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NEW CONGRUENCES INVOLVING TRIANGULAR NUMBERS

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Abstract

For $n = 0, 1, 2, \dots$ let T_n denote the nth triangular number $n(n+1)/2$. For any odd prime p, we prove that

$$\prod_{\substack{1 \le i < j \le (p-1)/2 \\ p \nmid T_i + T_j}} (T_i + T_j) \equiv \begin{cases} (-2)^{(p+3)/4} \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ 2^{(p+5)/4} \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Furthermore, we also prove the congruence

$$\sum_{\substack{1 \le i < j \le (p-1)/2 \\ p \nmid T_i + T_j}} \frac{1}{T_i + T_j} \equiv 2 + \frac{(-1)^{(p^2-1)/8}}{2} \pmod{p}$$

in the ring of p-adic integers.

1. Introduction

Let p be an odd prime. Then, Wilson's theorem states that

$$(p-1)! = \prod_{n=1}^{p-1} n \equiv -1 \pmod{p}.$$

Note that the integers

$$1^2,\ 2^2,\ \dots,\ \left(\frac{p-1}{2}\right)^2$$

are pairwise incongruent modulo p. Similar to Wilson's theorem, it is known that if $p \equiv 3 \pmod{4}$ then

$$\prod_{1 \le i < j \le (p-1)/2} (i^2 + j^2) \equiv (-1)^{\lfloor (p+1)/8 \rfloor} \pmod{p}$$

(cf. Problem N.2 of [4, pp. 364-365]). When $p \equiv 1 \pmod 4$, we can write p in a unique way as a^2+b^2 with $a, b \in \mathbb{Z}$ and $1 \le a < b \le (p-1)/2$, and recently Sun [3] proved that

$$\prod_{\substack{1\le i<j\le (p-1)/2\\ p\nmid i^2+j^2}} (i^2+j^2) \equiv (-1)^{\lfloor (p-5)/8\rfloor} \pmod p.$$

Recall that the triangular numbers are given by

$$T_n := \frac{n(n+1)}{2} \quad (n = 0, 1, 2, \ldots).$$

For any odd prime p, it is easy to see that the triangular numbers

$$T_1,\ T_2,\ \ldots,\ T_{(p-1)/2}$$

are also pairwise incongruent modulo p. Motivated by the above work and Sun's conjecture (cf. [2, Conjecture 4.4]) that

$$-\det\left[\left(\frac{2T_i+2T_j}{p}\right)\right]_{1\le i,j\le (p-1)/2}$$

is a quadratic nonresidue modulo any prime $p \equiv \pm 3 \pmod 8$ (where $(\frac{\cdot}{p})$ denotes the Legendre symbol), we obtain the following new result on a product involving triangular numbers.

Theorem 1.1. *Let p be an odd prime. Then*

$$\prod_{\substack{1\le i<j\le (p-1)/2\\ p\nmid T_i+T_j}} (T_i+T_j) \equiv \begin{cases} (-2)^{(p+3)/4} \pmod p & \text{if } p\equiv 1 \pmod 4,\\ 2^{(p+5)/4} \pmod p & \text{if } p \equiv 3 \pmod 4.\end{cases} \tag{1.1}$$

Corollary 1.1. *For any odd prime p, the product*

$$\prod_{\substack{1\le i<j\le (p-1)/2\\ p\nmid T_i+T_j}} (T_i+T_j)$$

is a quadratic residue modulo p.

Inspired by Theorem 1.1, we also establish the following result on a sum involving triangular numbers.

Theorem 1.2. *Let p be an odd prime. Then we have the congruence*

$$\sum_{\substack{1\le i<j\le (p-1)/2\\ p\nmid T_i+T_j}} \frac{1}{T_i+T_j} \equiv 2 + \frac{(-1)^{(p^2-1)/8}}{2} \pmod p. \tag{1.2}$$

in the ring of p-adic integers.

We will prove Theorem 1.1 and Corollary 1.1 in the next section. Theorem 1.2 will be proved in Section 3.

