CYCLEMASTER MATRICES AND COLLATZ CYCLES

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Received: 5/27/19, Revised: 5/6/20, Accepted: 9/17/20, Published: 10/9/20

Abstract

We introduce the cyclemaster matrix, which is defined in terms of the composition of an integer M into N parts, and prove that the existence of an N-cycle of index M for the "$3x$+1" problem implies the divisibility of the determinant of the cyclemaster matrix by $2^M - 3^N$. The cyclemaster matrix is shown to be a unifying concept for all "$ax + 1$" problems. Finally, we show that the non-existence of a solution to a polynomial equation of the form $F(x_1, \ldots, x_N) = 0$ in powers of 2 implies the non-existence of N-cycles for the original $3x + 1$ problem.

1. Introduction

Let $\mathcal{O}$ denote the set of all odd integers. For $r, q \in \mathcal{O}$, define the map $S\colon \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ by

$$S(r,q) = \frac{3r+q}{2^{n(r,q)}},$$

where the exponent $n(r, q)$ denotes the integer power of 2 which is necessary in order to ensure that $S(r, q)$ is an odd integer. This definition is often referenced in the literature as the "shortcut" definition of the Syracuse function.

We shall refer to the exponent $n(r, q)$ as the *indicial exponent* belonging to the lattice point (r, q). These exponents have an importance that has been overlooked in the literature. We introduce the cyclemaster matrix, which is defined in terms of the indicial exponents, and prove a divisibility property for its determinant. The cyclemaster matrix will be shown to have additional value as a unifying concept for all of the "$ax + 1$" problems. Surprisingly, we also show that the cyclemaster matrix is, in a very real sense, a bridge between all of the "$ax + 1$" problems and the theory of algebraic varieties.

To achieve these goals, we examine the entire collection of maps

$$T : \mathcal{O} \times \mathcal{O} \to \mathcal{O} \times \mathcal{O}$$

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defined by $T(r, q) = (S(r, q), q)$ and their iterations. If $r_2 = S(r_1, q)$, then we shall write $T(r_1, q) = (r_2, q)$ to denote a single interation of T, which maps the row of the lattice indexed by q into itself. Equivalently, we shall write

$$(r_1, q) \overset{n_1}{\to} (r_2, q),$$

where $n_1 = n(r_1, q)$. Iterations of T will be denoted by $T^k(r_1, q) = (r_{k+1}, q)$, or equivalently, as

$$(r_1, q) \overset{n_1}{\to} (r_2, q) \overset{n_2}{\to} \cdots \overset{n_k}{\to} (r_{k+1}, q).$$

The ultimate objective of our efforts is to illuminate the behavior of all sequences $\{T^k(r, q)\}_{k=1}^{\infty}$ of lattice points in $\mathcal{O} \times \mathcal{O}$, although we only discuss cyclic behavior in this paper.

Three possibilities for these sequences naturally arise:

1. *Some successor of* (r, q) *equals* (r, q). A *least* integer $N \geq 1$ exists such that $T^N(r, q) = (r, q)$, in which case we say that the set $\{(r, q), \ldots, T^{N-1}(r, q)\}$ of N lattice points constitutes an *N-cycle in the row of the lattice indexed by q.* This N-cycle can be represented more explicitly as

   $$(r_1, q) \overset{n_1}{\to} (r_2, q) \overset{n_2}{\to} \cdots \overset{n_{N-1}}{\to} (r_N, q) \overset{n_N}{\to} (r_1, q).$$

   We shall also say that this *N-cycle is of index M*, where $M = n_1 + \cdots + n_N$ is the sum of all of the indicial exponents belonging to that cycle, in the order in which they are generated.

2. *Some successor of* (r, q) *equals another successor of* (r, q). Integers N and k exist such that $T^k(r, q) = T^{N+k}(r, q)$, i.e., the sequence eventually enters into an N-cycle.

3. *No successor of* (r, q) *equals* (r, q) *or any other successor of* (r, q). In this case, the sequence is necessarily unbounded.

In particular, if $q = 1$, the special sequence

$$\{T^k(r, 1)\}_{k=1}^{\infty} = \{(S^k(r, 1), 1)\}_{k=1}^{\infty}$$

has been studied extensively by many authors. Since $S(r, 1)$ assumes the form

$$S(r, 1) = \frac{3r + 1}{2^{n(r,1)}},$$

the investigation of this special sequence has been referred to in the literature as the "$3x + 1$" problem. It has also been referred to as the Collatz problem, the Syracuse problem, Kakutani's problem, Ulam's problem, and Hasse's algorithm.

Here, we choose to refer to it as the original Collatz problem, since its inception is traditionally credited to Lothar Collatz in 1937. [3]

The original Collatz conjecture states that, for every odd positive integer r, there exists an integer N_r such that $S^{N_r}(r,1) = 1$. This conjecture can be interpreted in view of the three possibilities listed above.

1. The only N-cycle consisting of odd positive integers that exists in the row of the lattice indexed by $q = 1$ is the 1-cycle of index 2 which can be represented as $(1,1) \xrightarrow{2} (1,1)$. The existence of N-cycles of positive integers with $N \geq 2$ is explicitly excluded.
2. Every Collatz sequence of odd positive integers eventually enters the 1-cycle $(1,1) \xrightarrow{2} (1,1)$.
3. Every Collatz sequence is bounded.

After rephrasing, there are two conjectures of interest.

The Collatz Cycle Conjecture: For the original Collatz problem, there are no N-cycles of index M consisting of odd positive integers, other than the 1-cycle of index 2 which can be represented as $(1,1) \xrightarrow{2} (1,1)$.

The Collatz Entrance Conjecture: For the original Collatz problem, every Collatz sequence of positive integers eventually enters the 1-cycle of index 2 which can be represented as $(1,1) \xrightarrow{2} (1,1)$.

The Collatz Entrance Conjecture excludes the possibility that a Collatz sequence might be unbounded.

For the original Collatz problem, four known cycles have been discovered and are usually represented in the following way:

1. $+1 \to +1$
2. $-1 \to -1$
3. $-5 \to -7 \to -5$
4. $-17 \to -25 \to -37 \to -55 \to -41 \to -61 \to -91 \to -17$

However, these representations shed no light on their genesis.

In this paper, we show that these cycles, where $q = 2^M - 3^N$, are represented alternately as follows:

1. $(1,1) \xrightarrow{2} (1,1)$: This cycle is a 1-cycle of index 2, natural to the row indexed by $q = 2^2 - 3^1 = 1$.

2. $(1,-1) \xrightarrow{1} (1,-1)$: This cycle is a 1-cycle of index 1, natural to the row indexed by $q = 2^1 - 3^1 = -1$.

3. $(5,-1) \xrightarrow{1} (7,-1) \xrightarrow{2} (5,-1)$: This is a 2-cycle of index 3, natural to the row indexed by $q = 2^3 - 3^2 = -1$.

4. $(2363,-139) \xrightarrow{1} (3475,-139) \xrightarrow{1} (5143,-139) \xrightarrow{1} (7645,-139) \xrightarrow{2}$ $(5699,-139) \xrightarrow{1} (8479,-139) \xrightarrow{1} (12649,-139) \xrightarrow{4} (2363,-139)$: This is a 7-cycle of index 11, natural to the row indexed by $q = 2^{11} - 3^7 = -139$. If we divide the cycle elements by -139, we obtain $(-17,1) \xrightarrow{1} (-25,1) \xrightarrow{1}$ $(-37,1) \xrightarrow{1} (-55,1) \xrightarrow{2} (-41,1) \xrightarrow{1} (-61,1) \xrightarrow{1} (-91,1) \xrightarrow{4} (-17,1)$.

The main point here is that there is a significant difference between where a cycle is *created* and where it *exists*. These four cycles were *created* in the rows indexed by $q = 1, -1, -1, \text{and} -139$; by division or multiplication of q and the entries r_i, they can *exist* in infinitely many other rows.

When considering the more general "$3r + q$" problem, we will find that there exist infinitely many N-cycles of some index M, and that at least one cycle can be created in every row indexed by q. Thus, for this more general problem, there is really only one conjecture of interest which excludes unbounded cycles.

The Lattice Entrance Conjecture: For the "$3r + q$" problem, every sequence $\{(r_i, q)\}_{i=1}^{\infty}$ in the q^{th} row of the lattice $\mathcal{O} \times \mathcal{O}$ eventually enters some N-cycle of index M in that row.

A given row indexed by q may have many cycles within it. It would be of interest to determine into which of these cycles a sequence enters and why. For example, in the row indexed by $q = -1$, does a sequence terminate at the fixed point $(1,-1)$, or does it enter the 2-cycle $(5,-1) \xrightarrow{1} (7,-1) \xrightarrow{2} (5,-1)$? If a sequence enters the 2-cycle, does it enter at $(5,-1)$ or at $(7,-1)$, and why?

Since no N-cycles of positive integers ($N \geq 2$) have been observed when $q = 1$, it is rather difficult to formulate or prove any conjectures regarding their theoretical genesis, existence, structure, or behavior. We shall see that there is a distinct advantage in generalizing the original Collatz problem to the "$3r + q$" problem in order to understand these issues more fully. Indeed, it is the author's contention that the original Collatz problem can only be understood in this more general context.

At this point, we introduce the cyclemaster matrix, in order to understand better all Collatz sequences throughout the lattice $\mathcal{O} \times \mathcal{O}$. At first, this matrix may seem to be entirely irrelevant to the present discussion. However, we shall see that it is actually a central idea in the exposition to follow.

For $N \geq 2$, let $\{n_1, \ldots, n_N\}$ denote an ordered set of positive integers, with

$n_1 + \cdots + n_N = M$. If $N = 2$, then the 2×2 cyclemaster matrix is defined as

$$\mathcal{C}_{2,M}(n_1, n_2) = \begin{pmatrix} 1 & 1 \\ 2^{n_1} & 2^{n_2} \end{pmatrix}.$$

Note that $\det(\mathcal{C}_{2,M}(n_1, n_2)) = 0$ if and only if $n_1 = n_2$.

If $N = 3$, then the 3×3 cyclemaster matrix is defined as

$$\mathcal{C}_{3,M}(n_1, n_2, n_3) = \begin{pmatrix} 1 & 1 & 1 \\ 2^{n_1} & 2^{n_2} & 2^{n_3} \\ 2^{n_1+n_2} & 2^{n_2+n_3} & 2^{n_3+n_1} \end{pmatrix}.$$

Note that $\det(\mathcal{C}_{3,M}(n_1, n_2, n_3)) = 0$ if and only if $n_1 = n_2 = n_3$, since

$$\begin{aligned} \det(\mathcal{C}_{3,M}(n_1, n_2, n_3)) &= 3 \cdot 2^{n_1+n_2+n_3} - 2^{n_1+2n_2} - 2^{n_2+2n_3} - 2^{n_3+2n_1} \\ &= 3 \cdot 2^{n_1+n_2+n_3} \left(1 - \frac{2^{n_2-n_3} + 2^{n_3-n_1} + 2^{n_1-n_2}}{3}\right) \\ &\leq 0 \end{aligned}$$

by the Arithmetic-Geometric Mean Inequality.

If $N = 4$, then the 4×4 cyclemaster matrix is defined as

$$\mathcal{C}_{4,M}(n_1, n_2, n_3, n_4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2^{n_1} & 2^{n_2} & 2^{n_3} & 2^{n_4} \\ 2^{n_1+n_2} & 2^{n_2+n_3} & 2^{n_3+n_4} & 2^{n_4+n_1} \\ 2^{n_1+n_2+n_3} & 2^{n_2+n_3+n_4} & 2^{n_3+n_4+n_1} & 2^{n_4+n_1+n_2} \end{pmatrix}.$$

This determinant clearly vanishes if $n_1 = n_2 = n_3 = n_4$. Note that if $n_1 = n_3$ and $n_2 = n_4$, then $\det(\mathcal{C}_{4,M}(n_1, n_2, n_1, n_2)) = 0$, since the 3^{rd} row is a multiple of the 1^{st} row. Thus, periodicity in the set $\{n_1, n_2, n_3, n_4\}$ is significant in this regard.

These three matrices were defined explicitly in order to highlight the pattern evidenced by the exponents n_i. To define $\mathcal{C}_{N,M}(n_1, \ldots, n_N)$ generally, we introduce additional notation for the partial sums of indicial exponents in the order in which they are given. For $j \geq 1$, the *summary exponent* m_{ij} is the sum of j indicial exponents starting with n_i, and is defined formally as

$$m_{ij} = n_i + \cdots + n_{i+j-1} = \sum_{k=0}^{j-1} n_{i+k}.$$

For example, $m_{i1} = n_i$ for each i. If the subscript $i + k > N$, then n_{i+k} is replaced by n_{i+k-N}. Thus, $m_{33} = n_3 + n_4 + n_5$; but if $N = 4$, then $m_{33} = n_3 + n_4 + n_1$, as n_5 is replaced by n_1. Rephrased, if $i + k > N$, then the sum "wraps around" to the beginning. The summary exponents are important because they will also be used to define the elements of N-cycles.

In general, the cyclemaster matrix $\mathcal{C}_{N,M}(n_1,\ldots,n_N)$ is the $N\times N$ matrix whose ij^{th} entry is $C_{ij}=2^{m_{j,i-1}}$. Specifically,

$$\mathcal{C}_{N,M}(n_1,\ldots,n_N)=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2^{m_{11}} & 2^{m_{21}} & 2^{m_{31}} & \ldots & 2^{m_{N1}} \\ 2^{m_{12}} & 2^{m_{22}} & 2^{m_{32}} & \ldots & 2^{m_{N2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{m_{1,N-1}} & 2^{m_{2,N-1}} & 2^{m_{3,N-1}} & \ldots & 2^{m_{N,N-1}} \end{pmatrix}.$$

We have seen above that if $N=2,3,$ or 4, then the determinant of a cyclemaster matrix can vanish. More generally, suppose that the set $\{n_1,\ldots,n_N\}$ has period D, where D properly divides N. Observe that if the subset $\{n_1,\ldots,n_D\}$ is repeated N/D times, then the $(D+1)^{st}$ row of the matrix $\mathcal{C}_{N,M}(n_1,\ldots,n_D,n_{D+1},\ldots,n_N)$ is a scalar multiple of the first row, since every entry in the $(D+1)^{st}$ row is equal to 2^{M_D}, where $M_D=n_1+\cdots+n_D$. Consequently, $\det(\mathcal{C}_{N,M}(n_1,\ldots,n_N))=0$.

