ON THE x–COORDINATES OF PELL EQUATIONS THAT ARE PRODUCTS OF TWO PADOVAN NUMBERS

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Received: 6/26/19, Revised: 2/9/19, Accepted: 8/26/20, Published: 8/31/20

Abstract

Let $(P_n)_{n\geq 0}$ be the sequence of Padovan numbers defined by $P_0 = 0$, $P_1 = P_2 = 1$, and $P_{n+3} = P_{n+1} + P_n$ for all $n \geq 0$. In this paper, we find all positive square-free integers $d \geq 2$ such that the Pell equations $x^2 - dy^2 = \ell$, where $\ell \in \{\pm 1, \pm 4\}$, have at least two positive integer solutions (x, y) and (x', y') such that each of x and x' is a product of two Padovan numbers.

1. Introduction

Let $(P_n)_{n\geq 0}$ be the sequence of Padovan numbers given by

$$P_0 = 0,\ P_1 = 1,\ P_2 = 1, \quad \text{and} \quad P_{n+3} = P_{n+1} + P_n \text{ for all } n \geq 0.$$

This is sequence $A000931$ on the On-Line Encyclopedia of Integer Sequences (OEIS). The first few terms of this sequence are

$$(P_n)_{n\geq 0} = \{0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, \ldots\}.$$

In this paper, we let $\mathcal{U} := \{P_n P_m : n \geq m \geq 0\}$ be the sequence of products of two Padovan numbers. The first few members of $\mathcal{U}$ are

$$\mathcal{U} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 32, 35, \ldots\}.$$

Let $d \geq 2$ be a positive integer which is not a square. It is well-known that the Pell equations

$$x^2 - dy^2 = \ell, \tag{1}$$

[1]The author is supported by the Austrian Science Fund (FWF) projects: F5510-N26 – Part of the special research program (SFB), "Quasi-Monte Carlo Methods: Theory and Applications" and W1230 –"Doctoral Program Discrete Mathematics".

where $\ell \in \{\pm 1, \pm 4\}$, have infinitely many positive integer solutions (x, y). By putting (x_1, y_1) for the smallest positive solution to (1), all solutions are of the form (x_k, y_k) for some positive integer k, where

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k \qquad \text{for all} \quad k \geq 1, \quad \text{and} \quad \ell = \pm 1$$

and

$$\frac{x_k + y_k\sqrt{d}}{2} = \left(\frac{x_1 + y_1\sqrt{d}}{2}\right)^k \qquad \text{for all} \quad k \geq 1, \quad \text{and} \quad \ell = \pm 4.$$

Furthermore, the sequence $\{x_k\}_{k\geq 1}$ in both cases $\ell \in \{\pm 1, \pm 4\}$ is binary recurrent. In fact, the following formulas

$$x_k = \frac{(x_1 + y_1\sqrt{d})^k + (x_1 - y_1\sqrt{d})^k}{2}, \quad \text{for} \quad \ell = \pm 1,$$

and

$$x_k = \left(\frac{x_1 + y_1\sqrt{d}}{2}\right)^k + \left(\frac{x_1 - y_1\sqrt{d}}{2}\right)^k, \quad \text{for} \quad \ell = \pm 4,$$

hold for all positive integers k.

Recently, Kafle et al. [15] studied the Diophantine equation

$$x_l = F_m F_n, \tag{2}$$

where x_l are the x–coordinates of the solutions of the Pell equation (1) (in the case $\ell = \pm 1$) for some positive integer l and $\{F_n\}_{n\geq 0}$ is the sequence of Fibonacci numbers given by $F_0 = 0,\ F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. They proved that for each square-free integer $d \geq 2$, there is at most one positive integer l such that x_l admits the representation (2) for some nonnegative integers $0 \leq m \leq n$, except for $d \in \{2, 3, 5\}$. Furthermore, they explicitly stated all the solutions for these exceptional cases.

In the same spirit, Rihane et al. [22] studied the Diophantine equation

$$x_n = P_m, \tag{3}$$

where x_n are the x–coordinates of the solutions of the Pell equations (1), for some positive integers n, and $\{P_m\}_{m\geq 0}$ is the sequence of Padovan numbers. They proved that for each square-free integer $d \geq 2$, there is at most one positive integer x participating in the Pell equations (1), that is a Padovan number with a few exceptions of d that can be effectively computed. Furthermore, the exceptional cases in (3) were $d \in \{2, 3, 5, 6\}$ (for the case $\ell = \pm 1$) and $d \in \{5\}$ (for the case $\ell = \pm 4$). Several other related problems have been studied where x_l belongs to some interesting positive integer sequences. For example, see [2, 3, 7–11, 13, 16–20].

2. Main Results

In this paper, we study a problem related to that of Kafle et al. [15] but with the Padovan sequence instead of the Fibonacci sequence. We also extend the results from the Pell equation (1) in the case $\ell = \pm 1$ to the case $\ell = \pm 4$. In both cases we find that there are only finitely many solutions that we effectively compute.

Since $P_1 = P_2 = P_3 = 1$, we discard the situations when $n = 1$ and $n = 2$ and just count the solutions for $n = 3$. Similarly, since $P_4 = P_5 = 2$, we discard the situation when $n = 4$ and just count the solutions for $n = 5$. The main aim of this paper is to prove the following results.

Theorem 1. *For each integer $d \geq 2$ that is square-free, there is at most one positive integer k such that*

$$x_k \in \mathcal{U} \quad \textit{with} \quad \ell = \pm 1,$$

except when $d \in \{3, 6\}$ in the $+1$ case and $d \in \{2, 5\}$ in the -1 case.

Theorem 2. *For each integer $d \geq 2$ that is square-free, there is at most one positive integer k such that*

$$x_k \in \mathcal{U} \quad \textit{with} \quad \ell = \pm 4,$$

except when $d \in \{3, 5, 6, 77\}$ in the $+4$ case and $d \in \{2, 5, 13, 29, 65, 257\}$ in the -4 case.

