $m = 2$. □

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