Remark. At first glance, the cyclemaster matrix, which is defined here in terms of the composition of an integer M into N summands, appears to have nothing whatsoever to do with the Collatz problem or any generalization of it. However, we shall show that the divisibility of its determinant by 2^M-3^N is significant and relevant in any discussion of the existence of N-cycles of index M for the original Collatz problem. At this point, it is informative and motivational to mention three observations regarding divisibility.

1. If $N=2$, $M=3$, and $\{n_1,n_2\}=\{1,2\}$, then $2^3-3^2=-1$ divides $\det(\mathcal{C}_{2,3}(1,2))=2$. This observation corresponds to the known 2-cycle of index 3 which was noted above.

2. If $N=7$, $M=11$, and $\{n_1,n_2,n_3,n_4,n_5,n_6,n_7\}=\{1,1,1,2,1,1,4\}$, then $2^{11}-3^7=-139$ divides $\det(\mathcal{C}_{7,11}(1,1,1,2,1,1,4))=-2^{25}\cdot 5\cdot 79\cdot 139$. This observation corresponds to the known 7-cycle of index 11 that was noted above.

3. If $N=6$, $M=12$, and $\{n_1,n_2,n_3,n_4,n_5,n_6\}=\{1,1,1,6,1,2\}$, then $2^{12}-3^6=3367=7\cdot 13\cdot 37$ divides $\det(\mathcal{C}_{6,12}(1,1,1,6,1,2))=2^{17}\cdot 3^2\cdot 7\cdot 13^2\cdot 37^2$. This observation, due to Leonard K. Jones, does correspond to a 6-cycle of index 12, but does not correspond to a cycle for the original Collatz problem.

The main theorem of this paper, Theorem 1, explains the significance and relevance of these observations.

In Section 2, we discuss the genesis and structure of all N-cycles of index M in the lattice $\mathcal{O}\times\mathcal{O}$. We indicate how the cyclemaster matrix naturally arises from the generation of N-cycles and we establish a key divisibility property of the determinant of a cyclemaster matrix, namely that the gcd of the cycle elements r_i always divides this determinant.

In Section 3, we show how to construct all natural N-cycles of index M in the lattice $\mathcal{O} \times \mathcal{O}$, and then we prove the following result which is the main theorem in this paper. It illustrates a strong and direct connection between the existence of N-cycles for the original Collatz problem and the divisibility properties of the determinant of the cyclemaster matrix.

Theorem 1. *With $q = 2^M - 3^N$, let $\{(r_i, q)\}_{i=1}^N$ denote an N-cycle of index M with the associated indicial exponents $\{n_1, \ldots, n_N\}$ in the row indexed by q and let $\rho_N = \gcd\{r_1, \ldots, r_N\}$. Then the following statements are equivalent:*

1. *$\rho_N = q$;*
2. *each r_i/q is an integer;*
3. *$\{(r_i/q, 1)\}_{i=1}^N$ constitutes a completely reduced N-cycle of index M with the same indicial exponents.*

Furthermore, if these statements hold, then the cycle index $q = 2^M - 3^N$ divides the determinant $\det(\mathcal{C}_{N,M}(n_1, \ldots, n_N))$, but the converse does not hold.

We also introduce extremal N-cycles and discuss their properties. In a sense, an extremal N-cycle in a row will "bracket" all of the other N-cycles in that row. The divisibility properties of q, r_i, and the determinant of the cyclemaster matrix are treated. Exponential congruences, primitive roots, Mersenne primes, and the Beal Conjecture arise in an interesting way when discussing these properties.

In Section 4, we show how the cyclemaster matrix is a unifying concept for all of the "$ax + 1$" problems. In particular, we show how it is related to the computation of the known cycles for the "$5x + 1$" problem and the "$181x + 1$" problem.

In Section 5, we discuss the variable cyclemaster matrix, which is defined by replacing 2^{n_i} with the variable x_i, and show how the "$ax + 1$" problems are related to the theory of algebraic varieties. This will be established by showing that the existence of N-cycles implies the existence of solutions to polynomial equations or systems of polynomial equations in N variables.

In Section 6, we present some directions for further research.

2. The Genesis and Structure of N-cycles of Index M

Recall that an N-cycle of index M in the q^{th} row of the lattice $\mathcal{O} \times \mathcal{O}$ consists of N lattice points with the representation

$$(r_1, q) \xrightarrow{n_1} (r_2, q) \xrightarrow{n_2} \cdots \xrightarrow{n_{N-1}} (r_N, q) \xrightarrow{n_N} (r_1, q).$$

Lagarias [4] and other authors have previously discussed the structure of N-cycles. It is necessary to briefly discuss it again here in order to highight the genesis of the cyclemaster matrix and one of its main properties.

2.1. The Genesis and Structure of 1-Cycles of Index M

Theorem 2. *Let $(r, q) \in \mathcal{O} \times \mathcal{O}$. The point (r, q) constitutes a 1-cycle of index M if and only if there exists an integer $M \geq 1$ such that $\left(2^M - 3\right) r = q$.*

Proof. If (r, q) constitutes a 1-cycle of index M, then

$$(r, q) \stackrel{M}{\to} (r, q)$$

for some integer $M \geq 1$. Equivalently, we have

$$r = S(r, q) = \frac{3r + q}{2^M},$$

from which we obtain the necessary condition $\left(2^M - 3\right) r = q$.

This condition is also sufficient. For if the condition holds, then

$$S(r, q) = S(r, 2^M - 3) = \frac{3r + (2^M - 3)r}{2^M} = r. \qquad \square$$

Remark. A 1-cycle for T is also known as a *fixed point of* T. For example, if $M = 1$, then $q = -r$, so that $(r, -r)$ is a fixed point of T. If $M = 2$, then $q = r$, so that (r, r) is also a fixed point of T. In general, $\left(r, (2^M - 3)r\right)$ is a fixed point of T. A row may contain more than two fixed points. For example, $(-65, 65)$, $(5, 65)$, $(13, 65)$, and $(65, 65)$ are all fixed points in the 65^{th} row of the lattice.

2.2. The Genesis and Structure of 2-Cycles of Index M

The discussion continues with the following result.

Theorem 3. *Let $(r, q) \in \mathcal{O} \times \mathcal{O}$. Then (r_i, q) belongs to a 2-cycle of index M if and only if there exist integers M and m_1, with $M > m_1 \geq 1$, such that*

$$\left(2^M - 3^2\right) r = \left(2^{m_1} + 3\right) q.$$

Proof. If (r_1, q) and (r_2, q) belong to a 2-cycle of index M, then

$$(r_1, q) \stackrel{n_1}{\to} (r_2, q) \stackrel{n_2}{\to} (r_1, q),$$

where $M = n_1 + n_2$. Equivalently, we have the equations

$$r_2 = \frac{3r_1 + q}{2^{n_1}} \qquad \text{and} \qquad r_1 = \frac{3r_2 + q}{2^{n_2}}.$$

Eliminating r_2 from these equations yields

$$\left(2^{n_1+n_2} - 3^2\right) r_1 = \left(2^{n_1} + 3\right) q, \tag{2.1}$$

while eliminating r_1 from these equations yields

$$\left(2^{n_1+n_2} - 3^2\right) r_2 = \left(2^{n_2} + 3\right) q. \tag{2.2}$$

In these two equations, $m_1 = n_1$ or n_2 and $M = n_1 + n_2$. (Note that if $n_1 = n_2$, then $r_1 = r_2$, so that this 2-cycle is actually a repeated 1-cycle.)

Conversely, we must use the necessary condition given above to define the initial exponents n_i, for $i = 1$ and 2, in order to ensure that $S(r_i, q)$ is an odd integer. If we set $n_1 = m_1$, then

$$S(r_1, q) = \frac{3r_1 + q}{2^{n_i}} = \frac{3^2 r_1 + 3q}{3 \cdot 2^{m_i}} = \frac{2^M r_i - 2^{m_i} q}{3 \cdot 2^{m_i}} = \frac{2^{M-m_i} r_1 - q}{3},$$

where the necessary condition stated above is used here in the third equality. After cross multiplication, we have

$$3(3r_1 + q) = 2^{n_i}(2^{M-m_1} r_1 - q),$$

which implies that 3 divides the odd integer $2^{M-m_i} r_1 - q$. Thus, $r_2 = S(r_1, q)$ is an odd integer and we may set

$$r_2 = \frac{2^{M-m_1} r_1 - q}{3},$$

or equivalently,

$$r_1 = \frac{3r_2 + q}{2^{M-m_1}}.$$

Since r_1 is an odd integer, we define $n_2 = M - m_1 = M - n_1$. Explicitly, for $i = 1$ or 2, we have

$$S^2(r_i, q) = S(S(r_i, q), q) = \frac{3^2 r_i + (2^{n_i} + 3)q}{2^{n_1+n_2}} = \frac{3^2 r_i + (2^M - 3^2) r_i}{2^M} = r_i,$$

showing that (r_i, q) belongs to a 2-cycle of index M. □

Remark. It is advantageous here to introduce some definitions for the quantities that appeared in the proof of this theorem.

It is clear from equations (2.1) and (2.2) that every 2-cycle is determined by q and the indicial exponents n_1 and n_2. These essential equations are called the *cycle conditions of order* 2.

The integers $g_{2i} = 2^{n_i} + 3$ are called the *cycle generators of order* 2.

If we define [2]

$$\rho_2 = \gcd\{r_1, r_2\} \qquad \text{and} \qquad \gamma_2 = \gcd\{g_{21}, g_{22}\},$$

then the cycle conditions imply that

$$\left(2^M - 3^2\right)\rho_2 = \gamma_2 q,$$

where $M = n_1 + n_2$. This relation is called the *gcd condition of order* 2. Note that if $q = 2^M - 3^2$, then $\rho_2 = \gamma_2$.

Theorem 4. *If $q = 2^M - 3^2$, then γ_2 divides the determinant of the cyclemaster matrix $\mathcal{C}_{2,M}(n_1, n_2)$, where $M = n_1 + n_2$.*

Proof. If $q = 2^M - 3^2$, then the cycle conditions above can be recast explicitly in matrix form as

$$\begin{pmatrix} r_1 & r_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2^{n_1} & 2^{n_2} \end{pmatrix},$$

or more compactly as

$$\vec{\mathbf{r}}_2 = \vec{\mathbf{t}}_2\, \mathcal{C}_{2,M}(n_1, n_2),$$

where obvious assignments have been made. The matrix $\mathcal{C}_{2,M}(n_1, n_2)$ here is the 2×2 cyclemaster matrix that was defined in Section 1.

If $\det\left(\mathcal{C}_{2,M}(n_1, n_2)\right) = 0$, then it is trivially divisible by γ_2.

If $\det\left(\mathcal{C}_{2,M}(n_1, n_2)\right) \neq 0$, then $\mathcal{C}_{2,M}(n_1, n_2)$ is invertible, so that

$$\vec{\mathbf{r}}_2\, \mathcal{C}_{2,M}^{-1}(n_1, n_2) = \vec{\mathbf{t}}_2.$$

In terms of its adjoint and determinant, this equation becomes

$$\vec{\mathbf{r}}_2\, \mathrm{adj}\left(\mathcal{C}_{2,M}(n_1, n_2)\right) = \vec{\mathbf{t}}_2\, \det\left(\mathcal{C}_{2,M}(n_1, n_2)\right),$$

or more explicitly,

$$\begin{pmatrix} r_1 & r_2 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix} \det\left(\mathcal{C}_{2,M}(n_1, n_2)\right).$$

By equating the second entries on both sides of this equation, we get

$$r_1 c_{12} + r_2 c_{22} = \det\left(\mathcal{C}_{2,M}(n_1, n_2)\right).$$

Since $r_i = \rho_2\, h_i$ for $i = 1, 2$, this equation can be recast as

$$\rho_2\, \left(h_1 c_{12} + h_2 c_{22}\right) = \det\left(\mathcal{C}_{2,M}(n_1, n_2)\right).$$

If $q = 2^M - 3^2$, then $\rho_2 = \gamma_2$. Thus, $\rho_2 = \gamma_2$ divides $\det\left(\mathcal{C}_{2,M}(n_1, n_2)\right)$. $\square$

[2]Generally, the gcd is only defined for a set of positive integers and is therefore positive. Since the cycle elements r_i of an N-cycle are either all positive or all negative, it is reasonable to extend the definition of the gcd to be negative if all cycle elements are negative. Thus, ρ is positive if all of the cycle elements are positive and ρ is negative if all of the cycle elements are negative.

2.3. The Genesis and Structure of 3-Cycles of Index M

The discussion continues with the following result.

Theorem 5. *Let $(r, q) \in \mathcal{O} \times \mathcal{O}$. Then (r_i, q) belongs to a 3-cycle of index M if and only if there exist integers $m_3 > m_2 > m_1 \geq 1$ such that*

$$\left(2^{m_3} - 3^3\right) r = \left(2^{m_2} + 2^{m_1} \cdot 3 + 3^2\right) q. \tag{2.3}$$

Proof. If (r_1, q), (r_2, q), and (r_3, q) belong to a 3-cycle of index M, then

$$(r_1, q) \overset{n_1}{\to} (r_2, q) \overset{n_2}{\to} (r_3, q) \overset{n_3}{\to} (r_1, q),$$

where $M = n_1 + n_2 + n_3$. Equivalently, we have the equations

$$r_2 = \frac{3r_1 + q}{2^{n_1}}, \qquad r_3 = \frac{3r_2 + q}{2^{n_2}} \qquad \text{and} \qquad r_1 = \frac{3r_3 + q}{2^{n_3}}.$$

Eliminating r_2 and r_3 from these equations yields

$$\left(2^{n_1+n_2+n_3} - 3^3\right) r_1 = \left(2^{n_1+n_2} + 2^{n_1} \cdot 3 + 3^2\right) q. \tag{2.4}$$

Similarly, we obtain

$$\left(2^{n_2+n_3+n_1} - 3^3\right) r_2 = \left(2^{n_2+n_3} + 2^{n_2} \cdot 3 + 3^2\right) q \tag{2.5}$$

and

$$\left(2^{n_3+n_1+n_2} - 3^3\right) r_3 = \left(2^{n_3+n_1} + 2^{n_3} \cdot 3 + 3^2\right) q. \tag{2.6}$$

By employing the notation for summary exponents which were defined earlier, all three of these equations follow the prescription

$$\left(2^{m_{i3}} - 3^3\right) r_i = \left(2^{m_{i2}} + 2^{m_{i1}} \cdot 3 + 3^2\right) q, \tag{2.7}$$

and $m_{i3} > m_{i2} > m_{i1} \geq 1$ in every case. (Note that if $n_1 = n_2 = n_3$, then $r_1 = r_2 = r_3$, so that this 3-cycle is actually a repeated 1-cycle.)