For the exceptional values of d listed in Theorem 1 and Theorem 2, all solutions (k, n, m) are listed at the end of the proof of each result. The main tools used in this paper are the lower bounds for nonzero linear forms in logarithms of algebraic numbers and the Baker-Davenport reduction procedure, as well as the elementary properties of Padovan numbers and solutions to Pell equations. Computations are done with the help of a computer program in Mathematica.

3. Preliminary Results

3.1. The Padovan Sequence

Here, we recall some important properties of the Padovan sequence $\{P_n\}_{n \geq 0}$. The characteristic equation

$$x^3 - x - 1 = 0,$$

has roots $\alpha, \beta, \gamma = \bar{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \qquad \beta = \frac{-(r_1 + r_2) + \sqrt{-3}(r_1 - r_2)}{12}, \tag{4}$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}. \tag{5}$$

Furthermore, the Binet formula is given by

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \qquad \text{for all} \quad n \geq 0, \tag{6}$$

where

$$a = \frac{\alpha + 1}{(\alpha - \beta)(\alpha - \gamma)}, \quad b = \frac{\beta + 1}{(\beta - \alpha)(\beta - \gamma)}, \quad c = \frac{\gamma + 1}{(\gamma - \alpha)(\gamma - \beta)} = \bar{b}. \tag{7}$$

Numerically, the following estimates hold:

$$\begin{gathered} 1.32 < \alpha < 1.33, \\ 0.86 < |\beta| = |\gamma| = \alpha^{-\frac{1}{2}} < 0.87, \\ 0.54 < a < 0.55, \\ 0.28 < |b| = |c| < 0.29. \end{gathered} \tag{8}$$

From (4), (5), and (8), it is easy to see that the contribution the complex conjugate roots β and γ, to the right-hand side of equation (6), is very small. In particular, setting

$$e(n) := P_n - a\alpha^n = b\beta^n + c\gamma^n, \quad \text{we have} \quad |e(n)| < \frac{1}{\alpha^{n/2}}, \tag{9}$$

holds for all $n \geq 1$. Furthermore, by induction, one can prove that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 4. \tag{10}$$

3.2. Linear Forms in Logarithms

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without a reference:

$$h(\eta_1 \pm \eta_2) \leq h(\eta) + h(\eta_1) + \log 2,$$
$$h(\eta_1 \eta_2^{\pm 1}) \leq h(\eta_1) + h(\eta_2),$$
$$h(\eta^s) = |s| h(\eta) \quad (s \in \mathbb{Z}).$$

Here, we recall the result of Bugeaud, Mignotte, and Siksek (see [4], Theorem 9.4), which is a modified version of the result of Matveev [21]. This result is one of our main tools in this paper.

Theorem 3 (Matveev according to Bugeaud, Mignotte, Siksek)**.** *Let $\eta_1, \ldots, \eta_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K} \subset \mathbb{R}$ of degree D, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1,$$

is nonzero. Then

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{D h(\eta_i), |\log \eta_i|, 0.16\}, \qquad \text{for all} \qquad i = 1, \ldots, t.$$

3.3. Reduction Procedure

During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classical result in the theory of Diophantine approximation. It is called the Legendre criterion. For further details, we refer the reader to the books of Cohen [5, 6].

Lemma 1 (Legendre, [5, 6])**.** *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction expansion of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left| \tau - \frac{r}{s} \right| > \frac{1}{(a(M) + 2) s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [12], Lemma 5a). For a real number X, we write $||X|| := \min\{|X-n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 2 (Dujella, Pethő)**.** *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms in logarithms with bounded integer coefficients in three variables. In this case we use the LLL algorithm that we describe below. Let $\tau_1, \tau_2, \ldots \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \leq X_i.$$

We put $X := \max\{X_i\}$, $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \mathbf{e}_t \quad \text{for} \quad 1 \leq j \leq t-1 \quad \text{and} \quad \mathbf{b}_t := \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 3 (LLL algorithm, [6])**.** *Let $X_1, X_2, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ be a fixed sufficiently large constant. With the above notation on the lattice Ω, we consider a reduced base $\{\boldsymbol{b}_i\}$ to Ω and its associated Gram-Schmidt orthogonalization base $\{\boldsymbol{b}_i^*\}$. We set*

$$c_1 := \max_{1\leq i\leq t} \frac{||\boldsymbol{b}_1||}{||\boldsymbol{b}_i^*||}, \quad \theta := \frac{||\boldsymbol{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2, \quad \textit{and} \quad R := \frac{1}{2}\left(1 + \sum_{i=1}^{t} X_i\right).$$

If the integers x_i are such that $|x_i| \leq X_i$, for $1 \leq i \leq t$ and $\theta^2 \geq Q + R^2$, then we have

$$\left|\sum_{i=1}^{t} x_i\tau_i\right| \geq \frac{\sqrt{\theta^2 - Q} - R}{C}.$$

For the proof and further details, we refer the reader to the book of Cohen, (see [6], Proposition 2.3.20).

Finally, the following Lemma is also useful. It is Lemma 7 in [14].

Lemma 4 (Gúzman Sánchez, Luca)**.** *Let r, H, and L be positive real numbers. If $r \geq 1$, $H > (4r^2)^r$, and $H > L/(\log L)^r$, then*

$$L < 2^r H (\log H)^r.$$

4. Proof of Theorem 1

Let (x_1, y_1) be the smallest positive integer solution to the Pell equation (1) in the case $\ell = \pm 1$. We put

$$\delta := x_1 + y_1\sqrt{d} \quad \text{and} \quad \sigma := x_1 - y_1\sqrt{d}, \tag{11}$$

from which we get that

$$\delta \cdot \sigma = x_1^2 - dy_1^2 =: \epsilon, \quad \text{where} \quad \epsilon \in \{\pm 1\}. \tag{12}$$

Then,

$$x_k = \frac{1}{2}(\delta^k + \sigma^k). \tag{13}$$

Since $\delta \geq 1 + \sqrt{2} > \alpha^2$, it follows that the estimate

$$\frac{\delta^k}{\alpha^2} \leq x_k \leq \frac{\delta^k}{\alpha} \quad \text{holds for all} \quad k \geq 1. \tag{14}$$

