2-ADIC PROPERTIES OF GENERALIZED FIBONACCI NUMBERS

Paul Thomas Young
Department of Mathematics, College of Charleston, Charleston, South Carolina
paul@math.cofc.edu

Received: 6/24/19, Revised: 2/7/20, Accepted: 8/28/20, Published: 9/11/20

Abstract

Let T_n denote the generalized Fibonacci number of order k defined by the recurrence $T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}$ for $n \geq k$, with initial conditions $T_0 = 0$ and $T_i = 1$ for $1 \leq i < k$. In this paper we use the theory of 2-adic analytic functions to study T_n and the shifted sequence $T_n + 1$, focusing on properties of its 2-adic valuation.

1. Introduction

Let T_n denote the generalized Fibonacci number of order k defined by the recurrence $T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}$ for $n \geqslant k$, with initial conditions $T_0 = 0$ and $T_i = 1$ for $1 \leqslant i < k$. The determination of the 2-adic valuation $\nu_2(T_n)$ has been pursued by several authors [4, 5, 9, 11] in recent years, resulting in the following determinations:

Theorem 1 ([3]). *For order $k = 2$, the 2-adic valuation of the n-th Fibonacci number $T_n = F_n$ is given by*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

Theorem 2 (Marques and Lengyel [5], 2014). *For order $k = 3$, the 2-adic valuation of the n-th Tribonacci number T_n is given by*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, 3 \pmod 4, \\ 1, & \text{if } n \equiv 3 \pmod 8, \\ \nu_2(n) - 1, & \text{if } n \equiv 0 \pmod 8, \\ \nu_2(n+4) - 1, & \text{if } n \equiv 4 \pmod 8, \\ \nu_2((n+1)(n+17)) - 3, & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Theorem 3 (Sobolewski [9], 2017)**.** *For even order $k \geqslant 4$, the 2-adic valuation of the n-th generalized Fibonacci number T_n is given by*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{k+1}, \\ 1, & \text{if } n \equiv k+1 \pmod{2k+2}, \\ \nu_2(n) + \nu_2(k-2) + 1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

Theorem 4 ([11], 2018)**.** *For odd order $k \geqslant 5$, the 2-adic valuation of the n-th generalized Fibonacci number T_n is given by*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, k \pmod{k+1}, \\ \nu_2(k-1), & \text{if } n \equiv k \pmod{2k+2}, \\ \nu_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2}, \\ \nu_2(n) - \nu_2(k+1) + 1, & \text{if } n \equiv 0 \pmod{2k+2}, \\ 2 + \nu_2(m-z), & \text{if } n = (2k+2)m + k + 1, \end{cases}$$

where z is some 2-adic integer satisfying $z \equiv \frac{k-1}{4-2k} \pmod{2^{3\nu_2(k-1)-1}\mathbb{Z}_2}$.

Regardless of the parity of k, the description of the valuation of T_n depends essentially on the congruence class of n modulo $2k+2$, and is usually constant on such congruence classes. When k is even, there is only one such congruence class on which $\nu_2(T_n)$ is non-constant, while for odd $k \geqslant 5$ there are two such classes. In the case of odd $k \geqslant 5$, the formula for $\nu_2(T_n)$ is not completely explicit as the value of z is not known exactly, although for any such k it can be computed to reasonable 2-adic precision. Indeed, it was the conjecture of Lengyel and Marques [4] that $12z + 6 = -43266$ in the case $k = 5$ that inspired us [11] to prove Theorem 4. Eventually, our numerical computations revealed that -43266 is not the correct value, although the actual value is a 2-adic integer which agrees with -43266 to twenty 2-adic digits. Ruiz and Luca [7] then gave a proof that $12z + 6$ is not an integer for $k = 5$, but otherwise the nature of these values of z appears to be unknown, whether rational, algebraic, or transcendental. From the Binet formula of the recurrence and the analysis of [11], we know that each such value of z is the unique 2-adic zero of an algebraic linear combination of 2-adic exponential functions to algebraic bases, and the values do not appear to be integers based on numerical computations. It seems reasonable to conjecture for all odd $k \geqslant 5$, the values of z are transcendental 2-adic integers.