Conversely, suppose that the condition holds. If we define $n_i = m_{i1} = m_1$, $n_i + n_{i+1} = m_{i2} = m_2$ and $n_i + n_{i+1} + n_{i+2} = m_{i3} = m_3$, then equation (2.3) becomes equation (2.7) for each $i = 1, 2, 3$. A short argument similar to the one in the proof of Theorem 3 can be constructed to show that $S^k(r_i, q)$ is an odd integer for each value of k.

If $i = 1$, then

$$\begin{aligned} S^3(r_1, q) &= \frac{3^3\, r_1 + \left(2^{n_1+n_2} + 2^{n_1} \cdot 3 + 3^2\right) q}{2^{n_1+n_2+n_3}} \\ &= \frac{3^3\, r_1 + \left(2^{n_1+n_2+n_3} - 3^3\right) r_1}{2^{n_1+n_2+n_3}} \\ &= r_1, \end{aligned} \tag{2.8}$$

which shows that (r_1, q) belongs to a 3-cycle of index $M = m_{13} = m_3$. Similar calculations show that (r_2, q) and (r_3, q) belong to the same 3-cycle of index M. $\square$

Remark. Some additional definitions are appropriate.

It should be clear from equation (2.7) that every element of every 3-cycle is completely determined by q and the indicial exponents n_1, n_2, and n_3. These essential equations are called the *cycle conditions of order* 3.

The quantities

$$g_{3i} = 2^{m_{i2}} + 2^{m_{i1}} \cdot 3 + 3^2$$

will be called *cycle generators of order* 3.

If we define

$$\rho_3 = \gcd\{r_1, r_2, r_3\} \qquad \text{and} \qquad \gamma_3 = \gcd\{g_{31}, g_{32}, g_{33}\},$$

then the cycle conditions imply that

$$\left(2^M - 3^3\right) \rho_3 = \gamma_3 q,$$

where $M = n_1 + n_2 + n_3$. This relation is called the *gcd condition of order* 3. Note that if $q = 2^M - 3^3$, then $\rho_3 = \gamma_3$.

Theorem 6. *If $q = 2^M - 3^3$, then γ_3 divides $\det\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right)$, where $M = n_1 + n_2 + n_3$.*

Proof. If $q = 2^M - 3^3$, then the cycle conditions (2.4), (2.5), and (2.6) can be explicitly recast in matrix form as

$$\begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix} = \begin{pmatrix} 3^2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2^{n_1} & 2^{n_2} & 2^{n_3} \\ 2^{n_1+n_2} & 2^{n_2+n_3} & 2^{n_3+n_1} \end{pmatrix},$$

or more compactly as

$$\vec{\mathbf{r}}_3 = \vec{\mathbf{t}}_3 \, \mathcal{C}_{3,M}(n_1, n_2, n_3),$$

where obvious assignments have been made. The matrix $\mathcal{C}_{3,M}(n_1, n_2, n_3)$ here is the 3×3 cyclemaster matrix that was defined in Section 1.

If $\det\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right) = 0$, then it is trivially divisible by γ_3.

If $\det\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right) \neq 0$, then $\mathcal{C}_{3,M}(n_1, n_2, n_3)$ is invertible, so that

$$\vec{\mathbf{r}}_3 \, \mathcal{C}_{3,M}^{-1}(n_1, n_2, n_3) = \vec{\mathbf{t}}_3.$$

In terms of its adjoint and determinant, this equation becomes

$$\vec{\mathbf{r}}_3 \operatorname{adj}\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right) = \vec{\mathbf{t}}_3 \, \det\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right),$$

or more explicitly,

$$\begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 3^2 & 3 & 1 \end{pmatrix} \det\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right).$$

By equating the third entries on both sides of this equation, we get

$$r_1c_{13} + r_2c_{23} + r_3c_{33} = \det\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right).$$

Since $r_i = \rho_3\, h_i$ for $i = 1, 2, 3$, this equation can be recast as

$$\rho_3\ \left(h_1c_{13} + h_2c_{23} + h_3c_{33}\right) = \det\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right).$$

If $q = 2^M - 3^2$, then $\rho_3 = \gamma_3$. Thus, $\rho_3 = \gamma_3$ divides $\det\left(\mathcal{C}_{3,M}(n_1, n_2, n_3)\right)$. □

2.4. The Genesis and Structure of N-Cycles of Index M

We continue the discussion with the following general result.

Theorem 7. *Let $(r, q) \in \mathcal{O} \times \mathcal{O}$. Then (r_i, q) belongs to an N-cycle of index M if and only if there exist integers $m_N > \cdots > m_1 \geq 1$ with $m_0 = 0$ such that*

$$\left(2^{m_N} - 3^N\right) r = \left(\sum_{j=0}^{N-1} 2^{m_{N-j-1}} \cdot 3^j\right) q. \tag{2.9}$$

Proof. Without loss of generality, we may assume that the cycle element $(r, q) = (r_1, q)$ in the N-cycle

$$(r_1, q) \overset{n_1}{\to} (r_2, q) \overset{n_2}{\to} \cdots \overset{n_{N-1}}{\to} (r_N, q) \overset{n_N}{\to} (r_1, q),$$

where $M = n_1 + \cdots + n_N$. At each step of the iterative process, an equation of the form

$$2^{n_i}\, r_{i+1} = 3r_i + q$$

holds, with the convention that $r_{N+1} = r_1$. If we eliminate all variables except r_i from these N equations, we obtain

$$\left(2^{m_{iN}} - 3^N\right)\ r_i = \left(\sum_{j=0}^{N-1} 2^{m_{i,N-j-1}} \cdot 3^j\right) q, \tag{2.10}$$

where we have used the summary exponents m_{ik} to simplify the notation. If we set $m_k = m_{ik}$, then the sequence $\{m_k\}$ is increasing, and every lattice point (r_i, q) satisfies a cycle condition of the required form, where $m_{iN} = m_N = M$.

Conversely, suppose that equation (2.9) holds. Choose i and define $m_{ik} = m_k$ for all $k = 0, 1, \ldots, N$. Since m_{ik} is increasing, the indicial exponents for the N-cycle beginning with (r_i, q) can be defined by the relation $n_{i+k} = m_{i,k+1} - m_{i,k}$ for $k = 0, 1, \ldots, N - 1$. An argument similar to the one in the proof of Theorem 3 can be constructed to show that $S^k(r_i, q)$ is an odd integer for each value of k. After repeated iteration of the Syracuse function S, we observe that

$$\begin{aligned} S^N(r_i, q) &= \frac{3^N\, r_i + \left(\sum_{j=0}^{N-1} 2^{m_{i,N-j-1}} \cdot 3^j\right) q}{2^{m_N}} \\ &= \frac{3^N\, r_i + \left(2^{m_N} - 3^N\right) r_i}{2^{m_N}} \\ &= r_i, \end{aligned} \tag{2.11}$$

which shows that (r_i, q) belongs to an N-cycle of index $M = m_N$ for every i. □

Remark. Some additional definitions naturally generalize previously presented definitions.

It should be clear from the equation (2.9) that the elements of every N-cycle are completely determined by q and the indicial exponents $\{n_1, \ldots, n_N\}$. These essential equations are called the *cycle conditions of order* N.

The quantities

$$g_{Ni} = \sum_{j=0}^{N-1} 2^{m_{i,N-j-1}} \cdot 3^j$$

will be called *cycle generators of order* N. With this notation, equations (2.9) can be written in the simplified form

$$\left(2^M - 3^N\right) r_i = g_{Ni}\, q$$

for each i, where $M = m_N$.

If we define

$$\rho_N = \gcd\{r_1, \ldots, r_N\} \qquad \text{and} \qquad \gamma_N = \gcd\{g_{N1}, \ldots, g_{NN}\},$$

it follows that

$$\left(2^M - 3^N\right) \rho_N = \gamma_N\, q.$$

This condition is called the *gcd condition for the* N*-cycle of index* M.

Equations (2.10) can be rewritten in the form

$$r_i = q\, \frac{g_{Ni}}{2^M - 3^N},$$

from which we conclude that r_i belongs to an N-cycle of index M if and only if $q\, g_{Ni}$ is divisible by $2^M - 3^N$ for $i = 1, \ldots, N$. From this observation, Kaneda [2, pages 172-173] noted that the cycle conjectures for the "$3x + 1$" and "$3x + q$" problems can be interpreted as purely arithmetic conjectures involving divisibility, rather than as algorithmic conjectures.

Theorem 8. *If* $q = 2^M - 3^N$, *then* γ_N *divides* $\det(\mathcal{C}_{N,M}(n_1, \dots, n_N))$, *where* $M = n_1 + \cdots + n_N$.

Proof. If $q = 2^M - 3^N$, then the cycle conditions (2.10) for $i = 1, \dots, N$ can be recast in the form

$$\begin{aligned} &\begin{pmatrix} r_1 & r_2 & \cdots & r_N \end{pmatrix} \\ &= \begin{pmatrix} 3^{N-1} & 3^{N-2} & \cdots & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2^{m_{11}} & 2^{m_{21}} & 2^{m_{31}} & \cdots & 2^{m_{N1}} \\ 2^{m_{12}} & 2^{m_{22}} & 2^{m_{32}} & \cdots & 2^{m_{N2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{m_{1,N-1}} & 2^{m_{2,N-1}} & 2^{m_{3,N-1}} & \cdots & 2^{m_{N,N-1}} \end{pmatrix}, \end{aligned}$$

or more concisely as

$$\vec{\mathbf{r}}_N = \vec{\mathbf{t}}_N\, \mathcal{C}_{N,M}(n_1, \dots, n_N),$$

where obvious assignments have been made. The matrix $\mathcal{C}_{N,M}(n_1, \dots, n_N)$ here is the $N \times N$ cyclemaster matrix that was defined in Section 1.

If $\det(\mathcal{C}_{N,M}(n_1, \dots, n_N)) = 0$, then it is trivially divisible by γ_N.

If $\det(\mathcal{C}_{N,M}(n_1, \dots, n_N)) \neq 0$, then $\mathcal{C}_{N,M}(n_1, \dots, n_N)$ is invertible, so that

$$\vec{\mathbf{r}}_N\, \mathcal{C}^{-1}_{N,M}(n_1, \dots, n_N) = \vec{\mathbf{t}}_N.$$

In terms of its adjoint and determinant, this equation becomes

$$\vec{\mathbf{r}}_N \operatorname{adj}(\mathcal{C}_{N,M}(n_1, \dots, n_N)) = \vec{\mathbf{t}}_N \det(\mathcal{C}_{N,M}(n_1, \dots, n_N)),$$

or more explicitly, as

$$\begin{aligned} &\begin{pmatrix} r_1 & r_2 & \dots & r_N \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1N} \\ c_{21} & c_{22} & \dots & c_{2N} \\ c_{31} & c_{32} & \dots & c_{3N} \\ \dots & \dots & \ddots & \dots \\ c_{N1} & c_{N2} & \dots & c_{NN} \end{pmatrix} \\ &= \begin{pmatrix} 3^{N-1} & 3^{N-2} & \dots & 1 \end{pmatrix} \det(\mathcal{C}_{N,M}(n_1, \dots, n_N)). \end{aligned}$$

By equating the last entries on both sides of this equation, we get

$$\sum_{i=1}^{N} r_i c_{iN} = \det(\mathcal{C}_{N,M}(n_1, \dots, n_N)).$$

Since $r_i = \rho_N\, h_i$ for $i = 1, \dots, N$, this equation can be recast as

$$\rho_N \left(\sum_{i=1}^{N} h_i c_{iN} \right) = \det(\mathcal{C}_{N,M}(n_1, \dots, n_N)).$$

If $q = 2^M - 3^N$, then $\rho_N = \gamma_N$. Thus, $\rho_N = \gamma_N$ divides $\det(\mathcal{C}_{N,M}(n_1, \dots, n_N))$. $\square$

3. The Construction of All N-Cycles of Index M

In the previous section, we established the gcd condition

$$\left(2^M - 3^N\right) \rho_N = \gamma_N \, q,$$

where

$$\rho_N = \gcd\{r_1, \ldots, r_N\},$$

$$\gamma_N = \gcd\{g_{N1}, \ldots, g_{NN}\},$$

and

$$g_{Ni} = \sum_{j=0}^{N-1} 2^{m_{i,N-j-1}} \cdot 3^j.$$

The cycle generators g_{Ni} can be used to create all N-cycles of index M in the lattice $\mathcal{O} \times \mathcal{O}$.

3.1. Natural Cycles

A *natural N-cycle of index M* is an N-cycle of index M for which $q = 2^M - 3^N$. For natural N-cycles, $r_i = g_{Ni}$.

All natural N-cycles can be constructed by means of the following four-step process:

1. *Compute q:* Choose the cycle length N and the cycle index $M \geq N$, and then set $q = 2^M - 3^N$.

2. *Specify the Indicial Exponents:* Choose an ordered set $\{n_1, \ldots, n_N\}$ of N positive integers for which $n_1 + \cdots + n_N = M$.

3. *Compute the Summary Exponents:* Compute the necessary values of m_{ij} from the previously chosen indicial exponents.

4. *Compute the Cycle Generators:* Compute the values of g_{Ni} in terms of the summary exponents according to the prescription given in Section 2.

Illustrative Example. Suppose that we wish to construct a natural 3-cycle of index 11. This can be done in four steps:

1. With $N = 3$ and $M = 11$, the value of q will be

$$q = 2^{11} - 3^3 = 2048 - 27 = 2021 = 43 \times 47.$$

2. Choose three integers to serve as indicial exponents whose sum is 11. In this example, we choose $n_1 = 1$, $n_2 = 4$, and $n_3 = 6$.

3. Based upon this choice, we easily compute the summary exponents to be $m_{11} = 1$, $m_{12} = 5$, $m_{21} = 4$, $m_{22} = 10$, $m_{31} = 6$, and $m_{32} = 7$.