We assume that (k_1, n_1, m_1) and (k_2, n_2, m_2) are triples of integers such that

$$x_{k_1} = P_{n_1}P_{m_1} \quad \text{and} \quad x_{k_2} = P_{n_2}P_{m_2}. \tag{15}$$

We assume that $1 \leq k_1 < k_2$. We also assume that $3 \leq m_i < n_i$ for $i = 1, 2$. We set $(k, n, m) := (k_i, n_i, m_i)$, for $i = 1, 2$. Using the inequalities (10) and (14), we get from (15) that

$$\frac{\delta^k}{\alpha^2} \leq x_k = P_nP_m \leq \alpha^{n+m-2} \quad \text{and} \quad \alpha^{n+m-4} \leq P_nP_m = x_k \leq \frac{\delta^k}{\alpha}. \tag{16}$$

The above inequalities give

$$k \log \delta < (n + m) \log \alpha < k \log \delta + 3 \log \alpha.$$

Dividing through by $\log \alpha$ and setting $c_2 := 1/\log \alpha$, we get that

$$kc_2 \log \delta < n + m < kc_2 \log \delta + 3, \tag{17}$$

and since $\alpha^3 > 2$, we get

$$|n + m - c_2k \log \delta| < 3. \tag{18}$$

To fix ideas, we assume that

$$n \geq m \qquad \text{and} \qquad k_1 < k_2.$$

We also put

$m_3 := \min\{m_1, m_2\},\ m_4 := \max\{m_1, m_2\},\ n_3 := \min\{n_1, n_2\},\ n_4 := \max\{n_1, n_2\}.$

Inequality (17) together with the fact that $\delta > \alpha^2$ (so, $c_2 \log \delta > 2$), tells us that

$$2k_2 < c_2 k_2 \log \delta < 2n_2 \leq 2n_4,$$

so

$$k_1 < k_2 < n_4.$$

Besides, given that $k_1 < k_2$, we have by (10) and (15) that

$$\alpha^{n_1-2} \leq P_{n_1} \leq P_{n_1} P_{m_1} = x_{k_1} < x_{k_2} = P_{n_2} P_{m_2} \leq P_{n_2}^2 < \alpha^{2n_2-2}.$$

Thus, we get that

$$n_1 < 2n_2. \tag{19}$$

4.1. An Inequality for m, n, and k

Using the equations (6), (13), and (15), we get

$$\frac{1}{2}(\delta^k + \sigma^k) = P_n P_m = (a\alpha^n + e(n))(a\alpha^m + e(m)).$$

So,

$$\frac{1}{2}\delta^k - a^2\alpha^{n+m} = -\frac{1}{2}\sigma^k + a(e(m)\alpha^n + e(n)\alpha^m) + e(n)e(m),$$

and by (9), we have

$$\begin{aligned} \left|\delta^k (2a^2)^{-1}\alpha^{-(n+m)} - 1\right| &\leq \frac{1}{2\delta^k a^2 \alpha^{n+m}} + \frac{|e(m)\alpha^n + e(n)\alpha^m|}{a\alpha^{n+m}} + \frac{|e(n)e(m)|}{a^2\alpha^{n+m}} \\ &\leq \frac{1}{2\delta^k a^2 \alpha^{n+m}} + \frac{\alpha^{n/2} + \alpha^{m/2}}{a\alpha^{n+m}} + \frac{1}{a^2\alpha^{3(n+m)/2}} \\ &\leq \frac{1}{a^2\alpha^{(n+m)/2}}\left(\frac{1}{2\delta^k\alpha^{(n+m)/2}} + a + \frac{1}{\alpha^{n+m}}\right) \\ &< \frac{2}{\alpha^{(n+m)/2}}. \end{aligned}$$

Thus, we have

$$\left|\delta^k (2a^2)^{-1}\alpha^{-(n+m)} - 1\right| < \frac{2}{\alpha^{(n+m)/2}}. \tag{20}$$

Put

$$\Lambda_1 := \delta^k (2a^2)^{-1} \alpha^{-(n+m)} - 1$$

and

$$\Gamma_1 := k \log \delta - \log(2a^2) - (n+m) \log \alpha.$$

We assume that $n + m \geq 10$. $|\Lambda_1| = |e^{\Gamma_1} - 1| < \frac{1}{2}$ for $n + m \geq 10$ (because $2/\alpha^5 < 1/2$), since the inequality $|y| < 2|e^y - 1|$ holds for all $y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, it follows that $e^{|\Gamma_1|} < 2$ and so

$$|\Gamma_1| < e^{|\Gamma_1|} |e^{\Gamma_1} - 1| < \frac{4}{\alpha^{(n+m)/2}}.$$

Thus, we get that

$$\left| k \log \delta - \log(2a^2) - (n+m) \log \alpha \right| < \frac{4}{\alpha^{(n+m)/2}}. \tag{21}$$

We apply Theorem 3 on the left-hand side of (20) with the data:

$$t := 3, \quad \eta_1 := \delta, \quad \eta_2 := 2a^2, \quad \eta_3 := \alpha, \quad b_1 := k, \quad b_2 := -1, \quad b_3 := -(n+m).$$

Furthermore, we take the number field $\mathbb{K} := \mathbb{Q}(\sqrt{d}, \alpha)$ which has degree $D := 6$. Since $\max\{1, k, n+m\} \leq 2n$, we take $B := 2n$. First, we note that the left-hand side of (20) is non-zero, since otherwise,

$$\delta^k = 2a^2 \alpha^{n+m}.$$

The left-hand side belongs to the quadratic field $\mathbb{Q}(\sqrt{d})$, while the right-hand side belongs to the cubic field $\mathbb{Q}(\alpha)$. These fields only intersect when both sides are rational numbers. Since δ^k is a positive algebraic integer and a unit, we get that $\delta^k = 1$. Hence, $k = 0$, which is a contradiction. Thus, $\Lambda_1 \neq 0$. Now, we can apply Theorem 3.