When k is odd, it is also apparent that $\nu_2(T_n)$ takes every possible nonnegative integer value, distinguishing this case from the even orders k, for which $\nu_2(T_n)$ never takes the value 2. In general $\nu_2(T_n)$ takes all nonnegative integer values ν except for $2 \leqslant \nu \leqslant \nu_2(2k-4)$ when $k \geqslant 4$ is even. As a consequence, we observe that the sequence (T_n) cannot be dense in the ring $\mathbb{Z}_2$ of 2-adic integers when k is even. However, we will show (Theorem 7 below) that for even k the sequence (T_n) is in fact dense in the group of units of $\mathbb{Z}_2$. This is not true for odd k.

In this article, we consider also the shifted sequence $P_n = T_n + 1$ and determine, as explicitly as possible, its 2-adic valuation. We find that, as with (T_n), the behavior of the valuation is dependent on the congruence class of n modulo $2k+2$; the behavior differs depending on the parity of k; and in particular, when k is odd, the behavior depends on a single 2-adic integer z whose exact value and algebraic nature are unknown. The result is as follows:

Theorem 5. *If $k \geqslant 2$, then for all integers n we have*

$$\nu_2(P_n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{k+1}, \\ 1, & \text{if } n \equiv i \pmod{k+1}, \quad 2 \leqslant i \leqslant k-1 \\ \nu_2(k), & \text{if } n \equiv k \pmod{2k+2}, \\ \nu_2(k-4), & \text{if } n \equiv -1 \pmod{2k+2}, \quad (k \neq 4) \\ 1, & \text{if } n \equiv 1 \pmod{2k+2}. \end{cases}$$

When $k = 4$ and $n = 10m - 1$, we have $\nu_2(P_n) = 3 + \nu_2(m) + \nu_2(m+1)$. In the remaining case where $n = (2k+2)m - k$, we have

$$\nu_2(P_n) = \begin{cases} 2 + \nu_2(m), & \text{if } k \text{ is even}, \\ 3 + \nu_2(m) + \nu_2(m-z), & \text{if } k \text{ is odd}, \end{cases}$$

where $z \in \mathbb{Z}_2$ is some 2-adic integer satisfying $z \equiv \frac{k-3}{2k-4} \pmod{2^3\mathbb{Z}_2}$.

Remark. We observe from this theorem that when k is even, the valuation $\nu_2(P_n)$ always takes on every nonnegative integer value, a property which does not hold when k is odd; in particular, when k is odd the valuation $\nu_2(P_n)$ never takes the value 2. This is a reversal of the situation for $\nu_2(T_n)$.

While the sequences (T_n) are not 2-adically continuous functions of n except when the order k is of the form $k = 2^e - 1$, any lacunary subsequence $(T_{(2k+2)n+j})$ with gaps of length $2k+2$ is given by a 2-adic analytic function on a large 2-adic disc, that is, a power series which converges rapidly for n in the ring $\mathbb{Z}_2$ of 2-adic integers. This principle, developed in [11], allows us to determine much of the general behavior of (T_n) from relatively few values.

2. 2-adic Analytic Functions

We summarize the analytical results from [11] concerning the sequences (T_n) which we need. This first set of congruences was proved by induction from the recurrence $T_{n+1} = 2T_n - T_{n-k}$, which is implied by the defining recurrence for (T_n).