4. By following the prescriptions given in Section 2, compute

$$r_1 = 2^5 + 2^1 \cdot 3^1 + 3^2 = 47 = 1 \times 47$$
$$r_2 = 2^{10} + 2^4 \cdot 3^1 + 3^2 = 1081 = 23 \times 47$$
$$r_3 = 2^7 + 2^6 \cdot 3^1 + 3^2 = 329 = 7 \times 47.$$

The construction of the desired 3-cycle of index 11 is now complete and is given by

$$(47, 2021) \xrightarrow{1} (1081, 2021) \xrightarrow{4} (329, 2021) \xrightarrow{6} (47, 2021).$$

Note that by canceling a factor of 47 throughout the computed 3-cycle, we obtain

$$(1, 43) \xrightarrow{1} (23, 43) \xrightarrow{4} (7, 43) \xrightarrow{6} (1, 43),$$

which exists in the 43^{rd} row of the lattice $\mathcal{O} \times \mathcal{O}$. By Theorem 13, it will be shown that $43 \neq 2^m - 3^n$ for any admissible choices of m and n, so that this reduced 3-cycle is not, by definition, natural.

All possible 3-cycles are given here, arranged by increasing value of r_1.

$$(19, 2021) \xrightarrow{1} (1039, 2021) \xrightarrow{1} (2569, 2021) \xrightarrow{9} (19, 2021)$$
$$(23, 2021) \xrightarrow{1} (1045, 2021) \xrightarrow{2} (1289, 2021) \xrightarrow{8} (23, 2021)$$
$$(29, 2021) \xrightarrow{2} (527, 2021) \xrightarrow{1} (1801, 2021) \xrightarrow{8} (29, 2021)$$
$$(31, 2021) \xrightarrow{1} (1057, 2021) \xrightarrow{3} (649, 2021) \xrightarrow{7} (31, 2021)$$
$$(37, 2021) \xrightarrow{2} (533, 2021) \xrightarrow{2} (905, 2021) \xrightarrow{7} (37, 2021)$$
$$(47, 2021) \xrightarrow{1} (1081, 2021) \xrightarrow{4} (329, 2021) \xrightarrow{6} (47, 2021)$$
$$(49, 2021) \xrightarrow{3} (271, 2021) \xrightarrow{1} (1417, 2021) \xrightarrow{7} (49, 2021)$$
$$(53, 2021) \xrightarrow{2} (545, 2021) \xrightarrow{3} (457, 2021) \xrightarrow{6} (53, 2021)$$
$$(65, 2021) \xrightarrow{3} (277, 2021) \xrightarrow{2} (713, 2021) \xrightarrow{6} (65, 2021)$$
$$(79, 2021) \xrightarrow{1} (1129, 2021) \xrightarrow{5} (169, 2021) \xrightarrow{5} (79, 2021)$$
$$(85, 2021) \xrightarrow{2} (569, 2021) \xrightarrow{4} (233, 2021) \xrightarrow{5} (85, 2021)$$
$$(89, 2021) \xrightarrow{4} (143, 2021) \xrightarrow{1} (1225, 2021) \xrightarrow{6} (89, 2021)$$
$$(97, 2021) \xrightarrow{3} (289, 2021) \xrightarrow{3} (361, 2021) \xrightarrow{5} (97, 2021)$$
$$(121, 2021) \xrightarrow{4} (149, 2021) \xrightarrow{2} (617, 2021) \xrightarrow{5} (121, 2021)$$
$$(161, 2021) \xrightarrow{3} (313, 2021) \xrightarrow{4} (185, 2021) \xrightarrow{4} (161, 2021)$$

It was previously shown in Theorem 6 that the cycle gcd γ_3 always divides the determinant of the cyclemaster matrix. In this example, $\gamma_3 = 47$ and

$$\det\left(\mathcal{C}_{3,11}(1,4,6)\right) = \det\begin{pmatrix} 1 & 1 & 1 \\ 2^1 & 2^4 & 2^6 \\ 2^5 & 2^{10} & 2^7 \end{pmatrix} = -2^8 \cdot 5 \cdot 47,$$

showing that 47 is a factor of the determinant of the cyclemaster matrix.

Since there are

$$\binom{11-1}{3-1} = \binom{10}{2} = 45$$

solutions to the equation $n_1 + n_2 + n_3 = 11$ in positive integers and three of these are used to form a 3 cycle of index 11, a total of 15 such cycles can be constructed, so that the list given above is complete.

Remark. In general, prime numbers need not appear in N-cycles. However, it is curious to note in this example that at least one cycle element in each of the 14 additional 3-cycles is a prime number. Note that the smallest cycle element 19 is prime, the largest cycle element is $2569 = 7 \times 367$ is not prime, and that they both appear in the same 3-cycle.

3.2. Scalar Multiplication and Scalar Division

These two processes will permit the production of infinitely many N-cycles.

Let $\{(r_i, q) \colon i = 1, \ldots, N\}$ be an N-cycle of index M whose indicial exponents are given as $\{n_1, \ldots, n_N\}$. If a is an odd integer, then the N-cycle of index M with the same indicial exponents with the representation $\{(ar_i, aq) \colon i = 1, \ldots, N\}$ will be termed a *scalar multiple* of the given N-cycle. The process of multiplying each r_i and q by a will be called *scalar multiplication*.

Analogously, if b is an integer and each cycle element r_i/b is also an integer, then the N-cycle of index M with the same indicial exponents with the representation $\{(r_i/b, q/b) \colon i = 1, \ldots, N\}$ will be termed a *scalar quotient* of the given N-cycle. The process of dividing each r_i and q by the integer b, when possible, will be called *scalar division*. If scalar division is possible, the cycle will be called *reducible*, and when it is not possible the cycle will be called *irreducible*. If $b = q$, then the cycle is *completely reducible*.

Remark. The indicial exponents are invariant under these processes.

The N-cycle $\{(r_i, q)\}_{i=1}^N$ is always subject to scalar division by ρ_N. For if ρ_N divides r_i and r_{i+1}, then it divides q as well, since $2^{n_i} r_{i+1} = 3r_i + q$. Thus, the N-cycle $\{(r_i/\rho_N, q/\rho_N)\}_{i=1}^N$ with the same index M and same indicial exponents will belong to the row indexed by q/ρ_N. If $\rho_N = q$, then the N-cycle is completely reducible.

Theorem 9. *Every N-cycle of index M in the lattice $\mathcal{O} \times \mathcal{O}$ is either a natural N-cycle of index M or an N-cycle of index M obtained from a natural N-cycle of index M by scalar multiplication, scalar division, or both processes.*

Proof. Let $\{(r_i, q) \colon i = 1, \ldots, N\}$ be an N-cycle of index M in the q^{th} row of the lattice $\mathcal{O} \times \mathcal{O}$. The cycle generators g_{Ni} satisfy equations of the form

$$\left(2^M - 3^N\right) r_i = g_{Ni}\, q,$$

as shown in Section 2. Thus, q divides $\left(2^M - 3^N\right) r_i$ for each $i = 1, \ldots, N$.

If $q = 2^M - 3^N$, then the given cycle is natural by definition.

If $q \neq 2^M - 3^N$, then we may factor q as $q = q_1 q_2$, where q_1 divides $2^M - 3^N$ and q_2 divides r_i for all i. Let $a\, q_1 = 2^M - 3^N$ and $r_i = q_2\, b_i$, so that $(r_i, q) = (r_i, q_1\, q_2)$. After scalar multiplication by a, we get

$$(a\, r_i, a\, q_1\, q_2) = \left(a\, r_i, \left(2^M - 3^N\right) q_2\right).$$

After scalar division by q_2, we get

$$\left(a\, r_i / q_2, 2^M - 3^N\right) = \left(a\, b_i, 2^M - 3^N\right),$$

which shows that the given N-cycle was obtained from a natural N-cycle by scalar multiplication and division. □

The next result is the main theorem in this paper. It illustrates a strong and direct connection between the existence of N-cycles for the original Collatz problem and the divisibility properties of the determinant of the cyclemaster matrix.

Theorem 10. *With $q = 2^M - 3^N$, let $\{(r_i, q)\}_{i=1}^N$ denote an N-cycle of index M with the associated indicial exponents $\{n_1, \ldots, n_N\}$ in the row indexed by q and let $\rho_N = \gcd\{r_1, \ldots, r_N\}$. Then the following statements are equivalent:*

1. *$\rho_N = q$;*
2. *each r_i/q is an integer;*
3. *$\{(r_i/q, 1)\}_{i=1}^N$ constitutes a completely reduced N-cycle of index M with the same indicial exponents.*

Furthermore, if these statements hold, then the cycle index $q = 2^M - 3^N$ divides the determinant $\det(\mathcal{C}_{N,M}(n_1, \ldots, n_N))$, but the converse does not hold.

Proof. Given the definitions of the quantities involved, the equivalence statement is clear.

Suppose that the natural N-cycle $\{(r_i, q)\}_{i=1}^N$ of index M with its associated set of indicial exponents $\{n_1, \ldots, n_N\}$ can be completely reduced by scalar division to produce the N-cycle $\{(r_i/q,1)\}_{i=1}^N$ of index M with the same associated set of indicial exponents. Then, $\rho_N = q = 2^M - 3^N$. After noting that the gcd condition $\left(2^M - 3^N\right)\rho_N = \gamma_N\, q$ now implies that $\rho_N = \gamma_N$, the conclusion of the theorem follows by invoking Theorem 8, which states that $\gamma_N = \rho_N = 2^M - 3^N$ always divides the determinant $\det\left(C_{N,M}(n_1,\ldots,n_N)\right)$. The counterexample provided in Section 1 with $N = 6$, $M = 12$, and $\{n_1,\ldots,n_6\} = \{1,1,1,6,1,2\}$ shows that the converse does not hold. $\square$

Remark. If $2^M - 3^N$ does not divide the determinant of the cyclemaster matrix for every choice of indicial exponents, then no N-cycle of index M in the row of the lattice indexed by q can be reduced by scalar division to an N-cycle of index M in the row of the lattice indexed by 1. Thus, there will be no N-cycles of index M for the original Collatz problem.

3.3. Periodic Indicial Exponents

In this section, we will show that natural N-cycles of index M with an associated set of periodic indicial exponents can be neglected. It will also be noted by example that the determinant of a cyclemaster matrix with periodic indicial exponents vanishes.

Suppose that D is a proper divisor of N and that the subset $\{n_1,\ldots,n_D\}$ of $\{n_1,\ldots,n_N\}$ repeats $d = N/D$ times, so that $n_{i+D} = n_i$ for all i. Then $N = dD$ and $M = d\,M_D$, where $M_D = n_1 + \cdots + n_D$.

The periodicity of the indicial exponents is reflected in the summary exponents in the formula

$$m_{i,jD+k} = j\,M_D + m_{ik}.$$

Given this formula, the cycle generators can be factored as

$$\begin{aligned} r_i &= \left(2^{M_D(d-1)} + 2^{M_D(d-2)}\cdot 3^D + \cdots + 2^{M_D}\cdot 3^{(d-2)D} + 3^{(d-1)D}\right) \\ &\quad\times\left(2^{m_i,D-1} + 2^{m_i,D-2}\,3^1 + \cdots + 2^{m_i,1}\cdot 3^{D-2} + 3^{D-1}\right) \\ &= \left(\sum_{k=0}^{d-1} 2^{M_D(d-k-1)}\,3^{kD}\right)\left(\sum_{j=0}^{D-1} 2^{m_i,D-j-1}\cdot 3^j\right). \end{aligned}$$

But we also have

$$\begin{aligned} q &= 2^M - 3^N \\ &= \left(2^{M_D}\right)^d - \left(3^D\right)^d \\ &= \left(\sum_{k=0}^{d-1} 2^{M_D(d-k-1)}\,3^{kD}\right)\left(2^{M_D} - 3^D\right). \end{aligned}$$

Remark. Thus, the natural N-cycle of index M_N with periodic indicial exponents is essentially a natural D-cycle of index M_D that has been repeated N/D times.

A simple example will assist the reader to understand repeated cycles.

Illustrative Example. Consider the 6-cycle of index 12 whose indicial exponents are $\{1,2,3,1,2,3\}$. This natural cycle is created in the row indexed by $q = 2^{12} - 3^6 = (2^6 - 3^3)(2^6 + 3^3) = 37 \times 91$. By using the process described at the beginning of this section, the cycle elements are determined to be

$$r_1 = r_4 = 2093 = 23 \times 91,$$
$$r_2 = r_5 = 4823 = 53 \times 91,$$

and

$$r_3 = r_6 = 4459 = 49 \times 91.$$

By doing so, we have constructed a natural 6-cycle which consists of the repeated 3-cycle

$$(2093, 3367) \xrightarrow{1} (4823, 3367) \xrightarrow{2} (4459, 3367) \xrightarrow{3} (2093, 3367).$$

After scalar division by 91, we have also produced the natural 3-cycle of index 6 given by

$$(23, 37) \xrightarrow{1} (53, 37) \xrightarrow{2} (49, 37) \xrightarrow{3} (23, 37).$$

We can also observe that $\det(C_{6,12}(1,2,3,1,2,3)) = 0$, since every entry in the 4^{th} row of $C_{6,12}(1,2,3,1,2,3)$ is equal to 2^6, so that the 4^{th} row is a scalar multiple of the 1^{st} row. However, $\det(C_{3,6}(1,2,3)) = -2^7 \neq 0$.

Remark. It is possible to count the number of natural N-cycles of index M in a row of the lattice by employing elementary combinatorial principles.

If M and N are relatively prime, then the number $\mathcal{N}(N, M)$ of natural N-cycles of index M in the q^{th} row of the lattice $\mathcal{O} \times \mathcal{O}$, where $q = 2^M - 3^N$, is equal to

$$\mathcal{N}(N, M) = \frac{1}{N}\binom{M-1}{N-1}.$$

This follows because there are $\binom{M-1}{N-1}$ solutions to the equation $n_1 + \cdots + n_N = M$ in positive integers, and each solution $\{n_1, \ldots, n_N\}$ and its N rotations correspond to the N cycle generators used to prescribe one cycle.

If M and N are not relatively prime, then the situation can be considerably more complicated if M and N have many prime factors. The discussion above shows that repetition is possible when M and N are *not* relatively prime. For example, if $N = 10$, a 10-cycle may consist of two repeated 5-cycles, five repeated 2-cycles, or 10 repeated 1-cycles.

In any eventuality, the Inclusion/Exclusion Principle can be successfully used to count the number of repeated subcycles.

3.4. Extremal Natural N-cycles of index M

For $N \geq 2$, an *extremal* natural N-cycle of index M is a natural N-cycle of index M for which $n_1 = \cdots = n_{N-1} = 1$ and $n_N = M - N + 1$. Since extremality is determined by indicial exponents, all scalar multiples and scalar quotients of these cycles are also considered to be extremal. Rotations of extremal N-cycles are also considered to be extremal.