We have $h(\eta_1) = h(\delta) = \frac{1}{2} \log \delta$ and $h(\eta_3) = h(\alpha) = \frac{1}{3} \log \alpha$. Further,

$$a = \frac{\alpha(\alpha+1)}{2\alpha+3},$$

the mimimal polynomial of $2a^2$ is $529x^3 - 460x^2 + 100x - 8$, and has roots $\{2a^2, 2b^2, 2c^2\}$. Since $2a^2 < 1$ and $2|b|^2 = 2|c|^2 < 1$ (by (8)), then

$$h(\eta_2) = h(2a^2) = \frac{1}{3} \log 529.$$

Thus, we can take ,

$$A_1 := 3 \log \delta, \quad A_2 := 2 \log 529, \quad A_3 := 2 \log \alpha.$$

Now, Theorem 3 tells us that

$$\begin{aligned}\log|\Lambda_1| &> -1.4\times 30^6\times 3^{4.5}\times 6^2(1+\log 6)(1+\log(2n))(3\log\delta)\\ &\quad\times(2(\log 529)(2\log\alpha)\\ &> -1.06\times 10^{15}\log n\log\delta.\end{aligned}$$

Comparing the above inequality with (20), we get

$$\frac{n+m}{2}\log\alpha-\log 4<1.06\times 10^{15}\log n\log\delta.$$

Hence, we get that

$$m\le\frac{n+m}{2}<1.08\times 10^{15}\log n\log\delta. \tag{22}$$

Since $\delta^k\le\alpha^{n+m}$ (by (16)), we get that

$$k\log\delta\le(n+m)\log\alpha, \tag{23}$$

which together with estimate (22) gives

$$k<6.16\times 10^{14}\log n.$$

We now return to the equation $x_k=P_nP_m$ and rewrite it as

$$\frac{1}{2}\delta^k-aP_m\alpha^n=-\frac{1}{2}\sigma^k+e(n)P_m,$$

we obtain

$$\left|\delta^k(2aP_m)^{-1}\alpha^{-n}-1\right|\le\frac{1}{a\alpha^n}\left(1+\frac{1}{\delta^k\alpha^n}\right)<\frac{8}{\alpha^n}. \tag{24}$$

Put

$$\Lambda_2:=\delta^k(2aP_m)^{-1}\alpha^{-n}-1,\quad \Gamma_2:=k\log\delta-\log(2aP_m)-n\log\alpha.$$

We assume for technical reasons that $n\ge 10$. So $|e^{\Lambda_2}-1|<\frac{1}{2}$. It follows that

$$|k\log\delta-\log(2aP_m)-n\log\alpha|=|\Gamma_2|<e^{|\Lambda_2|}|e^{\Lambda_2}-1|<\frac{16}{\alpha^n}. \tag{25}$$

Furthermore, $\Lambda_2\ne 0$ (so $\Gamma_2\ne 0$), since $\delta^k\in\mathbb{Q}(\alpha)$ by the previous argument.

We now apply Theorem 3 to the left-hand side of (24) with the data

$$t:=3,\quad \eta_1:=\delta,\quad \eta_2:=2aP_m,\quad \eta_3:=\alpha,\quad b_1:=k,\quad b_2:=-1,\quad b_3:=-n.$$

Since

$$2a = \frac{2\alpha(\alpha+1)}{2\alpha+3},$$

the minimum polynomial of $2a$ over the integers is $23x^3 - 20x - 8$, with roots $\{2a, 2b, 2c\}$, where $2|b| = 2|c| < 1$ (by (8)). Thus, $h(2a) = \frac{1}{3}(\log 23 + \log(2a))$. So,

$$\begin{aligned} h(2aP_m) = h(2a) + h(P_m) &\leq \frac{1}{3}(\log 23 + \log(2a)) + \log P_m \\ &\leq \frac{1}{3}(\log 23 + \log(2a)) + (m-1)\log\alpha \\ &< 1.16 \times 10^{15} \log n \log \delta. \end{aligned}$$

Thus, we have $A_1 := 3\log\delta$, $\quad A_2 := 6.96 \times 10^{15} \log n \log \delta$, $\quad A_3 := 2\log\alpha$, as before. Then, by Theorem 3, we conclude that

$$\log|\Lambda| > -1.70 \times 10^{29} (\log n)^2 (\log \delta)^2.$$

By comparing with (24), we get

$$n < 1.80 \times 10^{29} (\log n)^2 (\log \delta)^2. \tag{26}$$

This was obtained under the assumption that $n \geq 10$, but if $n < 10$, then the inequality also holds as well.

We record what we have proved so far.

Lemma 5. *If $x_k \in \mathcal{U} := \{P_n P_m : n \geq m \geq 0\}$ with $k \geq 1$, and δ as defined in* (11)*, then*

$$m < 1.08 \times 10^{15} \log n \log \delta, \;\; k < 6.16 \times 10^{14} \log n, \;\; n < 1.80 \times 10^{29} (\log n)^2 (\log \delta)^2.$$

4.2. Absolute Bounds

We recall that $(k, n, m) = (k_i, n_i, m_i)$, where $3 \leq m_i < n_i$, for $i = 1, 2$ and $1 \leq k_1 < k_2$. Further, $n_i \geq 4$ for $i = 1, 2$. We return to (21) and rewrite

$$\left|\Gamma_1^{(i)}\right| := \left|k_i \log\delta - \log(2a^2) - (n_i + m_i)\log\alpha\right| < \frac{4}{\alpha^{(n_i+m_i)/2}}, \quad \text{for} \quad i = 1, 2.$$

We do a suitable cross product between $\Gamma_1^{(1)}$, $\Gamma_1^{(2)}$ and k_1, k_2 to eliminate the term involving $\log\delta$ in the above linear forms in logarithms:

$$\begin{aligned} |\Gamma_3| : &= |(k_1 - k_2)\log(2a^2) + (k_1(n_2 + m_2) - k_2(n_1 + m_1))\log\alpha| \\ &= |k_2\Gamma_1^{(1)} - k_1\Gamma_1^{(2)}| \leq k_2|\Gamma_1^{(1)}| + k_1|\Gamma_1^{(2)}| \\ &\leq \frac{4k_2}{\alpha^{(n_1+m_1)/2}} + \frac{4k_1}{\alpha^{(n_2+m_2)/2}} \leq \frac{8n_4}{\alpha^\lambda}, \end{aligned} \tag{27}$$

where $\lambda := \min_{1\le i\le 2}\left\{\frac{n_i+m_i}{2}\right\}$.

