A NOTE ON PARTITIONS WITH SHORT PROFILE STEPS

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Abstract
We consider partitions with parts repeated at most m times, with first differences and smallest part at most t. Generating functions are produced for such partitions with small t, and the question of a better form of these functions raised. Several divisibility properties are proved for the generating function for such partitions with largest part at most N.

1. Introduction

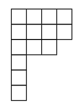
A *partition* of n is a weakly decreasing infinite sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\sum \lambda_i = n$, in which case we say $\lambda \vdash n$. For all basic terms in this subject see the reference [1]. The numbers of partitions of n, $p(n)$, are sequence A000041 in the On-line Encyclopedia of Integer Sequences [5], and are given by the coefficients of the generating function

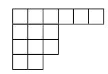
$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} (1-q^k)^{-1}.$$

Partitions are typically identified with their *Ferrers diagram*, which is the array of unit squares in the fourth quadrant, justified to the origin, in which the i-th row below the axis contains λ_i squares.

The generating function above was first produced by Euler in the paper "De Partitio Numerorum," the first scholarly look at the subject [4]. In that work he also proved the $m = 2$ case of a theorem later generalized by Sylvester, that the number of partitions of n in which nonzero parts are repeated less than m times, which we shall call *m-distinct* partitions, equals the number of partitions of n in which nonzero parts are not divisible by m, commonly called the *m-regular* partitions. A third class of partitions equal in number to these are partitions of n in which parts differ by strictly less than m, including the final positive difference before the infinite sequence of trailing zeros begins. These are called *m-flat.* When $m = 3$, for instance, the number of these is given by OEIS sequence A000726.

Sylvester's student Glaisher produced a combinatorial map which realized the first equality; the second is realized by *conjugation*, the map of reflecting the Ferrers diagram across the main diagonal:





$\lambda = (4, 4, 3, 1, 1, 1) \vdash 14 \quad \lambda' = (6, 3, 3, 2)$

This clearly maps m-distinct to m-flat partitions: short vertical segments along the southeast side of the diagram become short horizontal segments, and vice versa.

In a previous paper [3] the author considered partitions simultaneously possessing two or more of the named characteristics. A question of particular interest was the following:

Problem: Give a concise form of the generating function for partitions into parts simultaneously 3-distinct and 3-flat.

The reason for interest in these among the possible combinations of conditions is precisely because this set seems much less tractable than others. The easiest are partitions simultaneously m-distinct and t-regular: these are the fixed points of Glaisher's map on those two sets (and are a subset of the fixed points of a more general map on all partitions; see [2] for more on this), and possess the Euler product generating function

$$\sum_{n=0}^{\infty} p_{D,R}^{m,t}(n) q^n = \prod_{k=1}^{\infty} \frac{(1-q^{mk})(1-q^{tk})}{(1-q^k)(1-q^{mtk})}.$$

In contrast to this, the partitions considered in this note, those simultaneously m-distinct and t-flat, are much more difficult to handle. They are not the fixed points of the conjugation map, although it is easy to see that conjugation maps the class of m-distinct t-flat partitions to those t-flat and m-distinct, and the property of being m-distinct and m-flat is invariant under conjugation.

The generating function of the m-distinct t-flat partitions, however, has proven remarkably difficult to write down in any compact form. The very special case $t = 2$ is easy to state: it is

$$P_{D,F}^{(m,2)} = \sum_{k=0}^{\infty} q^{\binom{k+1}{2}} \frac{(q^{m-1}; q^{m-1})_k}{(q;q)_k}, \tag{1}$$

which is the generating function for partitions in which all parts from 1 to the largest part appear at least once. Here we have used the standard notation

$$(a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1}) \quad ,(a;q)_0 = 1.$$

In this note, we prove the following results for $P_{D,F}^{(m,3);\leq N}(q)$, the polynomial generating functions for the number of m-distinct, 3-flat partitions of n with largest part at most N.

First, we prove an interesting pattern in the divisibility properties for the case $m = t = 3$, the simplest nontrivial case above the $t = 2$ function given in (1). This is Theorem 1.