Technically, 1-cycles can, by definition, be extremal. If $N = 1$ and $n_1 = 1$, then $q = -1$ and $r_1 = 1$, yielding the fixed point $(1, -1)$, which can be reduced by scalar division by -1 to the fixed point $(-1, 1)$ in the row indexed by $q = 1$. If $n_1 = 2$, then $q = 1$ and $r_1 = 1$, yielding the fixed point $(1, 1)$.

If $N = 2$, then $\{n_1, n_2\} = \{1, a\}$, $M = 1 + a$, $q = 2^{1+a} - 3^2$, $r_1 = 5$, and $r_2 = 2^a + 3$, yielding the extremal natural 2-cycle of index $1 + a$

$$(5, 2^{1+a} - 9) \xrightarrow{1} (2^a + 3, 2^{1+a} - 9) \xrightarrow{a} (5, 2^{1+a} - 9).$$

For this extremal 2-cycle to be subject to scalar division by $2^{1+a} - 9$, thereby producing an extremal 2-cycle in the row indexed by $q = 1$, it is necessary either that $2^{1+a} - 9 = -5$, in which case $a = 1$, or that $2^{1+a} - 9 = -1$, in which case $a = 2$. If $a = 1$, then the 2-cycle is actually the fixed point $(5, -5)$, which can be reduced to $(-1, 1)$. If $a = 2$, then $M = 3$, $q = -1$, $r_1 = 5$, and $r_2 = 7$. After scalar division by -1, we obtain the known 2-cycle

$$(-5, 1) \xrightarrow{1} (-7, 1) \xrightarrow{2} (-5, 1).$$

In general, extremal natural cycles may be either reducible or irreducible. As previously shown, any natural cycle can be at least *partially* reduced by $\rho_N = \gcd\{r_1, \ldots, r_N\}$. For an extremal natural N-cycle of index M to *exist* in the row indexed by $q = 1$, it must be the scalar quotient of a *completely* reducible N-cycle of index M previously *created* in the row indexed by $q = 2^M - 3^N$.

Theorem 11. *For $q = 2^M - 3^N$, we define $\mathbb{C}(N, M)$ to be the set of all elements of all natural N-cycles of index M. Then the smallest r_{min} and largest r_{max} cycle generators have the following properties:*

1. *If $(r_i, q) \in \mathbb{C}(N, M)$ is any element of any natural N-cycle of index M, then $0 < r_{min} \leq r_i \leq r_{max}$.*
2. $r_{min} = 3^N - 2^N$
3. $r_{max} = 2^{M-N+1}\left(3^{N-1} - 2^{N-1}\right) + 3^{N-1}$
4. $r_{max} - r_{min} = 2\left(3^{N-1} - 2^{N-1}\right)\left(2^{M-N} - 1\right)$
5. *r_{min} and r_{max} belong to the same N-cycle and $S(r_{max}, q) = r_{min}$*

Proof.

1. This is clear from the definition.

2. From all choices of indicial exponents, choose the set $\{1, 1, \ldots, 1, M-N+1\}$. The generator

$$r_{min} = 2^{N-1} + 2^{N-2} \cdot 3 + \cdots + 2^1 \cdot 3^{N-2} + 3^{N-1} = 3^N - 2^N$$

is the smallest possible generator, since the choice of indicial and summary exponents is the smallest possible.

3. From all choices of indicial exponents, choose the set $\{M - N + 1, 1, \ldots, 1\}$. The generator

$$\begin{aligned} r_{max} &= 2^{M-1} + 2^{M-2} \cdot 3 + \cdots + 2^{M-N+1} \cdot 3^{N-2} + 3^{N-1} \\ &= 2^{M-N+1}\left(3^{N-1} - 2^{N-1}\right) + 3^{N-1} \end{aligned}$$

is the largest possible generator, since the summary exponents are the largest possible.

4. This follows by direct computation.

5. By direct computation, we have

$$r_{max} \overset{M-N+1}{\to} r_{min},$$

implying that r_{min} and r_{max} belong to the same cycle. □

Remark. The use of the term *extremal* natural N-cycle is justified by this theorem, since the N-cycle generated by choosing the indicial exponents $\{1, \ldots, 1, M-N+1\}$ contains the largest and smallest elements of $\mathbb{C}(N, M)$.

Extremal N-cycles have some interesting properties.

Theorem 12. *Let $\{(r_i, q) \colon i = 1, \ldots, N\}$ be an extremal natural N-cycle of index M with $M > N \geq 2$. Then the following properties of its cycle elements r_i are valid:*

1. $r_{min} = r_1 < r_2 < \cdots < r_N = r_{max}$
2. $\left(2^{M-N} - 1\right)$ *divides* $(r_{i+1} - r_i)$ *for all* $i = 1, \ldots, N-1$

Proof. First, note that

$$\begin{aligned} r_2 - r_1 &= \frac{3r_1 + q}{2} - r_1 \\ &= \frac{r_1 + q}{2} \\ &= \frac{\left(3^N - 2^N\right) + \left(2^M - 3^N\right)}{2} \\ &= 2^{N-1}\left(2^{M-N} - 1\right) \end{aligned}$$

This equation shows that $r_1 < r_2$ and that $\left(2^{M-N} - 1\right)$ divides the difference $r_2 - r_1$. Finally, note that

$$r_{i+1} - r_i = \sum_{k=1}^{i-1}\left(\frac{r_{k+1} - r_k}{2}\right) + \frac{r_1 + q}{2},$$

so that $r_i < r_{i+1}$ and $\left(2^{M-N} - 1\right)$ divides the difference $r_{i+1} - r_i$ for all i by an elementary induction argument. □

Remark. Although $\left(2^{M-N} - 1\right)$ divides each of the differences $r_{i+1} - r_i$, it does not necessarily divide each cycle element r_i. Consider the extremal natural 3-cycle of index 11 with indicial exponents $\{1, 1, 9\}$ that was presented in the Illustrative Example in Section 3.1:

$$(19, 2021) \xrightarrow{1} (1039, 2021) \xrightarrow{1} (2569, 2021) \xrightarrow{9} (19, 2021).$$

Here, $2^{M-N} - 1 = 2^{11-3} - 1 = 2^8 - 1 = 255$. Note that

$$\begin{aligned} r_2 - r_1 &= 1039 - 19 = 1020 = 4 \times 255 \\ r_3 - r_2 &= 2569 - 1038 = 1530 = 6 \times 255 \\ r_4 - r_3 = r_1 - r_3 &= 19 - 2569 = -2550 = -10 \times 255, \end{aligned}$$

but that 255 does not divide r_i for any i. This shows that there must be some minimal spacing between cycle elements, depending upon N and M.

3.5. Divisibility

Theorem 1 involves the divisibility of the determinant of the cyclemaster matrix by the cycle indicator $q = 2^M - 3^N$. The process of scalar division, whether partial or complete, is also concerned with the divisibility of the cycle elements r_i by q or its prime power factors.

In this subsection, we therefore investigate the prime power factors p^α of q, r_i, and the determinant of the cyclemaster matrix. Divisibility by p^α will involve the existence of primitive roots modulo p^α. Exponential congruences will also play an important role.

3.5.1. The Prime Power Factors of $q = 2^M - 3^N$.

A known result [5] details some of the elementary restrictions on the values of q.

Theorem 13. *Let $N \geq 1$, $M \geq 3$, and $I = 2^M - 3^N$. Then*

$$I \equiv \begin{cases} 5 \pmod{24} & \text{if } M \text{ is odd and } N \text{ is odd} \\ 7 \pmod{24} & \text{if } M \text{ is even and } N \text{ is even} \\ 13 \pmod{24} & \text{if } M \text{ is even and } N \text{ is odd} \\ 23 \pmod{24} & \text{if } M \text{ is odd and } N \text{ is even} \end{cases}$$

Proof. We prove the first case; the proofs of the other three cases are similar.

If M is odd and N is odd, then there exist non-negative integers k and j such that $M = 3 + 2k$ and $N = 1 + 2j$.

If $k = j = 0$, then $I = 2^3 - 3^1 = 5$.

If $k = 1$ and $j = 0$, then $M = 5$, $N = 1$, and $I = 2^5 - 3^1 = 29 \equiv 5 \pmod{24}$.

If $k = 0$ and $j = 1$, then $M = 3$, $N = 3$, and $I = 2^3 - 3^3 = -19 \equiv 5 \pmod{24}$.

If $k \geq 1$ and $j \geq 1$, then

$$I = 2^{3+2k} - 3^{1+2j} = (2^3 - 3^1) + 2^3(2^{2k} - 1) - 3^1(3^{2j} - 1) \equiv 5 \pmod{24}.$$

Since $2^{2k} - 1$ is divisible by 3 and $3^{2j} - 1$ is divisible by 2^3, these two terms taken together are a multiple of 24. $\square$

Remark. If $I \equiv 5, 7, 13,$ or $23 \pmod{24}$, then $I^{2k} \equiv 1 \pmod{24}$. Therefore, $I^{2k} \not\equiv 2^M - 3^N \pmod{24}$ for any choices of M and N, implying that no natural N-cycles of index M can be created in a row indexed by I^{2k}, unless $I = \pm 1$.

Note that $I \not\equiv 1, 11, 17,$ or $19 \pmod{24}$, unless $I = 1$. For these values, $I \not\equiv 2^M - 3^N$ for any choices of M and N. The same observation is true for the power of any integer congruent to these values of I, unless $I = 1$. For if k is odd, then $I^k \equiv I \pmod{24}$ and if k is even, then $I^k \equiv 1 \pmod{24}$.

There are additional restrictions on M and N if I is a prime number. The proofs of the following corollaries were previously published in [5]. This theorem and its two corollaries show that if the difference $2^M - 3^N$ is prime, then M and N are usually relatively prime, with very few exceptions. Note that if $M = N = d$, then the corresponding N-cycle is actually a 1-cycle that is repeated N times.

Corollary 1. *Suppose that $I = 2^M - 3^N > 0$ is a prime number.*

1. *If $I \equiv 5 \pmod{24}$, then M and N are relatively prime. The converse need not hold.*

2. *If $I \equiv 7 \pmod{24}$, then $I = 7$. ($M = 4$ and $N = 2$)*

3. *If $I \equiv 13 \pmod{24}$, then either M and N are relatively prime, or $M = 2d$ and $N = d$, where d is an odd prime. The converse need not hold.*

4. *If $I \equiv 23 \pmod{24}$, then M and N are relatively prime. The converse need not hold.*

Corollary 2. *Suppose that $-I = 2^M - 3^N < 0$ is a prime number.*

1. *If $-I \equiv 5 \pmod{24}$, then either M and N are relatively prime or $M = N = d$, where d is an odd prime. The converse need not hold.*

2. *If $-I \equiv 7 \pmod{24}$, then $-I = -17$. ($M = 6$ and $N = 4$)*

3. *If $-I \equiv 13 \pmod{24}$, then M and N are relatively prime. The converse need not hold.*

4. *If $-I \equiv 23 \pmod{24}$, then either M and N are relatively prime, or $M = 3d$ and $N = 2d$, where d is an odd prime. The converse need not hold.*

The next theorem, which appeared in [5], shows that there exist infinitely many exponents M and N for which a prime power p^α divides $2^M - 3^N$. In this theorem, the roles of the primitive roots of p^α and the Euler totient φ are critical.

Theorem 14. *Let p^α be a prime power factor of $2^M - 3^N$, with $p \geq 5$ and $\alpha \geq 1$. Let g be any of the $\varphi(\varphi(p^\alpha))$ primitive roots modulo p^α, and let y and x be the unique exponents such that $g^y \equiv 2 \pmod{p^\alpha}$ and $g^x \equiv 3 \pmod{p^\alpha}$. Also, let e_2 and e_3 be the exponents that 2 and 3 $\pmod{p^\alpha}$ belong to, respectively.* [3] *If $M = M(g) = e_2a + xk$ and $N = N(g) = e_3b + yk$, where a, b, and k are non-negative integers, then $2^M - 3^N \equiv 0 \pmod{p^\alpha}$.*

Proof. For the given values of M and N, we have

$$2^M \equiv 2^{e_2a+xk} \equiv 2^{xk} \equiv g^{yxk} \equiv 3^{yk} \equiv 3^{e_3b+yk} \equiv 3^N \pmod{p^\alpha}. \qquad \square$$

Remark. If $g = 2$, then $e_2 = \varphi(p^\alpha)$ and $y = 1$. If $g = 3$, then $e_3 = \varphi(p^\alpha)$ and $x = 1$.

If $2^M - 3^N = p_1^{\alpha_1} \dots p_n^{\alpha_n}$, then M and N must satisfy the requirements stated in the theorem for each of the prime power factors.

Illustrative Example. To illustrate this theorem, we choose $p = 19$, $\alpha = 1$, $a = 2$, $b = 1$, and $k = 1$. The $\varphi(\varphi(19)) = 6$ primitive roots of 19 are $g = 2, 3, 10, 13, 14,$ and 15. Note that $2^{18} \equiv 1$ and $3^{18} \equiv 1$ (mod 19) by Fermat's Little Theorem.

[3] That is, e_2 and e_3 are the smallest integers such that $2^{e_2} \equiv 1 \pmod{p^\alpha}$ and $3^{e_3} \equiv 1 \pmod{p^\alpha}$.

If $g = 2$, then $2^1 \equiv 2$ and $2^{13} \equiv 3 \pmod{19}$. With $M(2) = 36 + 13$ and $N(2) = 18 + 1$, we have

$$2^{M(2)} - 3^{N(2)} \equiv 2^{13} - 3^1 \equiv 8189 \equiv 19 \times 431 \equiv 0 \pmod{19}.$$

If $g = 3$, then $3^1 \equiv 2$ and $3^1 \equiv 3 \pmod{19}$. With $M(3) = 36 + 1$ and $N(3) = 18 + 7$, we have

$$2^{M(3)} - 3^{N(3)} \equiv 2^1 - 3^7 \equiv -2185 \equiv -5 \times 19 \times 23 \equiv 0 \pmod{19}.$$

If $g = 10$, then $10^{17} \equiv 2$ and $10^5 \equiv 3 \pmod{19}$. With $M(10) = 36 + 5$ and $N(10) = 18 + 17$, we have

$$2^{M(10)} - 3^{N(10)} \equiv 2^5 - 3^{17} \equiv -129,140,131 \equiv -19 \times 6,796,849 \equiv 0 \pmod{19}.$$

If $g = 13$, then $13^{11} \equiv 2$ and $13^{17} \equiv 3 \pmod{19}$. With $M(13) = 36 + 17$ and $N(13) = 18 + 11$, we have

$$2^{M(13)} - 3^{N(13)} \equiv 2^{17} - 3^{11} \equiv -46075 \equiv -5^2 \times 19 \times 97 \equiv 0 \pmod{19}.$$

If $g = 14$, then $14^{13} \equiv 2$ and $14^7 \equiv 3 \pmod{19}$. With $M(14) = 36 + 7$ and $N(14) = 18 + 13$, we have

$$2^{M(14)} - 3^{N(14)} \equiv 2^7 - 3^{13} \equiv -1,594,195 \equiv -5 \times 19 \times 97 \times 173 \equiv 0 \pmod{19}.$$

If $g = 15$, then $15^5 \equiv 2$ and $15^{11} \equiv 3 \pmod{19}$. With $M(15) = 36 + 11$ and $N(15) = 18 + 5$, we have

$$2^{M(15)} - 3^{N(15)} \equiv 2^{11} - 3^5 \equiv 1805 \equiv 5 \times 19^2 \equiv 0 \pmod{19}.$$

3.5.2. The Prime Power Factors of the Cycle Generators r_i

Previously, we have seen that the possible values of the length N and index M of a cycle are strongly dependent upon the prime power divisors of the cycle indicator $2^M - 3^N$, in the sense that the primitive roots of these prime power divisors play a crucial role in their values.