Proposition 1. *For all integers r we have*

$$T_{r(k+1)+i} \equiv 1 \pmod{2^i}, \quad 1 \leqslant i \leqslant k-1,$$

$$T_{r(k+1)+k} \equiv \begin{cases} k-1, & r \text{ even}, \\ 3-k, & r \text{ odd}, \end{cases} \pmod{2^k},$$

$$T_{r(k+1)} \equiv \begin{cases} 4r-2rk, & r \text{ even}, \\ 2rk-4r+2, & r \text{ odd}. \end{cases} \pmod{2^{k+1}}.$$

The main tool of [11] was the existence of 2-adic analytic functions which interpolate the sequence (T_n) in congruence classes modulo $2k+2$. We let $\mathbb{Z}_2$ denote the ring of 2-adic integers, $\mathbb{Q}_2$ the field of 2-adic numbers, and $\mathbb{C}_2$ the completion of an algebraic closure of $\mathbb{Q}_2$. The 2-adic valuation $\nu_2(n)$ of an integer n is equal to the highest power of 2 which divides n, with the convention that $\nu_2(0) = +\infty$. This valuation extends uniquely to $\mathbb{C}_2$, on which it takes rational values.

Theorem 6. *For each $j \in \mathbb{Z}$ there exists a function $g_j(x) = \sum_{m\geqslant 0} a_m x^m$ which is analytic on a large disc in $\mathbb{C}_2$, containing $\mathbb{Z}_2$, such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$. The coefficients a_m are 2-adic integers which satisfy*

$$\nu_2(a_m) \geqslant \begin{cases} m + S_2(m) - 1, & k \text{ odd}, \\ m + S_2(m), & k \text{ even}, \end{cases}$$

where $S_2(m)$ denotes the sum of the binary digits of m.

One may approximate the coefficients a_m of such an analytic function by computing $g_j(n)$ for several integers n and solving a system of linear equations. As a simple case which is suitable for our purposes here, for any exponent r, considering

$$g_j(2^r) - g_j(-2^r) = 2^{r+1}a_1 + 2^{3r+1}a_3 + 2^{5r+1}a_5 + \cdots \tag{2.1}$$

leads to the determination

$$a_1 \equiv \frac{g_j(2^r) - g_j(-2^r)}{2^{r+1}} \pmod{2^{2r+4}\mathbb{Z}_2}, \tag{2.2}$$

and similarly

$$a_2 \equiv \frac{g_j(2^r) + g_j(-2^r) - 2g_j(0)}{2^{2r+1}} \pmod{2^{2r+4}\mathbb{Z}_2}. \tag{2.3}$$

Taking $j = 1$, for example, by computing $g_1(1) = T_{2k+3} = 2^{k+2}(2k-3) - 12k + 21$ and $g_1(-1) = T_{-2k-1} = 13 - 4k$, we get an estimate

$$a_1 \equiv \frac{(-12k+21) - (13-4k)}{2} = -4k + 4 \pmod{2^4\mathbb{Z}_2} \tag{2.4}$$

for the linear coefficient of $g_1(x)$ (except when $k = 2$, where this congruence holds only modulo 2^3).

The other important tool for our analysis is the theory of Newton polygons. For a polynomial or power series $f(x) = \sum_{i\geqslant 0} a_i x^i \in \mathbb{C}_2[x]$, the *Newton polygon* of f is the upper convex hull of the set of points $\{(i, \nu_2(a_i)) : i \geqslant 0\}$. A basic property ([2], Ch. IV.3, Lemma 4; [6], Theorem 9.1) is that the Newton polygon of f has a side of slope m and horizontal run l if and only if f has l zeros (counted with multiplicity) $\alpha_i \in \mathbb{C}_2$ with $\nu_2(\alpha_i) = -m$.

3. 2-adic Behavior of (T_n) for Even k

In [11] we brought the theory of 2-adic analytic functions to bear on the determination of $\nu_2(T_n)$, focusing primarily on the case of odd order k. In this section we make some observations concerning the case of even k. We begin with an alternate proof of Sobolewski's Theorem 3 using 2-adic analytic functions.