We need to find an upper bound for λ. If $8n_4/\alpha^\lambda > 1/2$, we then get

$$\lambda < \frac{\log(16n_4)}{\log\alpha} < 4\log(16n_4). \tag{28}$$

Otherwise, $|\Gamma_3| < \frac{1}{2}$. So,

$$\left|e^{\Gamma_3}-1\right| = \left|(2a^2)^{k_1-k_2}\alpha^{k_1(n_2+m_2)-k_2(n_1+m_1)}-1\right| < 2|\Gamma_3| < \frac{16n_4}{\alpha^\lambda}. \tag{29}$$

We apply Theorem 3 with the data:

$$t := 2,\quad \eta_1 := 2a^2,\quad \eta_2 := \alpha,\quad b_1 := k_1-k_2,\quad b_2 := k_1(n_2+m_2)-k_2(n_1+m_1).$$

We take the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $D := 3$. We begin by checking that $e^{\Gamma_3}-1 \neq 0$ (so $\Gamma_3 \neq 0$). This is true because α and $2a^2$ are multiplicatively independent, since α is a unit in the ring of integers $\mathbb{Q}(\alpha)$ while the norm of $2a^2$ is $8/529$.

We note that $|k_1-k_2| < k_2 < n_4$. Further, from (27), we have

$$\begin{aligned}|k_2(n_1+m_1)-k_1(n_2+m_1)| &< (k_2-k_1)\frac{|\log(2a^2)|}{\log\alpha}+\frac{4k_2}{\alpha^{m_3}\log\alpha}\\ &< 25k_2 < 25n_4\end{aligned}$$

given that $m_3 \ge 1$. So, we can take $B := 25n_4$. By Theorem 3, with the same $A_1 := \log 529 + \log(2a^2)$ and $A_2 := \log\alpha$, we have that

$$\log|e^{\Gamma_3}-1| > -5.5\times 10^{11}(\log n_4)(\log\alpha).$$

By comparing this with (29), we get

$$\lambda < 5.6\times 10^{11}\log n_4. \tag{30}$$

Note that (30) is better than (28), so (30) always holds. Without loss of generality, we can assume that $\lambda = (n_i+m_i)/2$, for fixed $i = 1, 2$.

We set $\{i, j\} = \{1, 2\}$ and return to (21) to replace $(k, n, m) = (k_i, n_i, m_i)$:

$$|\Gamma_1^{(i)}| = \left|k_i\log\delta - \log(2a^2) - (n_i+m_i)\log\alpha\right| < \frac{4}{\alpha^{(n_i+m_i)/2}}, \tag{31}$$

and also return to (25), replacing with $(k, n, m) = (k_j, n_j, m_j)$:

$$|\Gamma_2^{(j)}| = \left|k_j\log\delta - \log(2aP_{m_j}) - n_j\log\alpha\right| < \frac{16}{\alpha^{n_j}}. \tag{32}$$

We perform a cross product on (31) and (32) in order to eliminate the term on $\log\delta$:

$$\begin{aligned}
|\Gamma_4| &:= \left|k_j\log(2a^2) - k_i\log(2aP_{m_j}) + (k_in_j - k_j(n_i+m_i))\log\alpha\right| \\
&\le |(k_j-k_i)\log(2a) + (k_j(n_i+m_i+1) - k_i(n_j+m_j-1))\log\alpha| \\
&\le \left|k_i\Gamma_2^{(j)} - k_j\Gamma_1^{(i)}\right| \le k_i\left|\Gamma_2^{(j)}\right| + k_j\left|\Gamma_1^{(i)}\right| \\
&< \frac{16k_i}{\alpha^{n_j}} + \frac{4k_j}{\alpha^{(n_i+m_i)/2}} < \frac{20n_4}{\alpha^{\nu}},
\end{aligned} \tag{33}$$

where $\nu := \min\limits_{1\le i,j\le 2}\left\{\dfrac{n_i+m_i}{2}, n_j\right\}$.

As before, we need to find an upper bound on ν. If $20n_2/\alpha^{\nu} > 1/2$, then we get

$$\nu < \frac{\log(40n_4)}{\log\alpha} < 4\log(40n_4). \tag{34}$$

Otherwise, $|\Gamma_4| < 1/2$, so we have

$$\left|e^{\Gamma_4} - 1\right| \le 2|\Gamma_4| < \frac{40n_4}{\alpha^{\nu}}. \tag{35}$$

In order to apply Theorem 3, first we check if $e^{\Gamma_4} = 1$, we obtain

$$(2a)^{k_i-k_j} = \alpha^{k_j(n_i+m_i+1)-k_i(n_j+m_j-1)}.$$

Since α is a unit, the right-hand side in above is an algebraic integer. This is a contradiction because $k_1 < k_2$ so $k_i - k_j \neq 0$, and neither $(2a)$ nor $(2a)^{-1}$ are algebraic integers. Hence, $e^{\Gamma_4} \neq 1$. By assuming that $\nu \ge 100$, we apply Theorem 3 with the data: $t := 2$,

$$\eta_1 := 2a, \quad \eta_2 := \alpha, \quad b_1 := k_j - k_i, \quad b_2 := k_j(n_i+m_i+1) - k_i(n_j+m_j-1),$$

and the inequalities (30) and (35). We get

$$\nu := \min_{1\le i,j\le 2}\left\{\frac{n_i+m_i}{2}, n_j\right\} < 7.2\times 10^{14}\lambda\log n_4 < 4.1\times 10^{25}(\log n_4)^2.$$