If q and r_i have no prime power divisors in common, then no scalar division by prime powers is possible. This is also true if r_i and r_{i+1} have no prime power factors in common. Here, we show that the primitive roots of prime powers play a crucial role in determining the prime power divisors of cycle generators r_i and the value of the index M of the cycles to which they belong. To illustrate the role of primitive roots with respect to this issue, we consider the possibility of scalar division of 2-cycles and 3-cycles, and then mention the general case.

Consider the process of scalar division for natural 2-cycles of index M. The natural 2-cycle of index $M = n_1 + n_2$ has the general form

$$(2^{n_1} + 3, 2^M - 3^2) \xrightarrow{n_1} (2^{n_2} + 3, 2^M - 3^2) \xrightarrow{n_2} (2^{n_1} + 3, 2^M - 3^2).$$

If scalar division by p^α is possible here, then there will simultaneously exist n_1 and n_2 such that

$$2^{n_1} \equiv -3 \pmod{p^\alpha} \qquad \text{and} \qquad 2^{n_2} \equiv -3 \pmod{p^\alpha}$$

and it will necessarily follow that

$$2^M \equiv 2^{n_1+n_2} \equiv (-3)^2 \equiv 3^2 \pmod{p^\alpha}.$$

If 2 is a primitive root modulo p^α, then the congruence equation

$$2^x \equiv -3 \pmod{p^\alpha} \tag{3.1}$$

always has solutions. If n_1 and n_2 are any two of these solutions, then we have $n_1 \equiv n_2 \pmod{\varphi(p^\alpha)}$, where φ is the Euler totient. Thus, $n_2 = n_1 + k\varphi(p^\alpha)$ and $M = 2n_1 + k\varphi(p^\alpha)$ for some non-negative integer k. If $k = 0$, then $2^M - 3^2 = (2^{n_1} + 3)(2^{n_1} - 3)$, and the generic 2-cycle above can be reduced by scalar division to the repeated 1-cycle

$$(1, 2^{n_1} - 3) \xrightarrow{n_1} (1, 2^{n_1} - 3)$$

as prescribed in Section 2.2.

If 2 is not a primitive root modulo p^α, then equation (3.1) may or may not have solutions.

If equation (3.1) does have solutions, then scalar division is still possible. For example, 2 is not a primitive root modulo 7, but $2^2 \equiv -3 \pmod 7$ and $2^5 \equiv -3 \pmod 7$. If $n_1 = n_2 = 2$, then the computed 2-cycle is just a repeated 1-cycle. If $n_1 = 2$ and $n_2 = 5$, then $M = 7$, $q = 2^7 - 3^2 = 119 = 7 \times 17$, and the computed 2-cycle

$$(7, 119) \xrightarrow{2} (35, 119) \xrightarrow{5} (7, 119)$$

is subject to scalar division by 7, yielding the reduced 2-cycle

$$(1, 17) \xrightarrow{2} (5, 17) \xrightarrow{5} (1, 17).$$

Note also that $\det(C_{2,7}(2, 5)) = 2^2 \cdot 7$, so that this determinant and q have the common factor of 7.

If equation (3.1) does not have solutions, then scalar division is not possible. This situation occurs if $p = 17, 23, 31, 41, 43, 47, \ldots$.

Primitive roots also play a significant role in the scalar division of 3-cycles of index M. As seen in Section 2.3, the natural 3-cycle of index $M = n_1 + n_2 + n_3$ has the general form

$$(r_1, q) \xrightarrow{n_1} (r_2, q) \xrightarrow{n_2} (r_3, q) \xrightarrow{n_3} (r_1, q),$$

where

$$r_1 = 2^{n_1+n_2} + 2^{n_1} \cdot 3 + 3^2 \tag{3.2}$$

$$r_2 = 2^{n_2+n_3} + 2^{n_2} \cdot 3 + 3^2 \tag{3.3}$$

and

$$r_3 = 2^{n_3+n_1} + 2^{n_3} \cdot 3 + 3^2. \tag{3.4}$$

If scalar division by the prime power p^α is possible, then the system of congruence equations

$$2^{n_1+n_2} \equiv -3(2^{n_1} + 3) \pmod{p^\alpha} \tag{3.5}$$

$$2^{n_2+n_3} \equiv -3(2^{n_2} + 3) \pmod{p^\alpha} \tag{3.6}$$

$$2^{n_3+n_1} \equiv -3(2^{n_3} + 3) \pmod{p^\alpha} \tag{3.7}$$

has a solution $\{n_1, n_2, n_3\}$, and $2^{n_1+n_2+n_3} \equiv 3^3 \pmod{p^\alpha}$. From these congruence equations, we deduce that if $2^{n_i} + 3 \equiv 0 \pmod{p^\alpha}$ for any i, then a solution to this system cannot exist. Hence, for a solution to exist, we require that $2^{n_i} \not\equiv -3 \pmod{p^\alpha}$ for all $i = 1, 2, 3$.

If 2 is a primitive root modulo p^α, then this system of congruence equations always has at least one solution. To see this, choose n_1 arbitrarily from all possible integers for which $2^{n_i} \not\equiv -3 \pmod{p^\alpha}$. Then there must exist a positive integer y which solves the congruence

$$2^{n_1+y} \equiv -3(2^{n_1} + 3) \pmod{p^\alpha}.$$

Let $n_2 = y$. Once n_2 is chosen, we may compute r_1 and rewrite the first congruence equation in the form

$$2^{n_1}(2^{n_2} + 3) + 3^2 \equiv 0 \pmod{p^\alpha}.$$

If $2^{n_2} + 3 \equiv 0 \pmod{p^\alpha}$, then $3^2 \equiv 0 \pmod{p^\alpha}$, an impossibility. Therefore, $2^{n_2} \not\equiv -3 \pmod{p^\alpha}$, so that the congruence

$$2^{z+n_2} \equiv -3(2^{n_2} + 3) \pmod{p^\alpha}$$

has a solution z. Let $n_3 = z$. Once n_3 is chosen, we may compute r_2 and $q = 2^M - 3^3$. Since $2^{n_1} r_2 = 3r_1 + q$, it follows that p^α divides q, from which it also follows that p^α divides r_3 as well. Hence, the given 3-cycle is subject to scalar division by p^α.

A well chosen example for 3-cycles will illustrate the process that we have just discussed.

Illustrative Example. Let $p = 13$. Since 2 is a primitive root modulo 13, we may choose any value of n_1 ($1 \leq n_1 \leq 12$), except $n_1 = 10$ since $2^{10} + 3 \equiv 0 \pmod{13}$.

Choosing $n = 1, 5, \text{or } 6$ leads to the 3-cycle of index 12

$$(143, 4069) \xrightarrow{1} (2249, 4069) \xrightarrow{6} (169, 4069) \xrightarrow{5} (143, 4069)$$

with $q = 2^{12} - 3^3 = 4069 = 13 \times 313$, which may be reduced by scalar division by 13 to

$$(11, 313) \xrightarrow{1} (173, 313) \xrightarrow{6} (13, 313) \xrightarrow{5} (11, 313).$$

Note also that $\det(\mathcal{C}_{3,12}(1, 6, 5)) = -2^7 \cdot 13 \cdot 37$.

Choosing $n_1 = 2, 3, \text{or } 7$ leads to the 3-cycle of index 12

$$(533, 4069) \xrightarrow{2} (1417, 4069) \xrightarrow{7} (65, 4069) \xrightarrow{3} (533, 4069)$$

which can be reduced by scalar division by 13 to

$$(41, 313) \xrightarrow{2} (109, 313) \xrightarrow{7} (5, 313) \xrightarrow{3} (41, 313).$$

Note also that $\det(\mathcal{C}_{3,12}(2, 7, 3)) = -2^7 \cdot 13 \cdot 37$.

Choosing $n_1 = 4, 9, \text{or } 11$ leads to the 3-cycle of index 24

$$(9737, 16777189) \xrightarrow{4} (1054729, 16777189) \xrightarrow{11} (32825, 16777189) \xrightarrow{9} (9737, 16777189)$$

which can be reduced by scalar division by 13 to

$$(749, 1290553) \xrightarrow{4} (81133, 1290553) \xrightarrow{11} (2525, 1290553) \xrightarrow{9} (749, 1290553).$$

Note also that $\det(\mathcal{C}_{3,24}(4, 11, 9)) = -2^{17} \cdot 5^2 \cdot 13^2$.

Choosing $n_1 = 8$ leads to a 3-cycle of index 24, which is actually the repeated 1-cycle of index 8

$$(1, 2^8 - 3) \xrightarrow{8} (1, 2^8 - 3).$$

Choosing $n_1 = 12$ leads to a 3-cycle of index 36, which is actually the repeated 1-cycle of index 12

$$(1, 2^{12} - 3) \xrightarrow{12} (1, 2^{12} - 3).$$

In this example, $M = n_1 + n_2 + n_3$ is always a multiple of 12. An application of Theorem 14 explains this observation. With $2^{12} \equiv 1 \pmod{13}$, $3^3 \equiv 1 \pmod{13}$, $2^1 \equiv 2 \pmod{13}$, and $2^4 \equiv 3 \pmod{13}$, M and N will have the representations $M = 12a + 4k$ and $N = 3b + k$. Since $N = 3$, we must have $b = 1$ and $k = 0$, which implies that $M = 12a$, or $b = 0$ and $k = 3$, which implies that $M = 12(a + 1)$.

We turn to the general case. The analysis given above suggests that a system of N exponential congruence equations modulo p^α needs to have a solution $\{n_1, \ldots, n_N\}$

consisting of indicial exponents if scalar division by p^α of the corresponding N-cycle if index M is to occur. To be more specific, if scalar division by p^α is possible for the natural N-cycle

$$(r_1, 2^M - 3^N) \to (r_2, 2^M - 3^N) \to \cdots \to (r_N, 2^M - 3^N),$$

where

$$r_i = g_{Ni} = \sum_{j=0}^{N-1} 2^{m_{i,N-j-1}} \cdot 3^j,$$

then there must exist positive integers $\{n_1, \ldots, n_N\}$ such that $g_{Ni} \equiv 0 \pmod{p^\alpha}$, or equivalently,

$$2^{m_{i,N-1}} \equiv -3 \left(\sum_{j=1}^{N-1} 2^{m_{i,N-j-1}} \cdot 3^{j-1} \right) \pmod{p^\alpha}$$

for all $i = 1, \ldots, N$, with $2^M \equiv 2^{n_1+\cdots+n_N} \equiv 3^N \pmod{p^\alpha}$. We have shown above that this is possible if $N = 2$ or $N = 3$ and 2 is a primitive root of p^α, but the general case is unresolved.

Remark. Suppose that $q = 2^M - 3^N = \pm p^\alpha$ for some integers $M > N \geq 2$, $p \geq 5$, and $\alpha \geq 1$, where p is an odd prime. Then either $2^M + p^\alpha = 3^N$ or $3^N + p^\alpha = 2^M$. A few solutions to these Diophantine equations are: $3^3 + 5^1 = 2^5$, $3^4 + 47^1 = 2^7$, $2^{11} + 139^1 = 3^7$, and $2^5 + 7^2 = 3^4$. In these examples, $\alpha = 1$ or 2.

The Beal Conjecture (a generalized version of Fermat's Last Theorem) states that the Diophantine equation $x^m + y^n = z^k$ has no solutions if x, y and z are positive coprime integers and m, n and k are greater than 2.

If the Beal Conjecture is true, then the Diophantine equations $2^M - 3^N = \pm p^\alpha$ have no solutions if $N \geq 3$, $M \geq N$ and $\alpha \geq 3$. Under this assumption, no natural N-cycles (with $N \geq 3$) of index M (with $M > N$) could be *created* in any row indexed by $q = \pm p^\alpha$, if $p \geq 5$ and $\alpha \geq 3$. Of course, N-cycles can still *exist* in these rows. Note that the solution $3^1 + 5^3 = 2^7$ is not a counterexample, as $N = 1$.

3.5.3. Some Factors of the Determinant of the Cyclemaster Matrix

Theorem 15. *Let $\{(r_i, q) \colon i = 1, \ldots, N\}$ be an extremal natural N-cycle of index M with the associated set of indicial exponents $\{1, \ldots, 1, M - N + 1\}$. Then*

$$\det(\mathcal{C}_{N,M}(1, \ldots, 1, M - N + 1)) = \pm 2^{N(N-1)/2} \left(2^{M-N} - 1\right)^{N-1}, \tag{3.8}$$

where the sign of the determinant is positive if $N \equiv 1 \pmod 4$ or $N \equiv 2 \pmod 4$, and is negative if $N \equiv 3 \pmod 4$ or $N \equiv 4 \pmod 4$.

The proof of this theorem follows by an application of elementary row operations.

