ON $(4,5)$-REGULAR PARTITIONS WITH ODD PARTS OVERLINED

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Abstract

Mahadeva Naika and Harishkumar defined $\overline{a}_5(n)$, the number of 5-regular partitions with the first occurrence of an odd number overlined. They proved many infinite families of congruences modulo powers of 2 for $\overline{a}_5(n)$. Let $\overline{a}_{4,5}(n)$ denote the number of $(4,5)$-regular partitions of n with the first occurrence of an odd number overlined. In this paper, we establish many infinite families of congruences modulo powers of 2 for $\overline{a}_{4,5}(n)$. For example, for all $n \geq 0$ and $\beta \geq 0$,

$$\overline{a}_{4,5}\left(16 \cdot 5^{2\beta+2}n + \frac{v_1 \cdot 5^{2\beta+1}+1}{3}\right) \equiv 0 \pmod{16},$$

where $v_1 \in \{46, 94\}$.

1. Introduction

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. For positive integer $\ell > 1$, a partition is an *ℓ-regular partition* of n if none of the parts are divisible by ℓ. Let $p_\ell(n)$ denote the number of ℓ-regular partitions of n with $p_\ell(0) = 1$; the generating function for $p_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} p_\ell(n) q^n = \frac{f_\ell}{f_1},$$

where

$$f_1 := (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$$

and

$$f_\ell := (q^\ell;q^\ell)_\infty = \prod_{m=1}^{\infty} (1-q^{m\ell}).$$

For more information about $p_\ell(n)$, one can see [3, 6, 15].

If $\ell, m > 1$, a partition is an (ℓ, m)-*regular partition* of a positive integer n if none of the parts are divisible by ℓ or m. For example, the $(4,5)$-regular partitions of 5 are

$$3+2,\ \ 3+1+1,\ \ 2+2+1,\ \ 2+1+1+1,\ \ 1+1+1+1+1.$$

In 2004, Corteel and Lovejoy [5] introduced overpartitions. An *overpartition* of a non-negative integer n is a non-increasing sequence of natural numbers whose sum is n, where the first occurrence of parts of each size may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n with $\overline{p}(0) = 1$; the generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots. \tag{1}$$

An extensive study on the overpartition function $\overline{p}(n)$ can be found in the work of Corteel and Lovejoy [5]. Later, Hirschhorn and Sellers [9] proved a number of arithmetic relations satisfied by $\overline{p}(n)$ and also obtained many Ramanujan-type congruences modulo powers of 2 for $\overline{p}(n)$. For more details about $\overline{p}(n)$, one can see [1, 10, 12, 13, 17, 20, 21].

Hirschhorn and Sellers [11] defined the partition function $\overline{p_o}(n)$, the number of overpartitions of n into odd parts. The generating function for $\overline{p_o}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p_o}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^{2n+1}}{1-q^{2n-1}} = 1 + 2q + 2q^2 + 4q^3 + 6q^4 + \cdots. \tag{2}$$

They proved a number of arithmetic results including several Ramanujan-type congruences satisfied by $\overline{p_o}(n)$ and some easily-stated characterizations of $\overline{p_o}(n)$ modulo small powers of 2. For example, for all $n \geq 1$,

$$\overline{p_o}(n) \equiv \begin{cases} 2 \pmod 4 & \textit{if } n \textit{ is square or } n \textit{ is twice a square,} \\ 0 \pmod 4 & \textit{otherwise.} \end{cases} \tag{3}$$

For more details about $\overline{p_o}(n)$, one can see [4, 19].

In [14], the authors defined $\overline{a}_5(n)$, the number of 5-regular partitions of n with the first occurrence of an odd number overlined and they obtained many infinite families of congruences modulo powers of 2 for $\overline{a}_5(n)$. For example, for all $n \geq 0$ and $\beta \geq 0$,

$$\overline{a}_5\left(16 \cdot 5^{2\beta+1}n + \frac{k_1 \cdot 5^{2\beta} - 1}{3}\right) \equiv 0 \pmod{16},$$

where $k_1 \in \{142, 238\}$.

By the motivation of the above work, in this paper, we define $\overline{a}_{4,5}(n)$, the number of $(4,5)$-regular partitions of n with the first occurrence of an odd number overlined. The generating function for $\overline{a}_{4,5}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}(n)\, q^n = \frac{f_2^2 f_5^2}{f_1^2 f_{10}^2}. \tag{4}$$

For example, $(4,5)$-regular partitions of 5 with the first occurrence of an odd number overlined are

$$3+2,\ \overline{3}+2,\ 3+1+1,\ \overline{3}+1+1,\ 3+\overline{1}+1,\ \overline{3}+\overline{1}+1,\ 2+2+1,\ 2+2+\overline{1},$$
$$2+1+1+1,\ 2+\overline{1}+1+1,\ 1+1+1+1+1,\ \overline{1}+1+1+1+1.$$

Also, we establish many infinite families of congruences modulo powers of 2 for $\overline{a}_{4,5}(n)$. For example, for all $n \geq 0$ and $\beta \geq 0$,

$$\overline{a}_{4,5}\left(16 \cdot 5^{2\beta+2}n + \frac{v_1 \cdot 5^{2\beta+1} + 1}{3}\right) \equiv 0 \pmod{16},$$

where $v_1 \in \{46, 94\}$.

2. Preliminary Results

In this section, we record some identities which are useful in proving our main results.

Lemma 1. *The following 2-dissection holds:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{5}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{6}$$

The identity (5) is the 2-dissection of $\phi(-q)$ [8, 1.9.4]. The identity (6) is the 2-dissection of $\phi(-q)^2$ [8, 1.10.1]. Also, one can see [2, p.40].