2. Proofs of Theorem 1.1 and Corollary 1.1

Lemma 2.1. *Let p be any odd prime. Then*

$$\left(\frac{p-1}{2}!\right)^2 \equiv (-1)^{(p+1)/2} \pmod p$$

and

$$2^{(p-1)/2} \equiv \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} \pmod p.$$

This is well known. Note that for any odd prime p we have

$$(p-1)! = \prod_{k=1}^{(p-1)/2} k(p-k) \equiv (-1)^{(p-1)/2} \left(\frac{p-1}{2}!\right)^2 \pmod p$$

and hence the first congruence in Lemma 2.1 follows from Wilson's theorem.

Lemma 2.2. *Let p be an odd prime, and let $b, c \in \mathbb{Z}$ with $p \nmid b^2 - 4c$. Then*

$$\sum_{x=0}^{p-1} \left(\frac{x^2+bx+c}{p}\right) = -1.$$

This is also known, see, e.g., [1, p. 58].

As usual, we denote the cardinality of a set S by $|S|$.

Lemma 2.3. *Let p be any odd prime and let*

$$r(n) := \left|\left\{(i,j) :\ 0 \le i < j \le \frac{p-1}{2} \text{ and } T_i + T_j \equiv n \pmod p\right\}\right|$$

for $n = 0, \ldots, p-1$. Then

$$r\left(\frac{p(2-(\frac{-1}{p}))-1}{4}\right) = \frac{1+(\frac{-1}{p})}{2} \cdot \frac{p-1}{4}. \tag{2.1}$$

Proof. Let $n = (p(2-(\frac{-1}{p}))-1)/4$. Then $4n+1 \equiv 0 \pmod p$. For $i, j \in \{0, \ldots, (p-1)/2\}$, we have

$$T_i + T_j \equiv n \pmod p \iff (2i+1)^2 + (2j+1)^2 \equiv 8n+2 \equiv 0 \pmod p.$$

Hence

$$(2i+1)^2 \equiv 0 \pmod p \iff i = \frac{p-1}{2}.$$

Thus

$$\begin{aligned}2r(n) &= \left|\left\{(i,j):\ 0 \le i,j < \frac{p-1}{2} \text{ and } (2i+1)^2 + (2j+1)^2 \equiv 0 \pmod p\right\}\right| \\ &= \left|\left\{(x,y):\ x,y \in \{1,\dots,p-1\},\ \left(\frac{x}{p}\right) = \left(\frac{y}{p}\right) = 1 \text{ and } p \mid x+y\right\}\right| \\ &= \left|\left\{1 \leqslant x \leqslant p-1:\ \left(\frac{x}{p}\right) = \left(\frac{-x}{p}\right) = 1\right\}\right| \\ &= \frac{1+(\frac{-1}{p})}{2}\left|\left\{1 \le x \le p-1:\ \left(\frac{x}{p}\right) = 1\right\}\right|\end{aligned}$$

and hence

$$r(n) = \frac{1+(\frac{-1}{p})}{2} \cdot \frac{p-1}{4}$$

as desired. $\square$

Proof of Theorem 1.1. For $n = 0, \dots, p-1$, we define $r(n)$ as in Lemma 2.3. For $n_0 = (p(2 - (\frac{-1}{p})) - 1)/4$, the value of $r(n_0)$ is given by (2.1). Now we compute $r(n)$ for any fixed $1 \leqslant n \leqslant p-1$ with $n \ne n_0$. Note that $4n+1 \not\equiv 0 \pmod p$ and

$$\begin{aligned}&\left|\left\{0 \le k \le \frac{p-1}{2}:\ 2T_k \equiv n \pmod p\right\}\right| \\ =&\left|\left\{0 \le k < \frac{p-1}{2}:\ (2k+1)^2 \equiv 4n+1 \pmod p\right\}\right| = \frac{1+(\frac{4n+1}{p})}{2}.\end{aligned}$$