Corollary 3. *If n is an odd integer, then n^{N-1} divides the determinant*

$$\det\left(\mathcal{C}_{N,M}(1,\ldots,1,1+\varphi(n))\right) = \pm 2^{N(N-1)/2}\left(2^{\varphi(n)}-1\right)^{N-1}.$$

In particular, if p is prime, then p^{N-1} divides the determinant

$$\det\left(\mathcal{C}_{N,M}(1,\ldots,1,p)\right) = \pm 2^{N(N-1)/2}\left(2^{p-1}-1\right)^{N-1}.$$

Proof. The Euler-Fermat Theorem states that if a and n are relatively prime, then $a^{\varphi(n)} \equiv 1 \pmod{n}$. Given this choice of indicial exponents, $M = N+\varphi(n)$. Choose $a = 2$. Also, if p is prime, then $\varphi(p) = p-1$. □

Corollary 4. *Let $M = N + p$.*

1. $\det\left(\mathcal{C}_{N,M}(1,\ldots,1,1+p)\right) = \pm 2^{N(N-1)/2}\left(2^{p}-1\right)^{N-1}$
2. *If $2^p - 1$ is a Mersenne prime and $N \geq 3$, then $2^M - 3^N$ does not divide this determinant.*
3. *If $2^p - 1$ is a Mersenne prime and $N \geq 3$, then there are no extremal N-cycles of index M for the original Collatz problem.*

Proof. Recall that a *Mersenne prime* is a prime of the form $2^p - 1$. If $2^p - 1$ is prime, then so is p.

1. Apply the theorem with $M - N = p$.
2. If $2^M - 3^N$ divides the determinant, then it also divides 2 or $2^p - 1$. But then $2^M - 3^N = \pm 1$ or $\pm(2^p - 1)$.

   If $2^M - 3^N = +1$, then $N = 1$, $M = 2$, $p = 1$, and $2^p - 1 = 1$.

   If $2^M - 3^N = -1$, then $N = 2$, $M = 3$, and $p = 1$.

   If $2^M - 3^N = 2^{N+p} - 3^N = +(2^p - 1)$, then $2^p(2^N - 1) = 3^N - 1$. If N is even, then 3 divides the left hand side of this equation, but not the right. If N is odd, then the largest power of 2 that divides the right hand side of this equation is 2^1, implying that $p = 1$, $2^p - 1 = 1$.

   If $2^M - 3^N = -(2^p - 1)$, then $2^p(2^N + 1) = 3^N + 1$. The highest power of 2 that divides the right hand side of this equation is 2^1 or 2^2, so that $p = 1$ or $p = 2$. If $p = 2$, then $2^p - 1 = 3$, and $2^M - 3^N = -3$, an impossibility.
3. This follows as a corollary to Theorem 1. □

Remark. The difference $M - N$ can be used to eliminate N-cycles for the original Collatz problem. If $M - N = 3$, then $\det\left(\mathcal{C}_{N,M}(1,\ldots,1,4)\right) = \pm 2^{N(N-1)/2} 7^{N-1}$. If $2^M - 3^N$ divides this determinant, then $2^M - 3^N = \pm 1$ or $\pm 7^d$, where d is an integer such that $1 \leq d \leq N-1$. If $d = 1$, then $N = 2$, $M = 4$, and $q = 7$, but $M - N = 2$, not 3. The other choices are similar.

4. The "$5x + 1$" Problem and Similar Problems

It is of interest to compare the strong similarities between the "$3x + 1$" problem and the "$5x + 1$" problem. The analysis presented in the previous sections can be repeated nearly verbatim to establish the following correspondences between these two problems:

1. The cycle indicator $2^M - 3^N$ corresponds to $2^M - 5^N$.
2. The cycle generators
$$g_{Ni} = 2^{m_{i,N-1}} + \cdots + 2^{m_{i1}} \cdot 3^{N-2} + 3^{N-1}$$
correspond to
$$g_{Ni} = 2^{m_{i,N-1}} + \cdots + 2^{m_{i1}} \cdot 5^{N-2} + 5^{N-1}.$$
3. The cyclemaster matrix is the **same** for both problems.
4. Theorem 1 analogizes appropriately, but the divisor $2^M - 3^N$ must be replaced by $2^M - 5^N$.
5. The known N-cycles are different.

To highlight the four known N-cycles of index M for the "$5x + 1$" problem, we discuss their genesis and structure here in terms of the vocabulary introduced in the previous sections.

1. The first known cycle is a 1-cycle of index 2, with $N = 1$, $n_1 = 2 = M$, and $q = 2^2 - 5^1 = -1$. The only element in this 1-cycle can be determined from the equation
$$\frac{5r_1 - 1}{2^2} = r_1$$
to be $r_1 = 1$. Thus, this natural cycle has the representation
$$(1, -1) \xrightarrow{2} (1, -1).$$
After scalar division by -1 the cycle becomes
$$(-1, 1) \xrightarrow{2} (-1, 1),$$
which is why it previously appeared in the literature simply as $-1 \to -1$. This cycle fits the technical definition of an extremal natural 1-cycle.

2. The second known cycle is an extremal natural 2-cycle of index 5, with $N = 2$, $M = 5$, $q = 2^5 - 5^2 = 7$, and the associated indicial exponents $\{n_1, n_2\} = \{1, 4\}$. The cycle elements are $r_1 = 2^1 + 5 = 7$ and $r_2 = 2^4 + 5 = 21$. Thus, this 2-cycle has the representation

$$(7, 7) \xrightarrow{1} (21, 7) \xrightarrow{4} (7, 7).$$

After scalar division by $q = 7$, we obtain the completely reduced extremal 2-cycle

$$(1, 1) \xrightarrow{1} (3, 1) \xrightarrow{4} (1, 1),$$

which is why it previously appeared in the literature as $1 \to 3 \to 1$. Note that $2^5 - 5^2 = 7$ divides the determinant

$$\det(\mathcal{C}_{2,5}(1, 4)) = 14,$$

in accordance with the obvious analog (with $a = 5$) to Theorem 1.

3. The third known cycle is an extremal natural 3-cycle of index 7, with $q = 2^7 - 5^3 = 3$ and the associated indicial exponents $\{n_1, n_2, n_3\} = \{1, 1, 5\}$. The cycle elements are computed to be

$$\begin{aligned} r_1 &= 2^2 + 2^1 \cdot 5 + 5^2 = 39 \\ r_2 &= 2^6 + 2^1 \cdot 5 + 5^2 = 99 \\ r_3 &= 2^6 + 2^5 \cdot 5 + 5^2 = 249 \end{aligned}$$

Thus, this 3-cycle can be represented as

$$(39, 3) \xrightarrow{1} (99, 3) \xrightarrow{1} (249, 3) \xrightarrow{5} (39, 3).$$

After scalar division by $q = 3$, we obtain the completely reduced 3-cycle

$$(13, 1) \xrightarrow{1} (33, 1) \xrightarrow{1} (83, 1) \xrightarrow{5} (13, 1),$$

which is why it previously appeared in the literature as $13 \to 33 \to 83 \to 13$. Note that $q = 3$ divides the determinant

$$\det(\mathcal{C}_{3,7}(1, 1, 5)) = \det \begin{pmatrix} 1 & 1 & 1 \\ 2^1 & 2^1 & 2^5 \\ 2^2 & 2^6 & 2^6 \end{pmatrix} = -2^3 \cdot 3^2 \cdot 5^2,$$

in accordance with the obvious analog to Theorem 1.

4. The fourth known cycle is a natural 3-cycle of index 7, with $q = 2^7 - 5^3 = 3$ and the associated indicial exponents $\{n_1, n_2, n_3\} = \{1, 3, 3\}$. After computing the cycle elements, this 3-cycle can be represented as

$$(51, 3) \xrightarrow{1} (129, 3) \xrightarrow{3} (81, 3) \xrightarrow{3} (51, 3).$$

After scalar division by $q = 3$, we obtain the completely reduced 3-cycle

$$(17, 1) \xrightarrow{1} (43, 1) \xrightarrow{3} (27, 1) \xrightarrow{3} (17, 1),$$

which is why it previously appeared in the literature as $17 \to 43 \to 27 \to 17$. Note that $q = 3$ divides the determinant $\det(\mathcal{C}_{3,7}(1, 3, 3)) = -2^6 \cdot 3^2$, in accordance with the obvious analog to Theorem 1.

Crandall [1] discovered a 2-cycle of index 15 for the "$181x + 1$" problem. With $N = 2$, $M = 15$, $q = 2^{15} - 181^2 = 7$, and the associated indicial exponents $\{n_1, n_2\} = \{3, 12\}$, the cycle elements are $r_1 = 2^3 + 181 = 189 = 7 \cdot 27$ and $r_2 = 2^{12} + 181 = 4277 = 7 \cdot 611$. Thus, this 2-cycle of index 15 has the representation

$$(189, 7) \xrightarrow{3} (4277, 7) \xrightarrow{12} (189, 7).$$

After scalar division by $q = 7$, we obtain the completely reduced 2-cycle of index 15

$$(27, 1) \xrightarrow{3} (611, 1) \xrightarrow{12} (27, 1),$$

which is why it previously appeared in the literature as $27 \to 611 \to 27$. Note that $2^{15} - 181^2 = 7$ divides the determinant

$$\det(\mathcal{C}_{2,15}(3, 12)) = 2^3 \cdot 7 \cdot 73,$$

in accordance with the obvious analog (with $a = 181$) to Theorem 1.

Remark. The essential observation here is that the cyclemaster matrix is the **same** for *all* of these problems. Thus, the cyclemaster matrix is an essential unifying feature in any approach to N-cycles for all of the "$ax + 1$" problems.

5. The Variable Cyclemaster Matrix and Algebraic Varieties

The *variable cyclemaster matrix* is the matrix that results by replacing 2^{n_i} with x_i in the cyclemaster matrix. For example,

$$\mathcal{V}_2(x_1, x_2) = \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix}, \qquad \mathcal{V}_3(x_1, x_2, x_3) = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1x_2 & x_2x_3 & x_3x_1 \end{pmatrix}$$

and

$$\mathcal{V}_4(x_1, x_2, x_3, x_4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1x_2 & x_2x_3 & x_3x_4 & x_4x_1 \\ x_1x_2x_3 & x_2x_3x_4 & x_3x_4x_1 & x_4x_1x_2 \end{pmatrix}.$$

In order to define $\mathcal{V}_N(x_1,\ldots,x_N)$ generally, we must introduce some notation for the partial products of the variables x_i in the given order. Let

$$u_{ij} = x_i x_{i+1} \cdots x_{i+j-1} = \prod_{k=0}^{j-1} x_{i+k}.$$

There are j terms in each of these products, starting with x_i. If the subscript $i+k > N$, then x_{i+k} is replaced by x_{i+k-N}; i.e., if $i+k > N$, the product wraps around to the beginning variables.

The variable cyclemaster matrix $\mathcal{V}_N(x_1,\ldots,x_N)$ is the $N \times N$ matrix whose ij^{th} entry is $v_{ij} = u_{j,i-1}$ for $i \geq 2$ and $v_{1j} = 1$.

The determinant of a variable cyclemaster matrix is a polynomial in N variables. Its degree is $N(N-1)/2$ and it enjoys the homogeneity property

$$\det\left(\mathcal{V}_N(dx_1,\ldots,dx_N)\right) = d^{N(N-1)/2} \det\left(\mathcal{V}_N(x_1,\ldots,x_N)\right).$$

For example,

$$\det\left(\mathcal{V}_2(x_1,x_2)\right) = x_2 - x_1,$$

$$\det\left(\mathcal{V}_3(x_1,x_2,x_3)\right) = 3x_1x_2x_3 - (x_1^1x_2^2 + x_2^1x_3^2 + x_3^1x_1^2),$$

and

$$\begin{aligned}\det\left(\mathcal{V}_4(x_1,x_2,x_3,x_4)\right) &= +2(x_1^1x_2^2x_3^1x_4^2 - x_1^2x_2^1x_3^2x_4^1)\\ &\quad + 4(x_1^1x_2^1x_3^2x_4^2 - x_1^1x_2^2x_3^2x_4^1 + x_1^2x_2^2x_3^1x_4^1 - x_1^2x_2^1x_3^1x_4^2)\\ &\quad + (x_1^1x_2^2x_3^3 - x_2^1x_3^2x_4^3 + x_3^1x_4^2x_1^3 - x_4^1x_1^2x_2^3).\end{aligned}$$

Note that the determinant of a variable cyclemaster matrix can vanish whenever the variables are periodic. For example, $\det\left(\mathcal{V}_4(x_1,x_2,x_1,x_2)\right) = 0$.

With the assignment $x_i = 2^{n_i}$, it follows that $x_1 \cdots x_N = 2^{n_1+\cdots+n_N} = 2^M$, so that

$$\frac{\det\left(\mathcal{C}_{N,M}(n_1,\ldots,n_N)\right)}{2^M - a^N} = \frac{\det\left(\mathcal{V}_N(x_1,x_2,\ldots,x_N)\right)}{x_1\cdots x_N - a^N}.$$

Remark. There are four relevant interpretations for this observation, if $a = 3$.

First, if the quotient is not an integer, then Theorem 1 implies that there does not exist an N-cycle of index M with this ordered set of indicial exponents for the original Collatz problem.

Second, if this quotient is an integer, then an N-cycle of index M may or may not exist for the original Collatz problem, as indicated by the examples presented in Section 1.

Third, if this quotient is not an integer, then the polynomial equation

$$P(x_1,\ldots,x_N) = \det\left(\mathcal{V}_N(x_1,x_2,\ldots,x_N)\right) - k(x_1\cdots x_N - a^N) = 0$$

does not have a solution in positive powers of 2 for any integer k.

Fourth, if this quotient has an integer value, then this polynomial equation has at least one solution for each such value. Some of these solutions may correspond to N-cycles of index M for the original Collatz problem, but some may not, again as indicated by the examples presented in Section 1.

An *algebraic variety* is the solution set for a multivariable polynomial equation or for a system of multivariable polynomial equations. Recently, there has been a great deal of activity in the theory of algebraic varieties. How many solutions to these polynomial equations or systems of polynomial equations are there? The answer to this question has consequences for the problem under consideration.

Of course, the polynomial equation above will always have the trival solution $x_1 = \cdots = x_N = a$, but there are non-trivial solutions as well. For a fixed value of M, the determinant of the cyclemaster matrix can only have a finite number of values corresponding to all choices of the indicial exponents. If a is large enough, the quotient cannot be an integer. Thus, there can only be a finite number of N-cycles of index M for all of the "$ax + 1$" problems.

Illustrative Examples.

1. If $k = -2$, $a = 3$, $N = 2$ and $M = 3$, then

$$P(2^1, 2^2) = \det\left(\mathcal{V}_2(2^1, 2^2)\right) + 2(2^3 - 3^2) = 2 + 2(-1) = 0.$$

This solution corresponds to the extremal natural 2-cycle of index 3

$$(5, -1) \xrightarrow{1} (7, -1) \xrightarrow{2} (5, -1)$$

for the "$3x + 1$" problem, as $q = -1$, $x_1 = 2^{n_1} = 2^1$ and $x_2 = 2^{n_2} = 2^2$.