Alternate proof of Theorem 3. All cases except for $n \equiv 0 \pmod{2k+2}$ follow directly from Proposition 1. For the remaining case, we give a proof featuring the analytic function $g_0(x) = \sum_m a_m x^m$ which interpolates $(T_{(2k+2)n})$. Put $a = \nu_2(k-2)$. It is clear that $a_0 = T_0 = 0$, and from Proposition 1

$$g_0(n) = T_{(2k+2)n} \equiv (8-4k)n \pmod{2^{k+1}} \tag{3.1}$$

for all integers n. Taking $r = a-1$ in (2.2) then shows that $\nu_2(a_1) = \nu_2(8-4k) = a+2$. It follows from (3.1) that $\nu_2(g_0(n)) = \nu_2(a_1 n)$ as long as $\nu_2(a_1 n) \leqslant k$, that is, $\nu_2(n) \leqslant k-a-2$. Therefore we suppose that $\nu_2(n) \geqslant k-a-1$; under this assumption it easily follows that for $i \geqslant 2$

$$\begin{aligned} \nu_2(a_i n^i) \quad &\geqslant 3 + 2\nu_2(n) \geqslant 3 + \nu_2(n) + (k-a-1) \\ &= \nu_2(n) + a + 2 + (k-2a) > \nu_2(n) + a + 2 = \nu_2(a_1 n), \end{aligned} \tag{3.2}$$

so that $\nu_2(a_i n^i) > \nu_2(a_1 n)$ for all $i > 1$. Therefore in all cases we have $\nu_2(g_0(n)) = \nu_2(a_1 n) = \nu_2(n) + a + 2$, proving the theorem. □

Since $\nu_2(T_n)$ never takes the value 2 when k is even, it follows that the sequence (T_n) cannot be dense in the ring $\mathbb{Z}_2$ of 2-adic integers. However, the sequence is dense in the unit group of this ring.

Theorem 7. *Let T_n denote the generalized Fibonacci number of even order k. Then the sequence (T_n) is dense in the unit group $\mathbb{Z}_2^\times$ of the ring of 2-adic integers. Consequently, for every odd integer N and every positive integer s there exists $n \in \mathbb{Z}$ such that $T_n \equiv N \pmod{2^s}$.*

Proof. Our proof will be based on the following principle of the analytic functions $g_j(x) = \sum_m a_m x^m$ which interpolate (T_n) in residue classes modulo $2k+2$: if $\nu_2(a_1) = 2$, then the image $g_j(\mathbb{Z}_2)$ equals the coset $a_0 + 4\mathbb{Z}_2$. To see this, we note that $\nu_2(a_i) > 2$ for all $i > 1$. Therefore if $y \in a_0 + 4\mathbb{Z}_2$, the Newton polygon of $g_j(x) - y$ has vertices $(0, \nu_2(a_0 - y))$ and $(1, 2)$ which determine its unique side of nonpositive integer slope, corresponding to its unique root of nonnegative valuation; all other sides have positive slope. By Hensel's Lemma [2, Theorem 3], this root lies in $\mathbb{Z}_2$; therefore there exists $x \in \mathbb{Z}_2$ such that $g_j(x) = y$.

The theorem then follows by observing that for $j = 1$, the analytic function $g_1(x)$ has $a_0 = 1$ and $a_1 \equiv 4 - 4k \pmod{2^3\mathbb{Z}_2}$ as in (2.4), while for $j = -k$ a similar calculation shows that the analytic function $g_k(x)$ has $a_0 = -1$ and $a_1 \equiv 12 - 4k \pmod{2^3\mathbb{Z}_2}$. Therefore the image of (T_n) is dense in both $1 + 4\mathbb{Z}_2$ and $-1 + 4\mathbb{Z}_2$, so it is dense in $\mathbb{Z}_2^\times$. □

4. 2-adic Valuation of $T_n + 1$

In our proof of Theorem 5 we will use $g_j(x)$ to denote the analytic function which interpolates $P_{(2k+2)x+j}$, a notation which differs from Theorem 6 only by changing the value of the constant coefficient a_0 by one.

Proof of Theorem 5. All cases of the theorem except for $n \equiv 1 \pmod{k+1}$ and the one exceptional case where $k = 4$ follow immediately from Proposition 1.