Thus

$$\begin{aligned}&2r(n) + \frac{1+(\frac{4n+1}{p})}{2} \\ =&\left|\left\{(i,j):\ 0 \le i,j \le \frac{p-1}{2} \text{ and } T_i + T_j \equiv n \pmod p\right\}\right| \\ =&\left|\left\{(i,j):\ 0 \le i,j \le \frac{p-1}{2} \text{ and } (2i+1)^2 + (2j+1)^2 \equiv 8n+2 \pmod p\right\}\right| \\ =&\left|\left\{(x,y):\ 0 \le x,y \le p-1,\ \left(\frac{x}{p}\right) \ge 0,\ \left(\frac{y}{p}\right) \ge 0,\ x+y \equiv 8n+2 \pmod p\right\}\right| \\ =&\frac{1+(\frac{8n+2}{p})}{2} + \left|\left\{1 \le x \le p-1:\ \left(\frac{x}{p}\right) = 1 \text{ and } \left(\frac{8n+2-x}{p}\right) \ge 0\right\}\right|\end{aligned}$$

and hence

$$\begin{aligned}
&2r(n)+\frac{1-(\frac{2}{p})}{2}\left(\frac{4n+1}{p}\right)\\
=&\sum_{x=1}^{p-1}\frac{1+(\frac{x}{p})}{2}\cdot\frac{1+(\frac{8n+2-x}{p})}{2}+\sum_{\substack{x=1\\ p\mid 8n+2-x}}^{p-1}\frac{1+(\frac{x}{p})}{2}\cdot\frac{1}{2}\\
=&\frac{p-1}{4}+\frac{1}{4}\sum_{x=1}^{p-1}\left(\frac{x}{p}\right)+\frac{1}{4}\sum_{x=1}^{p-1}\left(\frac{8n+2-x}{p}\right)\\
&+\frac{1}{4}\left(\frac{-1}{p}\right)\sum_{x=0}^{p-1}\left(\frac{x^2-(8n+2)x}{p}\right)+\frac{1+(\frac{8n+2}{p})}{4}\\
=&\frac{p+(\frac{8n+2}{p})}{4}-\frac{1}{4}\left(\frac{8n+2}{p}\right)-\frac{1}{4}\left(\frac{-1}{p}\right)=\frac{p-(\frac{-1}{p})}{4}
\end{aligned}$$

with the aid of Lemma 2.2. Note that

$$\frac{1}{2}\left(\frac{p-(\frac{-1}{p})}{4}+\frac{1-(\frac{2}{p})}{2}\right)=\left\lfloor\frac{p+5}{8}\right\rfloor.$$

Therefore

$$r(n)=\left\lfloor\frac{p+5}{8}\right\rfloor+\frac{1+(\frac{4n+1}{p})}{2}\cdot\frac{(\frac{2}{p})-1}{2}. \tag{2.2}$$

Now we know the value of $r(n)$ for each $n=1,\ldots,p-1$. Observe that

$$\begin{aligned}
\prod_{\substack{0\le i<j\le(p-1)/2\\ p\nmid T_i+T_j}}(T_i+T_j)&=\prod_{n=1}^{p-1}n^{r(n)}\\
&=n_0^{r(n_0)-\lfloor\frac{p+5}{8}\rfloor}\prod_{n=1}^{p-1}n^{\lfloor\frac{p+5}{8}\rfloor}\times\prod_{\substack{n=1\\ (\frac{4n+1}{p})=1}}^{p-1}n^{\frac{1}{2}((\frac{2}{p})-1)}.
\end{aligned}$$

By Lemma 2.3,

$$n_0^{r(n_0)-\lfloor\frac{p+5}{8}\rfloor}\equiv\left(-\frac{1}{4}\right)^{(1+(\frac{-1}{p}))\frac{p-1}{8}-\lfloor\frac{p+5}{8}\rfloor}\pmod p.$$

Also,

$$\prod_{n=1}^{p-1}n^{\lfloor\frac{p+5}{8}\rfloor}=((p-1)!)^{\lfloor\frac{p+5}{8}\rfloor}\equiv(-1)^{\lfloor\frac{p+5}{8}\rfloor}\pmod p$$

by Wilson's theorem. For each $n = 1, \ldots, p-1$, clearly

$$\begin{aligned}\left(\frac{4n+1}{p}\right) = 1 \ &\Leftrightarrow (2k+1)^2 \equiv 4n+1 \pmod p \text{ for some } 0 \leqslant k < \frac{p-1}{2} \\ &\Leftrightarrow n \equiv k(k+1) \pmod p \text{ for some } k = 1, \ldots, \frac{p-3}{2}.\end{aligned} \tag{2.3}$$