2. If $k = 2^{25} \cdot 5 \cdot 79$, $a = 3$, $N = 7$ and $M = 11$, then

$$\begin{aligned} &P(2^1, 2^1, 2^1, 2^2, 2^1, 2^1, 2^4) \\ &= \det\left(\mathcal{V}_7(2^1, 2^1, 2^1, 2^2, 2^1, 2^1, 2^4)\right) - 2^{25} \cdot 5 \cdot 79(2^{11} - 3^7) \\ &= 0. \end{aligned}$$

This solution corresponds to the natural 7-cycle of index 11 given in Section 1 for the "$3x + 1$" problem, where $x_1 = x_2 = x_3 = x_5 = x_6 = 2^1$, $x_4 = 2^2$, and $x_7 = 2^4$.

3. If $k = 2^{17} \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 37^2$, $a = 3$, $N = 6$ and $M = 12$, then

$$\begin{aligned} &P(2^1, 2^1, 2^1, 2^6, 2^1, 2^2) \\ &= \det\left(\mathcal{V}_6(2^1, 2^1, 2^1, 2^6, 2^1, 2^2)\right) - 2^{17} \cdot 3^2 \cdot 13 \cdot 37(2^{12} - 3^6) \\ &= 0. \end{aligned}$$

This solution *does not correspond* to a 6-cycle of index 12 for the original Collatz problem.

4. If $k = 2$, $a = 5$, $N = 2$, and $M = 5$, then

$$P(2^1, 2^2) = \det\big(\mathcal{V}_2(2^1, 2^4)\big) - 2(2^5 - 5^2) = 14 - 2 \cdot 7 = 0.$$

This solution corresponds to the extremal natural 2-cycle of index 5

$$(7,7) \xrightarrow{1} (21,7) \xrightarrow{4} (7,7)$$

given in Section 4 for the “$5x+1$” problem, with $q = 7$, $x_1 = 2^{n_1} = 2^1$ and $x_2 = 2^{n_2} = 4$. After scalar division by 7, this 2-cycle can be reduced to the cycle

$$(1,1) \xrightarrow{1} (3,1) \xrightarrow{4} (1,1).$$

5. If $k = -600$, $a = 5$, $N = 3$ and $M = 7$, then

$$P(2^1, 2^1, 2^5) = \det\big(\mathcal{V}_3(2^1, 2^1, 2^5)\big) + 600(2^7 - 5^3) = -1800 + 600 \cdot 3 = 0.$$

This solution corresponds to the extremal natural 3-cycle of index 7

$$(39,3) \xrightarrow{1} (99,3) \xrightarrow{1} (249,3) \xrightarrow{5} (39,3)$$

given in Section 4 for the “$5x+1$” problem, with $q = 2^7 - 5^3 = 3$, $x_1 = 2^{n_1} = 2^1$, $x_2 = 2^{n_2} = 2^1$, and $x_3 = 2^{n_3} = 2^5$. After scalar division by 3, this 3-cycle can be reduced to the cycle

$$(13,1) \xrightarrow{1} (33,1) \xrightarrow{1} (83,1) \xrightarrow{5} (13,1).$$

6. If $k = -192$, $a = 5$, $N = 3$ and $M = 7$, then

$$P(2^1, 2^3, 2^3) = \det\big(\mathcal{V}_3(2^1, 2^3, 2^3)\big) + 192(2^7 - 5^3) = -576 + 192 \cdot 3 = 0.$$

This solution corresponds to the natural 3-cycle of index 7

$$(51,3) \xrightarrow{1} (129,3) \xrightarrow{3} (81,3) \xrightarrow{3} (51,3)$$

given in Section 4 for the “$5x+1$” problem, which can be reduced to the 3-cycle

$$(17,1) \xrightarrow{1} (43,1) \xrightarrow{3} (27,1) \xrightarrow{3} (17,1).$$

7. If $k = 2^3 \cdot 73$, $a = 181$, $N = 2$ and $M = 15$, then

$$P(2^3, 2^{12}) = \det\big(\mathcal{V}_2(2^3, 2^{12})\big) - 2^3 \cdot 73(2^{15} - 181^2) = 0.$$

This solution corresponds to the natural 2-cycle of index 15

$$(189, 7) \xrightarrow{3} (4277, 7) \xrightarrow{12} (189, 7)$$

given in Section 4 for the "$181x+1$" problem, where $q = 7$, $x_1 = 2^{n_1} = 2^3$ and $x_2 = 2^{n_2} = 2^{12}$. After scalar division by $q = 7$, this 2-cycle can be reduced to

$$(27, 1) \xrightarrow{3} (611, 1) \xrightarrow{12} (27, 1)$$

Remark. These seven examples illustrate how the solutions to the polynomial equation $P(x_1, \dots, x_N) = 0$ correspond in some cases to the existence of N-cycles for the "$ax + 1$" problems. Since the graph of this equation is a level surface in N-dimensional space, the existence of N-cycles for some of the "$ax + 1$" problems also corresponds in some cases to the existence of points on that level surface with coordinates that are positive powers of 2.

Another formulation in terms of a simple system of polynomial equations is possible.

If $\{(r_i, q) \colon i = 1, \dots, N\}$ is a natural N-cycle, then $r_i = g_{Ni}$ for all i. If this N-cycle is subject to scalar division by $q = 2^M - a^N$, thereby yielding an N-cycle in the row indexed by $q = 1$, then

$$\frac{g_{Ni}}{q} = \frac{\sum_{j=0}^{N-1} 2^{m_{i,N-j-1}} \cdot a^j}{2^M - a^N} = k_i$$

for all $i = 1, \dots, N$, where k_i is an odd integer.

If we replace 2^{n_i} with x_i in this quotient, then we obtain a variable version of these N requirements. For example, if $N = 3$, then the generators g_{Ni} become

$$\begin{aligned} h_{N1}(x_1, x_2, x_3) &= x_1x_2 + ax_1 + a^2 \\ h_{N2}(x_1, x_2, x_3) &= x_2x_3 + ax_2 + a^2 \\ h_{N3}(x_1, x_2, x_3) &= x_3x_1 + ax_3 + a^2 \end{aligned}$$

and $q(x_1, x_2, x_3) = x_1x_2x_3 - a^3$. It follows that the solutions to the system

$$\begin{aligned} P_1(x_1, x_2, x_3) &= x_1x_2 + ax_1 + a^2 - k_1(x_1x_2x_3 - a^3) = 0 \\ P_2(x_1, x_2, x_3) &= x_2x_3 + ax_2 + a^2 - k_2(x_1x_2x_3 - a^3) = 0 \\ P_3(x_1, x_2, x_3) &= x_3x_1 + ax_3 + a^2 - k_3(x_1x_2x_3 - a^3) = 0 \end{aligned}$$

in positive powers of 2 give rise to candidates for indicial exponents which can be used to create 3-cycles in some cases for an "$ax+1$" problem, if there are any. The graphs of these equations are level surfaces in $\mathbb{R}^3$. The intersections of all of these surfaces may give rise to points in $\mathbb{R}^3$ which belong simultaneously to all three level surfaces. The coordinates of these points of intersection are positive powers of 2, if they correspond to 3-cycles. There may be other points of intersections that are not positive powers of 2.

6. Further Research

There are several promising areas for further research.

1. A *Mersenne prime* is a prime number of the form $2^M - 1$. Mersenne primes have been generalized in various ways. Given the results presented in Corollary 1 and Corollary 2, it is of independent interest to study prime numbers of the form $2^M - 3^N$, when M and N are relatively prime with $N \geq 1$.

2. It was noted in Section 1 that if the set $\{n_1, \ldots, n_N\}$ has period D, where D is a proper divisor of N, then $\det(\mathcal{C}_{N,M}(n_1, \ldots, n_N)) = 0$. The converse does not hold. This determinant vanishes if $N = 4$, $M = 4n_1$, $n_1 = n_3$, and $2n_1 = n_2 + n_4$. For example, $\det(\mathcal{C}_{4,12}(3, 2, 3, 4)) = 0$. It follows from Theorem 15 that the determinant of the cyclemaster matrix does not vanish if the indicial exponents correspond to an extremal N-cycle with $M > N$. What can be said in general?

3. The non-existence of N-cycles of index M for all "$ax + 1$" problems depends upon the divisibility properties of $\det(\mathcal{C}_{N,M}(n_1, \ldots, n_N))$ by the cycle indicator $2^M - a^N$. Since the divisibility properties of this determinant depend completely upon the composition $n_1 + \cdots + n_N = M$, a study of the divisibility properties of this determinant can be conducted *independently*.

In addition, the cyclemaster matrix has interesting properties in its own right which deserve attention. For instance, the product of all elements in the second row of the cyclemaster matrix is 2^M; the product of all elements in the third row is 2^{2M}, and so forth.

4. Let

$$\mathcal{N}_{N,M} = \{\{n_1, \ldots, n_N\} : n_i \geq 1 \text{ and } n_1 + \cdots + n_N = M\}$$

denote the set of all indicial exponents for all N-cycles of index M.

For $2 \leq N < M$, let

$$\delta_{N,M} = \min_{\mathcal{N}_{N,M}} |\det(\mathcal{C}_{N,M}(n_1, \ldots, n_N))| .$$

If $n_i = a$ for all $i = 1, \ldots, N$, then $M = aN$ and $\delta_{N,aN} = 0$. However, if $\delta_{N,M} > 0$, then $\delta_{N,M} \geq 2^{N(N-1)/2}$. To see this, simply note that a factor of 2^1 can be factored from the second row of the cyclemaster matrix; a factor of 2^2 from the third row, etc. All of these factors result in an aggregate factor of $2^{1+2+\cdots+(N-1)} = 2^{N(N-1)/2}$ being removed. By Theorem 15, equality holds for every extremal N-cycle of index M for which $M = N + 1$. Determine $\delta_{N,M}$ for all values of N and M.

For $2 \leq N < M$, let

$$\Delta_{N,M} = \max_{\mathcal{N}_{N,M}} |\det(\mathcal{C}_{N,M}(n_1, \ldots, n_N))| .$$

It was mentioned in Section 4 that the cyclemaster matrix is the same for all of the "$ax+1$" problems. If a is so large that $|2^M - a^N| > \Delta_{N,M}$, then complete scalar division of any N-cycle of index M is not possible. For smaller values of a, complete scalar division may be possible. It is for this reason that it is important to determine $\Delta_{N,M}$ in terms of N and M only.

5. The problem of determining $\Delta_{N,M}$ is clearly equivalent to the problem of determining the set(s) of indicial exponents in $\mathcal{N}_{N,M}$ for which the maximum is attained. Consequently, maps from $\mathcal{N}_{N,M}$ to $\mathcal{N}_{N,M}$ are of interest.

It was noted previously that the *rotation maps*

$$\{n_1, \ldots, n_N\} \to \{n_2, \ldots, n_N, n_1\} \quad \text{and} \quad \{n_1, \ldots, n_N\} \to \{n_N, n_1, \ldots, n_{N-1}\}$$

do not generate different N-cycles, but merely relabel existing N-cycles. Also, the magnitude of the determinant of the corresponding N-cycles does not change, but its sign may change.

However, *transposition maps*, that is maps of the form

$$\{n_1, \ldots, n_i, n_{i+1}, \ldots, n_N\} \to \{n_1, \ldots, n_{i+1}, n_i, \ldots, n_N\}$$

generate different N-cycles and the determinant of the cyclemaster matrix can change in magnitude as a result. Due to the action of the rotation maps, it is sufficient only to consider the action of transposition maps of the form

$$\{n_1, \ldots, n_{N-1}, n_N\} \to \{n_1, \ldots, n_N, n_{N-1}\}.$$

When does the action of a transposition map lead to an increase (decrease) in the value of the determinant of the corresponding cyclemaster matrix?

Adjacency maps, that is maps of the form

$$\{n_1, \ldots, n_i, n_{i+1}, \ldots, n_N\} \to \{n_1, \ldots, n_i + 1, n_{i+1} - 1, \ldots, n_N\}$$

and

$$\{n_1, \ldots, n_i, n_{i+1}, \ldots, n_N\} \to \{n_1, \ldots, n_i - 1, n_{i+1} + 1, \ldots, n_N\},$$

generate different N-cycles, and the determinant of the cyclemaster matrix can change in magnitude as a result. It is sufficient to consider adjacency maps of the form

$$\{n_1, \ldots, n_{N-1}, n_N\} \to \{n_1, \ldots, n_{N-1} + 1, n_N - 1\}$$

or

$$\{n_1, \ldots, n_{N-1}, n_N\} \to \{n_1, \ldots, n_{N-1} - 1, n_N + 1\},$$

due to the action of rotation maps. When does the action of an adjacency map lead to an increase (decrease) in the value of the determinant of the corresponding cyclemaster matrix?

6. All of the elements of $\mathcal{N}_{N,M}$ can be considered to be points in the N-dimensional plane $x_1 + \cdots + x_N = M$. Every element of $\mathcal{N}_{N,M}$ belongs to the closed convex hull $\overline{co}(\mathcal{N}_{N,M})$ of $\mathcal{N}_{N,M}$ in that plane; thus, all of its elements can be written as convex combinations of its extreme points. Also, every element of $\mathcal{N}_{N,M}$ is the image of the element $\{1, \ldots, 1, M-N-1\}$ under a composition of the maps defined above. Are the points $\{1, \ldots, 1, M-N-1\}$ and all of its rotations the extreme points of $\overline{co}(\mathcal{N}_{N,M})$? Is the maximum value $\Delta_{N,M}$ attained there?

7. It was observed in Section 3.5.2 that if $N = 2$ or $N = 3$, then an N-cycle which is subject to scalar division by p^α can be constructed, where p^α is a prime power divisor of $2^M - 3^N$ and 2 is a primitive root of p^α. Is this construction possible for all $N \geq 4$? Do primitive roots always play a role in the construction of N-cycles?

8. How do the indicial exponents $\{n_1, \ldots, n_N\}$ determine the factorization of $\det(\mathcal{C}_{N,M}(n_1, \ldots, n_N))$ into prime powers?

Acknowledgements. The author would like to thank the referee for providing an extremely painstaking review of this paper. I am also grateful to Professor Jeffrey C. Lagarias for providing some very helpful comments regarding an earlier draft of this paper. The author would also like to acknowledge Emeritus Professor Leonard K. Jones for the example provided in Section 1. *Mathematica* was used extensively to compute cycles and determinants.

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