For the case $n \equiv 1 \pmod{2k+2}$, we consider the analytic function $g_1(x) = \sum_{m \geqslant 0} a_m x^m$ which interpolates $P_{(2k+2)x+1}$. We have the estimate $a_1 \equiv 4 - 4k \pmod{2^4}$ from (2.4) for the linear coefficient of $g_1(x)$. Since the constant coefficient $a_0 = 2$, and $\nu_2(a_i) > 1$ for $i > 0$, we have $\nu_2(g_1(x)) = 1$ for all $x \in \mathbb{Z}$, giving the result for $n \equiv 1 \pmod{2k+2}$.

For the case $k = 4$ and $n = 10m - 1$, we calculate that $P_{-1} = 0$, $P_{-11} = 0$, $P_{-21} = 16$, and $P_9 = 80$. Consider the analytic functions $g_j(x) = \sum_i a_i x^i$ which satisfy $g_j(m) = P_{10m+j}$ for $m \in \mathbb{Z}$, whose coefficients a_i satisfy $\nu_2(a_i) \geqslant i + S_2(i)$ for all $i > 0$. First take $m = 2x$, $j = -1$ and consider the function $f(x)$ given by

$$P_{20x-1} = f(x) = g_{-1}(2x) = 2a_1 x + 4a_2 x^2 + \cdots$$

which satisfies $f(0) = 0$ and $f(-1) = 16$. Since all terms of the series except the first term $2a_1$ have valuation at least 5, we must have $\nu_2(a_1) = 3$ and $\nu_2(f(x)) = 3 + \nu_2(2x)$ for all $x \in \mathbb{Z}$, proving the stated result when m is even. Next take $m = 2x - 1$ and consider the function

$$P_{20x-11} = h(x) = g_{-1}(2x-1) = g_{-11}(2x) = 2a_1 x + 4a_2 x^2 + \cdots$$

which satisfies $h(0) = 0$ and $h(1) = 80$. Since all terms of the series except the first term $2a_1$ have valuation at least 5, we must have $\nu_2(a_1) = 3$ and $\nu_2(h(x)) =$

$3+\nu_2(2x)$ for all $x \in \mathbb{Z}$, proving the stated result when m is odd. Therefore in any case we have $\nu_2(P_{10m-1}) = 3+\nu_2(m)+\nu_2(m+1)$.

Suppose k is even and $n = (2k+2)m-k$. The analytic function $g_{-k}(x) = \sum_i a_i x^i$ satisfies $g_{-k}(0) = 0$, $g_{-k}(1) = 4k-4$, and $g_{-k}(-1) = 12k-28$. We see from (2.2) that $a_0 = 0$ and $a_1 \equiv -4k+12 \pmod{2^3\mathbb{Z}_2}$, so that $\nu_2(a_1) = 2$ when k is even. Therefore $\nu_2(g_{-k}(m)) = \nu_2(a_1 m) = 2+\nu_2(m)$ for $m \in \mathbb{Z}$.

Finally, suppose k is odd and $n = (2k+2)m-k$. The analytic function $g_{-k}(x) = \sum_i a_i x^i$ satisfies $g_{-k}(0) = 0$, $g_{-k}(1) = 4k-4$, $g_{-k}(-1) = 12k-28$, $g_{-k}(-2) = 40k-88$, and $g_{-k}(2) \equiv 24k-40 \pmod{2^{k+2}}$. We see from (2.2), (2.3) that $a_0 = 0$, $a_1 \equiv -4k+12 \pmod{2^6}$, $a_2 \equiv 8k-16 \pmod{2^6}$, and $a_3 \equiv a_4 \equiv 0 \pmod{2^6}$. Thus $\nu_2(a_1) \geqslant 3$, the first two vertices of the Newton polygon are $(1, \nu_2(a_1))$ and $(2,3)$, and all other sides have slope at least 1. Therefore $g_{-k}(x)$ has a zero at $x = 0$ and exactly one other zero $x = z$ which satisfies $\nu_2(z) = \nu_2(a_1)-3$ and

$$0 = g(z) \equiv a_1 z + a_2 z^2 \pmod{2^6 z^3 \mathbb{Z}_2},$$

which implies $z \equiv \frac{k-3}{2k-4} \pmod{2^3\mathbb{Z}_2}$.