Therefore

$$\begin{aligned}&\prod_{\substack{0\le i<j\le (p-1)/2 \\ p\nmid T_i+T_j}} (T_i+T_j) \\ \equiv& \left(-\frac{1}{4}\right)^{(1+(\frac{-1}{p}))\frac{p-1}{8}-\lfloor\frac{p+5}{8}\rfloor} (-1)^{\lfloor\frac{p+5}{8}\rfloor} \left(\prod_{k=1}^{(p-3)/2} k(k+1)\right)^{\frac12((\frac{2}{p})-1)} \\ =&(-1)^{(1+(\frac{-1}{p}))\frac{p-1}{8}} 2^{2\lfloor\frac{p+5}{8}\rfloor-(1+(\frac{-1}{p}))\frac{p-1}{4}} \left(\frac{2}{p-1}\left(\frac{p-1}{2}!\right)^2\right)^{\frac12((\frac{2}{p})-1)} \\ \equiv&(-1)^{(1+(\frac{-1}{p}))\frac{p-1}{8}} 2^{2\lfloor\frac{p+5}{8}\rfloor-(1+(\frac{-1}{p}))\frac{p-1}{4}} \left((-1)^{(p-1)/2}2\right)^{\frac12((\frac{2}{p})-1)} \pmod p\end{aligned}$$

with the aid of Lemma 2.1. Note that

$$(-1)^{(1+(\frac{-1}{p}))\frac{p-1}{8}+\frac{p-1}{4}((\frac{2}{p})-1)} = \left(\frac{2}{p}\right)$$

and

$$2^{2\lfloor\frac{p+5}{8}\rfloor-(1+(\frac{-1}{p}))\frac{p-1}{4}+\frac12((\frac{2}{p})-1)} = 2^{\frac14(1-p(\frac{-1}{p}))},$$

which can be easily checked by considering all the cases $p \equiv 1, 3, 5, 7 \pmod 8$. So we have

$$\prod_{\substack{0\le i<j\le (p-1)/2 \\ p\nmid T_i+T_j}} (T_i+T_j) \equiv \left(\frac{2}{p}\right) 2^{\frac14(1-p(\frac{-1}{p}))} \pmod p. \tag{2.4}$$

Observe that

$$\begin{aligned}\prod_{\substack{j=1 \\ p\nmid T_0+T_j}}^{(p-1)/2} (T_0+T_j) = \prod_{j=1}^{(p-1)/2} \frac{j(j+1)}{2} &= 2^{-(p-1)/2}\,\frac{p+1}{2}\left(\frac{p-1}{2}!\right)^2 \\ &\equiv \left(\frac{2}{p}\right)\frac{(-1)^{(p+1)/2}}{2} \pmod p\end{aligned}$$

in view of Lemma 2.1. Combining this with (2.4) we see that

$$\begin{aligned}\prod_{\substack{1\le i<j\le (p-1)/2 \\ p\nmid T_i+T_j}} (T_i+T_j) \equiv& \frac{(\frac{2}{p})2^{\frac14(1-p(\frac{-1}{p}))}}{(\frac{2}{p})\frac{(-1)^{(p+1)/2}}{2}} = (-1)^{(p+1)/2}2^{1+\frac14(1-p(\frac{-1}{p}))} \\ \equiv& \begin{cases} (-2)^{(p+3)/4} \pmod p & \text{if } 4 \mid p-1, \\ 2^{(p+5)/4} \pmod p & \text{if } 4 \mid p+1. \end{cases}\end{aligned}$$

The proof of Theorem 1.1 is now complete. □

Proof of Corollary 1.1. If $p \equiv 1 \pmod 4$, then

$$\left(\frac{(-2)^{(p+3)/4}}{p}\right) = \left(\frac{2}{p}\right)^{(p+3)/4} = (-1)^{\frac{p-1}{4}\cdot\frac{p+3}{4}} = 1.$$

If $p \equiv 3 \pmod 4$, then

$$\left(\frac{2^{(p+5)/4}}{p}\right) = \left(\frac{2}{p}\right)^{(p+5)/4} = (-1)^{\frac{p+1}{4}\cdot\frac{p+5}{4}} = 1.$$

Therefore, applying (1.1) we immediately get the desired result. □

3. Proof of Theorem 1.2

To prove Theorem 1.2, we utilize some ideas in the proof of Theorem 1.1.