Theorem 1. *If $N \equiv i \pmod 3$, $i \in \{1, 2\}$, then*

$$(1+q+q^2)^i | \sum_{j=0}^{N} P_{D,F}^{3,3;j}(q) =: P_{D,F}^{3,3;\leq N}(q)$$

where divisibility is determined in $\mathbb{Z}[q]$. If instead $N \equiv 0 \pmod 3$, say $N = 3k$, then we have $P_{D,F}^{3,3;\leq N}(q) \equiv (-2)^k \pmod{(1+q+q^2)}$.

A more detailed conjecture on strict divisibility appears in the final section.

Addressing the motivating problem, we can establish one form for the generating function for the case of $t = 3$ and general m.

Theorem 2. *The number $p_{D,F}^{m,3;N}(n)$ of partitions of n in which parts of any nonzero size appear less than m times, parts differ by less than 3, and the largest part is exactly N, has the following generating functions.*

If $N \equiv 1 \pmod 2$, say $N = 2k+1$, then, letting $\vec{s}$ run over (possibly empty) increasing sequences in $\{1, 2, \dots, k-1\}$ with entries differing by at least 2,

$$P_{D,F}^{m,3;N} = \sum_{n=0}^{\infty} p_{D,F}^{m,3;N}(n)q^n = \frac{q^N - q^{mN}}{1-q^N}\left(\prod_{i=1}^{k}\left(\frac{(1-q^{(2i-1)m})(1-q^{(2i)m})}{(1-q^{2i-1})(1-q^{2i})} - 1\right)\right)\times$$
$$\sum_{\vec{s}=(s_1,\dots,s_j)}\prod_{r=1}^{j}\left((-1)\frac{(q^{2s_i-1}+\dots+q^{(2s_i-1)(m-1)})(q^{2s_i+2}+\dots+q^{(2s_i+2)(m-1)})}{\left(\frac{(1-q^{(2s_i-1)m})(1-q^{(2s_i)m})}{(1-q^{2s_i-1})(1-q^{2s_i})}-1\right)\left(\frac{(1-q^{(2s_i+1)m})(1-q^{(2s_i+2)m})}{(1-q^{2s_i+1})(1-q^{2s_i+2})}-1\right)}\right).$$

If $N = 2k$, then

$$P_{D,F}^{m,3;N} = \left(\prod_{i=1}^{k}\left(\frac{(1-q^{(2i-1)m})(1-q^{(2i)m})}{(1-q^{2i-1})(1-q^{2i})} - 1\right)\right)\times$$
$$\sum_{\vec{s}=(s_1,\dots,s_j)}\prod_{r=1}^{j}\left((-1)\frac{(q^{2s_i-1}+\dots+q^{(2s_i-1)(m-1)})(q^{2s_i+2}+\dots+q^{(2s_i+2)(m-1)})}{\left(\frac{(1-q^{(2s_i-1)m})(1-q^{(2s_i)m})}{(1-q^{2s_i-1})(1-q^{2s_i})}-1\right)\left(\frac{(1-q^{(2s_i+1)m})(1-q^{(2s_i+2)m})}{(1-q^{2s_i+1})(1-q^{2s_i+2})}-1\right)}\right)$$
$$- P_{D,F}^{m,3;N-1}.$$

Summed over all N, this gives the generating function for m-distinct t-flat partitions with unrestricted largest part.

As an answer to the Problem, we briefly discuss how "concise" this might be considered. We conclude by considering further directions. The question for general t is still open, and a better form of the generating functions involved is entirely possible; a combinatorial proof of the divisibility theorems would be desirable.

2. Proof of Theorem 1

During investigation an interesting phenomenon was observed. Whatever the generating function $P_{D,F}^{(m,t);N}(q)$ might be, its evaluation at $q = 1$ counts the number of such partitions of any n which satisfy the specified conditions. In the case $m = t = 3$, as N varies, we find that we obtain the OEIS sequence A077846.