Now write $g_{-k}(x) = \sum_m a_m x^m = x(x-z)\sum_m b_m x^m$. Then $b_0 = -a_1/z$ has $\nu_2(b_0) = 3$, and $b_m = -(a_1+a_2z+\cdots+a_{m+1}z^m)/z^{m+1}$ for $m \geqslant 1$. Since $g_{-k}(z) = 0$ we have $\nu_2(b_1) \geqslant 6$ and $\nu_2(b_m) \geqslant m+2$ for all $m > 0$. Therefore $\sum_m b_m x^m$ also converges on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$. Since $\nu_2(b_m) > 3$ for all $m > 0$, we have $\nu_2(\sum_m b_m x^m) = 3$ for all $x \in \mathbb{Z}_2$. It follows that $\nu_2(g_{-k}(x)) = \nu_2(x)+\nu_2(x-z)+3$ for all $x \in \mathbb{Z}_2$, completing the proof. □

As with the case of even order k, the generalized Fibonacci sequences (T_n) of odd order k are 2-adically dense in two of the four congruence classes modulo 4.

Theorem 8. *Let T_n denote the generalized Fibonacci number of odd order k. Then the sequence (T_n) is dense in $4\mathbb{Z}_2$ but not in $2\mathbb{Z}_2$. The sequence (T_n) is also dense in $1+4\mathbb{Z}_2$ but not in $1+2\mathbb{Z}_2$. Consequently, for every integer $N \equiv 0, 1 \pmod 4$ and every positive integer s there exists $n \in \mathbb{Z}$ such that $T_n \equiv N \pmod{2^s}$.*

Proof. Consider the analytic function $g_0(x) = \sum_{m \geqslant 0} a_m x^m$ which interpolates $T_{(2k+2)m}$. Since $g_0(0) = 0$, $g_0(1) = 2^{k+1}(2k-3)-4k+8$, and $g_0(-1) = 4k-8$, we determine from (2.2), (2.3) that $\nu_2(a_1) = 2$ and $\nu_2(a_i) > 2$ for $i > 1$. Therefore, as in the proof of Theorem 7, for every $y \in 4\mathbb{Z}_2$ there exists $x \in \mathbb{Z}_2$ such that $g_0(x) = y$. Thus (T_n) is dense in $4\mathbb{Z}_2$. However, from Proposition 1 we observe that the values of (T_n) which are even but not multiples of 4 all lie in a single congruence class modulo 2^k, and therefore cannot be dense in $2+4\mathbb{Z}_2$. Thus (T_n) is not dense in $2\mathbb{Z}_2$.

A similar calculation using the analytic function $g_{1-k}(x) = \sum_{m \geqslant 0} a_m x^m$ which interpolates $T_{(2k+2)x+1-k}$ shows that $a_0 = 1$, $a_1 \equiv 8k-20 \pmod{2^4\mathbb{Z}_2}$, $\nu_2(a_1) = 2$, and $\nu_2(a_i) > 2$ for $i > 1$, so that (T_n) is dense in $1+4\mathbb{Z}_2$. However, since $\nu_2(P_n)$ never takes the value 2, (T_n) cannot be dense in $1+2\mathbb{Z}_2$. □

5. Small Values of k

We conclude with some illustrations of these theorems for specific small values of k, including some numerical computations of the z values.

Corollary 1. *If $k = 2$, then for all integers n we have*

$$\nu_2(P_n) = \nu_2(F_n + 1) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3, \\ 1, & \text{if } n \equiv 2, \pm 1 \pmod 6, \\ 2 + \nu_2(m), & \text{if } n = 6m - 2. \end{cases}$$

Corollary 2. *If $k = 3$, then for all integers n we have*

$$\nu_2(P_n) = \begin{cases} 0, & \text{if } n \equiv 0, 3 \pmod 4, \\ 1, & \text{if } n \equiv 1, \pm 2 \pmod 8, \\ 3 + \nu_2(m) + \nu_2(m - z), & \text{if } n = 8m - 3, \end{cases}$$

where z is some 2-adic integer satisfying $z \equiv -601592 \pmod{2^{20}\mathbb{Z}_2}$.