Proof of Theorem 1.2. Since

$$\sum_{j=1}^{(p-1)/2} \frac{1}{T_j} = 2\sum_{j=1}^{(p-1)/2}\left(\frac{1}{j} - \frac{1}{j+1}\right) = 2\left(1 - \frac{2}{p+1}\right) \equiv -2 \pmod p,$$

the congruence (1.2) has the following equivalent form:

$$\sum_{\substack{0\le i<j\le (p-1)/2\\ p\nmid T_i+T_j}} \frac{1}{T_i+T_j} \equiv \frac{1}{2}\left(\frac{2}{p}\right) \pmod p. \tag{3.1}$$

Note that

$$\sum_{\substack{0\le i<j\le (p-1)/2\\ p\nmid T_i+T_j}} \frac{1}{T_i+T_j} = \sum_{n=1}^{p-1}\frac{r(n)}{n} = \frac{r(n_0)}{n_0} + \sum_{\substack{n=1\\ n\ne n_0}}^{p-1}\frac{r(n)}{n},$$

where $r(n)$ is defined as in Lemma 2.3 and $n_0 := (p(2-(\frac{-1}{p}))-1)/4$. In view of (2.1), we have

$$\frac{r(n_0)}{n_0} = \frac{1+(\frac{-1}{p})}{2}\cdot\frac{p-1}{p(2-(\frac{-1}{p}))-1} \equiv \frac{1+(\frac{-1}{p})}{2} \pmod p. \tag{3.2}$$

For each $n = 1, \ldots, p-1$ with $n \ne n_0$, by (2.2) we have

$$\frac{r(n)}{n} = \left\lfloor\frac{p+5}{8}\right\rfloor\frac{1}{n} + \frac{(\frac{2}{p})-1}{2}\cdot\frac{1+(\frac{4n+1}{p})}{2n}.$$

Thus

$$\begin{aligned}\sum_{\substack{n=1\\ n\neq n_0}}^{p-1}\frac{r(n)}{n}=&\left\lfloor\frac{p+5}{8}\right\rfloor\left(\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k}+\frac{1}{p-k}\right)-\frac{1}{n_0}\right)+\frac{(\frac{2}{p})-1}{2}\sum_{\substack{n=1\\ (\frac{4n+1}{p})=1}}^{p-1}\frac{1}{n}\\ \equiv&-\frac{1}{n_0}\left\lfloor\frac{p+5}{8}\right\rfloor+\frac{(\frac{2}{p})-1}{2}\sum_{k=1}^{(p-3)/2}\frac{1}{k(k+1)}\quad\text{(by (2.3))}\\ \equiv&4\left\lfloor\frac{p+5}{8}\right\rfloor+\frac{(\frac{2}{p})-1}{2}\sum_{k=1}^{(p-3)/2}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\ =&4\left\lfloor\frac{p+5}{8}\right\rfloor+\frac{(\frac{2}{p})-1}{2}\left(1-\frac{2}{p-1}\right).\ \pmod p.\end{aligned}$$

Combining this with (3.2), we obtain

$$\begin{aligned}\frac{r(n_0)}{n_0}+\sum_{\substack{n=1\\ n\not\equiv 0}}^{p-1}\frac{r(n)}{n}\equiv&\frac{1+(\frac{-1}{p})}{2}+4\left\lfloor\frac{p+5}{8}\right\rfloor+\frac{3}{2}\left(\left(\frac{2}{p}\right)-1\right)\\ \equiv&\frac{1}{2}\left(\frac{2}{p}\right)\pmod p\end{aligned}$$

and hence the desired (3.1) follows.

The proof of Theorem 1.2 is now complete. □

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