This sequence has formula $\sum_{i,j=0}^{N}(m-1)^j\binom{j}{i-j}$, which matches partitions of the type under investigation by the following procedure. Let j sizes of part exist, starting with the assumption that they are 1 through j. Let there be from 1 to $m-1$ repetitions of the various sizes. Insert $i-j$ spaces consisting of additional missing part sizes, at most one below the first or between each occupied part size, in $\binom{j}{i-j}$ possible ways, with $i-j$ ranging from 0 to $N-j$, i.e., i ranging from j to N. (Lower i gives zero terms.)

This sequence is well-divisible by increasing powers of 3; here we establish a partial q-analogue of this divisibility property for $P_{D,F}^{3,3;\leq N}(q)$, those partitions of this type with largest part at most N.

Proof. The proof is by induction. First, check the theorem computationally for small N to establish base cases. Next, we observe the recurrence, for $N \geq 3$,

$$\begin{aligned} P_{D,F}^{3,3;\leq N}(q) &= (q^N + q^{2N})\left(P_{D,F}^{3,3;\leq N-1}(q) - P_{D,F}^{3,3;\leq N-3}(q)\right) + P_{D,F}^{3,3;\leq N-1}(q) \\ &= (1 + q^N + q^{2N})P_{D,F}^{3,3;\leq N-1}(q) - (q^N + q^{2N})P_{D,F}^{3,3;\leq N-3}(q). \end{aligned}$$

That is, if the largest part is at most N, then either 1 or 2 parts of size N exist, in which case the remaining parts form a 3-distinct, 3-flat partition with largest part at most $N-1$ and more than $N-3$, or parts of size N do not exist, so parts are at most $N-1$.

Now assume that the theorem holds for all values smaller than N.

When $N \equiv 1 \pmod 3$, then

$$P_{D,F}^{3,3;\leq N}(q) = (1 + q^N + q^{2N})(X) - (q^N + q^{2N})Y$$

where $(1 + q + q^2)$ does not divide X and $(1 + q + q^2)|Y$.

Next observe that when $N \equiv 1 \pmod 3$, say $N = 3k+1$, then we have

$$1 + q^N + q^{2N} = (1+q+q^2) \cdot \left(\sum_{j=0}^{2k} q^{3j} - \sum_{j=0}^{k-1} (q^{3j+1} + q^{3k+2+3j}) \right).$$

Therefore when $N \equiv 1 \pmod 3$, we have $P_{D,F}^{3,3;\leq N}(q)$ as a sum of two terms both divisible by $(1+q+q^2)$.

When $N \equiv 2 \pmod 3$, we have $(1+q+q^2)|X$ and $(1+q+q^2)^2|Y$. Likewise,

$$1 + q^{3k+2} + q^{6k+4} = (1+q+q^2) \cdot \left(\sum_{j=0}^{k} q^{3j}(1+q^{3k+2}) - \sum_{j=0}^{k} q^{1+3j} - \sum_{j=0}^{k-1} q^{3k+4+3j} \right).$$

Hence the theorem holds for $N \equiv 2 \pmod 3$.

Finally, when $N = 3k$, we have $(1+q+q^2)|X$ and $Y \equiv (-2)^{k-1} \pmod{(1+q+q^2)}$. Since $(1+q+q^2)|(1-q^3)$ and $(1-q^3)|(1-q^{3k})$, we have $(q^{3k}+q^{6k}) \equiv -2 \pmod{(1+q+q^2)}$ and the theorem holds for $N \equiv 0 \pmod 3$. □

3. Proof of Theorem 2

Proof. Begin with the odd N case. The factor $\frac{q^N - q^{mN}}{1-q^N}$ expresses the existence of anywhere from 1 to $m-1$ copies of part size N.

Each term

$$\frac{(1-q^{(2i-1)m})(1-q^{(2i)m})}{(1-q^{2i-1})(1-q^{2i})} - 1,$$

which we may call bin i, expresses a choice of at most $m-1$ copies of parts of size $2i-1$ and at most $m-1$ copies of parts of size $2i$, with the exception that we cannot choose zero copies of *both*. In essence, this term forbids the existence of one type of violation of the "difference less than 3" condition, that in which parts $2i-1$ and $2i$ do not appear but some higher part does.