The above inequality also holds when $\nu < 100$. Further, it also holds when the inequality (34) holds. So the above inequality holds in all cases. Note that the case $\{i,j\} = \{2,1\}$ leads to $n_1 < 2n_2 \le 2n_4$ whereas $\{i,j\} = \{1,2\}$ leads to $\nu = \min\{(n_1+m_1)/2, n_2\}$. Hence, either the minimum is $(n_1+m_1)/2$, so

$$n_1 \le \frac{n_1+m_1}{2} < 1.36\times 10^{25}(\log n_4)^2, \tag{36}$$

or the minimum is n_j and from the inequality (30) we get that

$$n_3 := \min_{1\le j\le 2}\{n_j\} < 4.1\times 10^{25}(\log n_4)^2. \tag{37}$$

By the inequality (17),

$$\log \delta \leq k_1 \log \delta \leq 2n_1 \log \alpha < 2.33 \times 10^{25} (\log n_4)^2.$$

By substituting this into Lemma 5, we get $n_4 < 9.77 \times 10^{79} (\log n_4)^6$. Also, by Lemma 4, with the data $r := 6$, $H := 9.77 \times 10^{79}$, and $L := n_4$, we get that $n_2 \leq n_4 < 2.44 \times 10^{95}$. This immediately gives that $n_1 \leq n_3 < 6.56 \times 10^{29}$ and $m_1 \leq m_3 < 6.50 \times 10^{29}$.

We record what we have proved.

Lemma 6. *Let (k_i, n_i, m_i) be a solution to $x_{k_i} = P_{n_i} P_{m_i}$, with $3 \leq m_i < n_i$ for $i \in \{1, 2\}$ and $1 \leq k_1 < k_2$, then*

$$\max\{k_1, m_1\} < n_1 < 10^{30} \quad \textit{and} \quad \max\{k_2, m_2\} < n_2 < 10^{96}.$$

5. Reducing the Bounds for n_1 and n_2

In this section, we reduce the upper bounds for n_1 and n_2 given in Lemma 6 reasonably enough so that these can be treated computationally. For this, we return to the inequalities for Γ_3, Γ_4, and Γ_5.

5.1. The First Reduction

We divide both sides of the inequality (27) by $(k_2 - k_1) \log \alpha$. We get that

$$\left| \frac{\log(2a^2)}{\log \alpha} - \frac{k_2(n_1 + m_1) - k_1(n_2 + m_2)}{k_2 - k_1} \right| < \frac{8n_2}{\alpha^\lambda (k_2 - k_1) \log \alpha}. \tag{38}$$

We assume that $\lambda \geq 10$. Below we apply Lemma 1. We put $\tau := \frac{\log(2a^2)}{\log \alpha}$ is irrational. We compute its continued fraction expansion

$$[a_0, a_1, a_2, \ldots] = [1, 23, 3, 2, 1, 1, 2, 1, 2, 3, 1, 1, 8, 1, 1, 3, 1, 2, 3, 7, 5, 48, 2, 3, 1, 2, 18, \ldots]$$

and its convergents

$$\left[\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots \right] = \left[1, \frac{24}{23}, \frac{73}{70}, \frac{170}{163}, \frac{243}{233}, \frac{413}{396}, \frac{1069}{1025}, \frac{1482}{1421}, \frac{4033}{3867}, \frac{13581}{13022}, \ldots \right].$$

Furthermore, we note that taking $M := 10^{96}$ (by Lemma 6), it follows that

$$q_{182} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \leq i \leq 182\} = a_{144} = 204.$$

Thus, by Lemma 1, we have that

$$\left| \tau - \frac{k_2(n_1 + m_1) - k_1(n_2 + m_2)}{k_2 - k_1} \right| > \frac{1}{206(k_2 - k_1)^2}. \tag{39}$$

Hence, combining the inequalities (38) and (39), we obtain

$$\alpha^{\lambda} < 722 n_2 (k_2 - k_1) < 7.22 \times 10^{194},$$

so $\lambda \leq 1573$. This was obtained under the assumption that $\lambda \geq 10$, Otherwise, $\lambda < 10 < 1573$ holds as well.

Now, for each $m_j \leq \dfrac{n_i + m_i}{2} = \lambda \in [1, 1573]$ we estimate a lower bound $|\Gamma_4|$, with

$$|\Gamma_4| := \left| k_j \log(2a^2) - k_i \log(2aP_{m_j}) + (k_i n_j - k_j(n_i + m_i)) \log \alpha \right| \tag{40}$$

given in the inequality (33), via the procedure described in Subsection 3.3 (LLL algorithm). We recall that $\Gamma_4 \neq 0$.

We apply Lemma 3 with the data:

$$t := 3, \quad \tau_1 := \log(2a^2), \quad \tau_2 := \log(2aP_{m_j}), \quad \tau_3 := \log \alpha,$$
$$x_1 := k_j, \quad x_2 := -k_i, \quad x_3 := k_i n_j - k_j(n_i + m_i).$$

We set $X := 25 \times 10^{96}$ as an upper bound to $|x_i| < 25 n_2$ for all $i = 1, 2, 3$, and $C := (5X)^5$. A computer in Mathematica search allows us to conclude, together with the inequality (33), that

$$2 \times 10^{-280} < \min_{1 \leq \lambda \leq 1573} |\Gamma_4| < 20 n_2 \alpha^{-\nu}, \quad \text{with} \quad \nu := \min\left\{\frac{n_i + m_i}{2}, n_j\right\}$$

which leads to $\nu \leq 3043$. As we have noted before, $\nu = n_1$ (so $n_1 \leq 3043$). But we also know that

$$\log \delta \leq (n_1 + m_1) \log \alpha \leq 2\left(\frac{n_1 + m_1}{2}\right) \log \alpha < 1735.$$

Substituting for $\log \delta$ in the inequality involving n in Lemma 5, we get that

$$n_2 < 5.42 \times 10^{35} (\log n_2)^2.$$

An application of Lemma 4 with the data $r = 2$, $H := 5.42 \times 10^{35}$, and $L := n_2$, gives that $n_2 < 1.5 \times 10^{40}$.