Proof. To estimate z, we used the lacunary recurrence

$$T_{n+16} = 131T_{n+8} - 3T_n + T_{n-8} \tag{5.1}$$

[10, eq. (3.6)] to calculate (T_n) for negative integers in the congruence class $n \equiv -3 \pmod 8$, determining that for $m = -601592$, $n = 8m - 3 = -4812739$ we have $\nu_2(P_n) = 26$, showing that $\nu_2(-601592 - z) = 20$ by comparison with the formula from Theorem 5. □

Remark. A referee has observed that a previously published paper [1] contains an incorrect formula for $\nu_2(T_n + 1)$ in the case $k = 3$. Lemma 2 of [1] stated that the valuation is given by

$$\begin{cases} 0, & \text{if } n \equiv 0, 3 \pmod 4, \\ 1, & \text{if } n \equiv 1, \pm 2 \pmod 8, \\ 3 + \nu_2(m) + \nu_2(m - 8), & \text{if } n = 8m - 3 \text{ and } m \neq 8, \\ 15, & \text{if } n = 8m - 3 \text{ and } m = 8. \end{cases} \tag{5.2}$$

This formula is incorrect because it would imply that (T_n) is not continuous as a 2-adic function of n on $\mathbb{Z}_2$, having a discontinuity at $n = 61$, which contradicts [11, Theorem 3]. The above formula (5.2) involves $\nu_2(m - 8)$ where the correct formula from Corollary 2 has $\nu_2(m - z)$ where $z \equiv -601592 \pmod{2^{20}\mathbb{Z}_2}$; since $\nu_2(8 - z) = 9$, the above formula (5.2) will give an incorrect valuation whenever $n > 61$ satisfies $n \equiv 61 \pmod{2^{12}}$. The smallest counterexample to (5.2) occurs when $n = 4157$, for which $\nu_2(T_{4157} + 1) = 18$ and the above formula predicts the valuation to be 15.

We also observe that the above corollary implies that the Tribonacci sequence (T_n) has the property that $\nu_2(T_n+1)$ never takes the values 2,4,6,8, or 9, although it takes all other nonnegative integer values. If $n = 8m-3$ and $\nu_2(m) \in \{0,1,2\}$ then $\nu_2(m-z) = \nu_2(m)$ and thus $\nu_2(P_n) \in \{3,5,7\}$. If $\nu_2(m) > 3$ then $\nu_2(m-z) = 3$, while if $\nu_2(m) = 3$ then $\nu_2(m-z) > 3$, so in either of these cases $\nu_2(P_n) \geqslant 10$.

Corollary 3. *If $k = 4$, then for all integers n we have*

$$\nu_2(P_n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{5}, \\ 1, & \text{if } n \equiv 1, \pm 2, \pm 3 \pmod{10}, \\ 2, & \text{if } n \equiv 4 \pmod{10}, \\ 2 + \nu_2(m), & \text{if } n = 10m - 4, \\ 3 + \nu_2(m) + \nu_2(m+1), & \text{if } n = 10m - 1. \end{cases}$$

Corollary 4. *If $k = 5$, then for all integers n we have*

$$\nu_2(P_n) = \begin{cases} 0, & \text{if } n \equiv 0, 5 \pmod{6}, \\ 1, & \text{if } n \equiv 1, \pm 2, \pm 3, \pm 4 \pmod{12}, \\ 3 + \nu_2(m) + \nu_2(m-z), & \text{if } n = 12m - 5, \end{cases}$$

where z is some 2-adic integer satisfying $z \equiv -953013 \pmod{2^{20}\mathbb{Z}_2}$.