With those factors alone violations are still possible, but only if two in adjacent bins s_i and s_i+1 we choose some parts of size $2s_i-1$, and parts of size $2s_i+2$, but neither of the two intervening sizes. We wish to subtract off those terms of the product in which exactly this choice occurs. The benefit of this construction is that errors cannot "overlap": if a partition has several errors, they must occur in distinct pairs of bins. Therefore, we select a vector $\vec{s}$ of places s_i where errors occur, choosing as our s_i the smaller of the two part sizes bounding the error; an inclusion-exclusion argument yields the resulting product.

For the $N = 2k$ case, the first term is simply built with this argument without guaranteeing the existence of a part of size N; instead, the inclusion of bin k means that the largest part is either N or $N-1$. The final subtraction removes any terms in which the largest part is exactly $N-1$. Of course, most factors are common to the two products, and considerable combining of terms could occur. $\square$

3.1. Remarks on Concision

Clearly this generating function is far more complicated than the simple expressions noted earlier. It may reasonably be asked how good it is as a counting function. Readers of Stanley [6] or Wilf [7] generally agree with the criterion that a product form is a gold standard, and if a summation must be employed, it should have as few terms as possible.

A very straightforward form of the generating function would be to sum products of $(q^i + \cdots + q^{(m-1)i})$ over all allowable collections of part sizes. For partitions with largest size N, these collections are enumerated by choices of parts from 1 through N to drop, which are sequences differing by at least 2 from the set $\{1, \ldots, N-1\}$. These are well known to be enumerated by the Fibonacci numbers, and hence we would require approximately $\frac{1}{\sqrt{5}}\phi^{N-1}$ summands, where ϕ is the golden ratio.

With the generating function given in the theorem, the number of summands is the square root of this, for we enumerate errors by pairs of bins, and thus remove nonconsecutive sequences in $\{1, \ldots, k\}$ with $N \approx 2k$. Although the number of summands is still exponential, a root reduction is certainly a great savings.

A pure product form is exceedingly unlikely. Recall that $P_{D,F}^{(3,3);\leq N}(1)$ is given by OEIS sequence A077846. The $N = 14$ case then yields $1605717 = 3^3 \cdot 59471$ as the prime factorization; likewise, the prime 110771 divides $P_{D,F}^{3,3;18}(1)$. This strongly suggests the lack of a compact product form. Still, perhaps other variations might yield better results.

The problem, then, certainly still has room for advancement. A form of the generating function with one or two simple linear indices would be a great improvement over the present case.

4. Further Questions

Regarding Theorem 1, the following higher power congruences appear from computation to be the case but the calculations become substantially more involved.

Conjecture 1. The divisibilities in Theorem 1 are strict, and in fact

$$P_{D,F}^{3,3;\leq 3k} \equiv (-2)^{k-1}\left(3(1-q^3)\cdot\binom{k+1}{2} - 2\right) \pmod{(1+q+q^2)^2},$$

$$P_{D,F}^{3,3;\leq 3k+1} \equiv (1+q+q^2)\left(\sum_{j=0}^{k}(3j+1)(-2)^j\right) \pmod{(1+q+q^2)^2}.$$

Conjecture 2. We have $(1+q^3+q^6)||P_{D,F}^{3,3;\leq N}(q)$ if and only if $N \equiv 7 \pmod 9$, and for any $k \geq 0$,

$$P_{D,F}^{3,3;\leq 3k}(q) \equiv P_{D,F}^{3,3;\leq 3k+3}(q) \equiv P_{D,F}^{3,3;\leq 3k+6}(q) \pmod{(1+q^3+q^6)}.$$

(For identifying the coefficients in these conjectures the author, as is often the case, is indebted to the OEIS.)

The following question also seems natural, given the divisibility theorems:

Problem: Give a combinatorial proof of Theorem 1, i.e., a triple matching between partitions with largest part at most N involving sizes $3j$, $3j+1$ and $3j+2$ for all valid j when $N \equiv 1 \pmod 3$, or nonary matching when $N \equiv 2 \pmod 3$.

This matching cannot simply be adding 0, 1 or 2 parts of size 1, since some 3-distinct 3-flat partitions have smallest parts 1 and 3, and therefore cannot be members of a class constructed by removing the 1s.

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