We note that the upper bound for n_2 represents a very good reduction of the bound given in Lemma 6. Hence, it is expected that if we start our reduction cycle with the new bound on n_2, then we even get a better bound on n_1. Indeed, returning to (38), we take $M := 1.5 \times 10^{40}$ and computationally verify that $q_{77} > M > n_2 > k_2 - k_1$ and $a(M) := \max\{a_i : 0 \leq i \leq 77\} = a_{13} = 149$, from which it follows that $\lambda \leq 666$. We now return to (40), where putting $X := 3.75 \times 10^{41}$ and $C := (5X)^5$, we apply the LLL algorithm to $\lambda \in [1, 666]$. In this case we get that

$$2 \times 10^{-142} < \min_{1 \leq \lambda \leq 666} |\Gamma_4| < 20 n_2 \alpha^{-\nu}, \quad \text{with} \quad \nu := \min\left\{\frac{n_i + m_i}{2}, n_j\right\},$$

which implies that $\nu \leq 1487$. Thus, $n_1 \leq 1487$. Also, $\log\delta < 848$. By a similar substitution in Lemma 6 for $\log\delta$ for the inequality involving n, we get that

$$n_2 < 8.18 \times 10^{30} (\log n_2)^2.$$

By Lemma 4, we have that $n_2 < 1.66 \times 10^{35}$, a better bound than that obtained in the previous step of the reduction cycle. We record what we have proved.

Lemma 7. *If (k_i, n_i, m_i) is a solution to $x_{k_i} = P_{n_i} P_{m_i}$, with $3 \leq m_i < n_i$ for $i = 1, 2$ and $1 \leq k_1 < k_2$, then*

$$k_1 < m_1 < n_1 \leq 1487 \qquad \textit{and} \quad k_2 < m_2 < n_2 \leq 1.66 \times 10^{35}.$$

5.2. The Final Reduction

Returning to (11) and (13) and using the fact that (x_1, y_1) is the smallest positive solution to the Pell equation (1), we obtain

$$\begin{aligned} x_k = \frac{1}{2}(\delta^k + \sigma^k) \quad &= \quad \frac{1}{2}\left(\left(x_1 + y_1\sqrt{d}\right)^k + \left(x_1 - y_1\sqrt{d}\right)^k\right) \\ = \frac{1}{2}&\left(\left(x_1 + \sqrt{x_1^2 \mp 1}\right)^k + \left(x_1 - \sqrt{x_1^2 \mp 1}\right)^k\right) := Q_k^{\pm}(x_1). \end{aligned}$$

Thus, we return to the Diophantine equation $x_{k_1} = P_{n_1} P_{m_1}$ and consider the equations

$$Q_{k_1}^{+}(x_1) = P_{n_1} P_{m_1} \quad \text{and} \quad Q_{k_1}^{-}(x_1) = P_{n_1} P_{m_1}, \tag{41}$$

with $k_1 \in [1, 1487]$, $m_1 \in [3, 1487]$, and $n_1 \in [m_1 + 1, 1487]$.

Besides the trivial case $k_1 = 1$, with the help of a computer search in Mathematica on the above equations in (41), we list the only nontrivial solutions in Table 1.

$Q_{k_1}^{+}(x_1)$				
k_1	x_1	y_1	d	δ
2	2	1	3	$2+\sqrt{3}$
2	5	2	6	$5+2\sqrt{6}$
2	23	4	33	$23+4\sqrt{33}$

$Q_{k_1}^{-}(x_1)$				
k_1	x_1	y_1	d	δ
2	1	1	2	$1+\sqrt{2}$
2	2	1	5	$2+\sqrt{5}$

Table 1: Solutions to $Q_{k_1}^{\pm}(x_1) = P_{n_1} P_{m_1}$

From the above tables, we set each $\delta := \delta_t$ for $t = 1, 2, \ldots, 5$. We then work on the linear forms in logarithms Γ_1 and Γ_2, in order to reduce the bound on n_2 given in Lemma 7. From the inequality (21), for $(k, n, m) := (k_2, n_2, m_2)$, we write

$$\left| k_2 \frac{\log\delta_t}{\log\alpha} - (n_2 + m_2) + \frac{\log(2a^2)}{\log(\alpha^{-1})} \right| < \left(\frac{4}{\log\alpha}\right) \alpha^{-(n_2+m_2)/2}, \tag{42}$$

for $t = 1, 2, \ldots, 5$.

We put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \quad \mu_t := \frac{\log(2a^2)}{\log(\alpha^{-1})}, \quad \text{and} \quad (A_t, B_t) := \left(\frac{4}{\log \alpha}, \alpha\right).$$

We note that τ_t is transcendental by the Gelfond-Schneider Theorem (see [1], Theorem 2.1). Thus, τ_t is irrational. We can rewrite the inequality (42) as

$$0 < |k_2\tau_t - (n_2 + m_2) + \mu_t| < A_t B_t^{-(n_2+m_2)/2}, \quad \text{for} \quad t = 1, 2, \ldots, 5. \tag{43}$$

We take $M := 1.66 \times 10^{35}$, which is the upper bound on n_2 according to Lemma 7, and apply Lemma 2 to the inequality (43). As before, for each τ_t with $t = 1, 2, \ldots, 5$, we compute its continued fraction $[a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, \ldots]$ and its convergents $p_0^{(t)}/q_0^{(t)}, p_1^{(t)}/q_1^{(t)}, p_2^{(t)}/q_2^{(t)}, \ldots$. For each case, by means of a computer search in Mathematica, we find an integer s_t such that

$$q_{s_t}^{(t)} > 9.96 \times 10^{35} = 6M \qquad \text{and} \qquad \epsilon_t := \|\mu_t q^{(t)}\| - M\|\tau_t q^{(t)}\| > 0.$$

We finally compute all the values of $b_t := \lfloor \log(A_t q_{s_t}^{(t)}/\epsilon_t)/\log B_t \rfloor$. The values of b_t correspond to the upper bounds on $m_2 \leq \dfrac{n_2 + m_2}{2}$, for each $t = 1, 2, \ldots, 5$, according to Lemma 2. The results of the computation for each t are recorded in Table 2.