Proof. To estimate z, we used multisection as in [10] to determine that the lacunary recurrences (T_{12n+j}) all have characteristic polynomial

$$p(x) = x^5 - 3333x^4 - 758x^3 - 10x^2 + 5x - 1. \tag{5.3}$$

We then used this lacunary recurrence to calculate (T_n) for negative values of n congruent to $-5 \pmod{12}$, determining that for $m = -953013$, $n = 12m - 5 = -11436161$, we have $\nu_2(P_n) = 23$, showing that $\nu_2(-953013-z) = 20$ by comparison with the formula from Theorem 5. □

Remark. This corollary shows that for $k = 5$, $\nu_2(P_n)$ takes on every nonnegative integer value except 2 and 3.

Corollary 5. *If $k = 6$, then for all integers n we have*

$$\nu_2(P_n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{7}, \\ 1, & \text{if } n \equiv 6, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5 \pmod{14}, \\ 2 + \nu_2(m), & \text{if } n = 14m - 6. \end{cases}$$

Corollary 6. *If $k = 7$, then for all integers n we have*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, 7 \pmod{8}, \\ 1, & \text{if } n \equiv 7 \pmod{16}, \\ 2, & \text{if } n \equiv -1 \pmod{16}, \\ \nu_2(n) - 2, & \text{if } n \equiv 0 \pmod{16}, \\ 2 + \nu_2(m-z), & \text{if } n = 16m + 8, \end{cases}$$

where z is some 2-adic integer satisfying $z \equiv -11687 \pmod{2^{22}\mathbb{Z}_2}$.

Corollary 7. *If $k = 7$, then for all integers n we have*

$$\nu_2(P_n) = \begin{cases} 0, & \text{if } n \equiv 0, 7 \pmod{8}, \\ 1, & \text{if } n \equiv 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6 \pmod{16}, \\ 3 + \nu_2(m) + \nu_2(m - z), & \text{if } n = 16m - 7, \end{cases}$$

where z is some 2-adic integer satisfying $z \equiv -558438 \pmod{2^{21}\mathbb{Z}_2}$.

Remark. This corollary shows that for $k = 7$, $\nu_2(P_n)$ takes on every nonnegative integer value except 2, 4 and 5.

Acknowledgement. The author extends his thanks to the referees of this article for their helpful suggestions. All numerical computation was done using the PARI-GP calculator created by C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

References

[1] V. Facó and D. Marques, Tribonacci numbers and the Brocard-Ramanujan equation, *J. Integer Seq.* **19** (2016), Article 16.4.4, 7pp.

[2] N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta Functions*, Second Edition, Springer-Verlag, New York, 1984.

[3] T. Lengyel, The order of the Fibonacci and Lucas numbers, *Fibonacci Quart.* **33** (1995), 234–239.

[4] T. Lengyel and D. Marques, The 2-adic order of some generalized Fibonacci numbers, *Integers* **17** (2017), Article #A5, 10pp.

[5] D. Marques and T. Lengyel, The 2-adic order of the Tribonacci numbers and the equation $T_n = m!$, *J. Integer Seq.* **17** (2014), Article 14.10.1, 1–8.

[6] M. R. Murty, *Introduction to p-adic Analytic Number Theory*, AMS Studies in Advanced Mathematics Vol. 27, American Mathematical Society, Providence, 2002.

[7] C. A. G. Ruiz and F. Luca, On the zero-multiplicity of a fifth-order linear recurrence, *Int. J. Number Theory* **15.3** (2019), 585–595.

[8] W. H. Schikhof, *Ultrametric calculus. An introduction to p-adic analysis*, Cambridge University Press, London, 1984.

[9] B. Sobolewski, The 2-adic valuation of generalized Fibonacci sequences with an application to certain Diophantine equations, *J. Number Theory* **180**, 730–742, 2017.

[10] P. T. Young, On lacunary recurrences, *Fibonacci Quart.* **41.1** (2003), 41–47.

[11] P. T. Young, 2-adic valuations of generalized Fibonacci numbers of odd order, *Integers* **18** (2018), Article #A1, 13pp.