t	δ_t	s_t	q_{s_t}	$\epsilon_t >$	b_t
1	$2 + \sqrt{3}$	81	2.32528×10^{36}	0.118103	316
2	$5 + 2\sqrt{6}$	67	1.11311×10^{37}	0.128740	320
3	$23 + 4\sqrt{33}$	83	2.65107×10^{36}	0.181168	314
4	$1 + \sqrt{2}$	64	1.35690×10^{37}	0.009827	330
5	$2 + \sqrt{5}$	79	1.35905×10^{36}	0.073971	312

Table 2: First reduction computation results

By replacing (k, n, m) with (k_2, n_2, m_2) in the inequality (25), we can write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - n_2 + \frac{\log(2aP_{m_2})}{\log(\alpha^{-1})} \right| < \left(\frac{16}{\log \alpha}\right) \alpha^{-n_2}, \tag{44}$$

for $t = 1, 2, \ldots, 5$.

We now put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \quad \mu_{t,m_2} := \frac{\log(2aP_{m_2})}{\log(\alpha^{-1})}, \quad \text{and} \quad (A_t, B_t) := \left(\frac{16}{\log \alpha}, \alpha\right).$$

With the above notations, we can rewrite (44) as

$$0 < |k_2\tau_t - n_2 + \mu_{t,m_2}| < A_t B_t^{-n_2}, \quad \text{for} \quad t = 1, 2, \dots, 5. \tag{45}$$

We again apply Lemma 2 to the above inequality (45), for

$$t = 1, 2, \dots, 5, \quad m_2 = 1, 2, \dots, b_t, \quad \text{with} \quad M := 1.66 \times 10^{35}.$$

We take

$$\epsilon_{t,m_2} := ||\mu_t q^{(t,m_2)}|| - M||\tau_t q^{(t,m_2)}|| > 0,$$

and

$$b_t = b_{t,m_2} := \lfloor \log(A_t q_{s_t}^{(t,m_2)}/\epsilon_{t,m_2})/\log B_t \rfloor.$$

With the help of Mathematica, we obtain the results in Table 3.

t	1	2	3	4	5
b_{t,m_2}	330	351	336	345	332

Table 3: Final reduction computation results

Thus, $\max\{b_{t,n_2-m_2} : t = 1, 2, \dots, 5 \quad \text{and} \quad m_2 = 1, 2, \dots, b_t\} \leq 351$. Thus, by Lemma 2, we have that $n_2 \leq 351$, for all $t = 1, 2, \dots, 5$, and by the inequality (19) we have that $n_1 \leq 2n_2$. From the fact that $\delta^k \leq \alpha^{n+m}$, we can conclude that $k_1 < k_2 \leq 194$. Collecting everything together, our problem is reduced to search for the solutions for (15) in the following range:

$$1 \leq k_1 < k_2 \leq 194, \quad 3 \leq m_1 < n_1 \in [3, 351], \quad \text{and} \quad 3 \leq m_2 < n_2 \in [3, 351].$$

After a computer search on the equation (15) on the above ranges, we obtained the following solutions, which are the only solutions for the exceptional d cases we have stated in Theorem 1.

For the $+1$ case:

$$(d = 3) \qquad x_1 = 2 = P_5P_3,\ x_2 = 7 = P_9P_3;$$
$$(d = 6) \qquad x_1 = 5 = P_8P_3,\ x_2 = 49 = P_{16}P_3 = P_9P_9.$$

For the -1 case:

$$(d = 2) \qquad x_1 = 1 = P_3P_3, \quad x_2 = 3 = P_6P_3, \quad x_3 = 7 = P_9P_3;$$
$$(d = 5) \qquad x_1 = 2 = P_5P_3, \quad x_2 = 9 = P_{10}P_3 = P_6P_6.$$

This completes the proof of Theorem 1. $\square$

6. Proof of Theorem 2

The proof of Theorem 2 follows similar arguments as in the proof of Theorem 1. So, we do not give the details here. We leave it as an easy exersise to the reader.

Below, we give the exceptional d cases we have stated in Theorem 2.

For the $+4$ case:

$$\begin{aligned}
(d=3) \qquad & x_1 = 4 = P_7P_3 = P_5P_5, \quad x_2 = 14 = P_9P_5;\\
(d=5) \qquad & x_1 = 3 = P_6P_3, \quad x_2 = 7 = P_9P_3, \quad x_3 = 18 = P_{10}P_5;\\
(d=6) \qquad & x_1 = 10 = P_8P_5, \quad x_2 = 98 = P_{16}P_5;\\
(d=77) \qquad & x_1 = 9 = P_{10}P_3 = P_6P_6, \quad x_3 = 702 = P_{23}P_5.
\end{aligned}$$

For the -4 case:

$$\begin{aligned}
(d=2) \qquad & x_1 = 2 = P_5P_3, \quad x_2 = 6 = P_6P_5, \quad x_3 = 14 = P_9P_5;\\
(d=5) \qquad & x_1 = 1 = P_3P_3, \quad x_2 = 3 = P_6P_3, \quad x_3 = 4 = P_7P_3 = P_5P_5,\\
& x_4 = 7 = P_9P_3, \quad x_6 = 18 = P_{10}P_5;\\
(d=13) \qquad & x_1 = 3 = P_6P_3, \quad x_2 = 36 = P_{11}P_6 = P_{10}P_7;\\
(d=29) \qquad & x_1 = 5 = P_8P_3, \quad x_2 = 27 = P_{10}P_6, \quad x_3 = 140 = P_{14}P_8;\\
(d=65) \qquad & x_1 = 16 = P_{12}P_3 = P_7P_7, \quad x_2 = 258 = P_{18}P_6;\\
(d=257) \qquad & x_1 = 32 = P_{12}P_5, \quad x_2 = 1026 = P_{19}P_{10}.
\end{aligned}$$

□

Acknowledgements. The author thanks the anonymous referee and the editor for the careful reading of the manuscript and the useful comments and recommendations that greatly improved the quality of presentation of the current paper.

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