Lemma 2. *The following 2-dissections hold:*

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2} \tag{7}$$

and

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{8}$$

Equation (7) was proved by Hirschhorn and Sellers [7]; see also [18]. Replacing q by $-q$ in (7) and using the fact that $(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}$, we obtain (8).

Lemma 3. *We have*

$$\frac{1}{f_1^3 f_5} = \frac{f_4^4}{f_2^7 f_{10}} - 2q\frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q\frac{f_4^3 f_{20}}{f_2^8} + 2q^2\frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}, \tag{9}$$

$$f_1 f_5^3 = f_2^3 f_{10} - q\frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3\frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2}, \tag{10}$$

$$\frac{1}{f_1 f_5^3} = \frac{f_4 f_{20}^3}{f_{10}^8} + q\frac{f_{20}^4}{f_2 f_{10}^7} + 2q^2\frac{f_4^2 f_{20}^6}{f_2^3 f_{10}^9} + 2q^3\frac{f_4^5 f_{20}^3 f_{40}^2}{f_2^4 f_8^2 f_{10}^8} \tag{11}$$

and

$$f_1^3 f_5 = \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3 + 2q^2\frac{f_4^6 f_{10} f_{40}^2}{f_2 f_8^2 f_{20}^2}. \tag{12}$$

For proofs, see [16].

Lemma 4. *[8, p. 85, 8.1.1] We have the following 5-dissection formula*

$$f_1 = f_{25}(a(q^5) - q - q^2/a(q^5)), \tag{13}$$

where

$$a := a(q) := \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}. \tag{14}$$

Lemma 5. *For any positive integers k and m, we have*

$$f_k^{2m} \equiv f_{2k}^m \pmod 2, \tag{15}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod 4 \tag{16}$$

and

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod 8. \tag{17}$$

3. Congruences for $\overline{a}_{4,5}(n)$

In this section, we prove many infinite families of congruences modulo powers of 2 for $\overline{a}_{4,5}(n)$.

Theorem 1. *Let $v_1 \in \{46, 94\}, v_2 \in \{92, 188\}, v_3 \in \{368, 752\}$ and $v_4 \in \{184, 376\}$. Then for all $n \geq 0$ and $\alpha, \beta \geq 0$, we have, modulo* 16,

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16 \cdot 5^{2\beta} n + \frac{38 \cdot 5^{2\beta} + 1}{3}\right) q^n \equiv 8 f_2^7 f_5, \tag{18}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16 \cdot 5^{2\beta+1} n + \frac{46 \cdot 5^{2\beta+1} + 1}{3}\right) q^n \equiv 8 q^2 f_1 f_{10}^7, \tag{19}$$

$$\overline{a}_{4,5}\left(16 \cdot 5^{2\beta+2} n + \frac{v_1 \cdot 5^{2\beta+1} + 1}{3}\right) \equiv 0, \tag{20}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32 \cdot 5^{2\beta} n + \frac{92 \cdot 5^{2\beta} + 1}{3}\right) q^n \equiv 8 q^2 f_1 f_{10}^7, \tag{21}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32 \cdot 5^{2\beta+1} n + \frac{76 \cdot 5^{2\beta+1} + 1}{3}\right) q^n \equiv 8 f_2^7 f_5, \tag{22}$$

$$\overline{a}_{4,5}\left(32 \cdot 5^{2\beta+1} n + \frac{v_2 \cdot 5^{2\beta} + 1}{3}\right) \equiv 0, \tag{23}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(128 \cdot 5^{2\beta} n + \frac{368 \cdot 5^{2\beta} + 1}{3}\right) q^n \equiv 8 q^2 f_1 f_{10}^7, \tag{24}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(128 \cdot 5^{2\beta+1} n + \frac{304 \cdot 5^{2\beta+1} + 1}{3}\right) q^n \equiv 8 f_2^7 f_5, \tag{25}$$

$$\overline{a}_{4,5}\left(128 \cdot 5^{2\beta+1} n + \frac{v_3 \cdot 5^{2\beta} + 1}{3}\right) \equiv 0, \tag{26}$$

$$\overline{a}_{4,5}\left(2^{4\alpha+6} n + \frac{2^{4\alpha+7} + 1}{3}\right) \equiv \overline{a}_{4,5}\left(4n + 3\right), \tag{27}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(64 \cdot 5^{2\beta} n + \frac{152 \cdot 5^{2\beta} + 1}{3}\right) q^n \equiv 8 f_2^7 f_5, \tag{28}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(64 \cdot 5^{2\beta+1} n + \frac{184 \cdot 5^{2\beta+1} + 1}{3}\right) q^n \equiv 8 q^2 f_1 f_{10}^7, \tag{29}$$

$$\overline{a}_{4,5}\left(64 \cdot 5^{2\beta+2} n + \frac{v_4 \cdot 5^{2\beta+1} + 1}{3}\right) \equiv 0. \tag{30}$$

Proof. Employing (8) in (4) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(2n+1)\,q^n = 2\frac{f_2^3 f_{10}}{f_1^3 f_5}. \tag{31}$$

Using (9) in (31) and then collecting the even and odd terms from both sides, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(4n+1)\,q^n = 2\frac{f_2^4}{f_1^4} + 4q\frac{f_2^9 f_{20}^2}{f_1^7 f_4^2 f_5 f_{10}} \tag{32}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(4n+3)\,q^n = 10\frac{f_2^3 f_5 f_{10}}{f_1^5} - 4\frac{f_2^6 f_{10}^2}{f_1^6 f_5^2}. \tag{33}$$

Invoking (16) and (17) in (32) and (33), we get, modulo 16,

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(4n+1)\,q^n \equiv 2f_1^4 + 4q\frac{f_1 f_2 f_{10}^3}{f_5} \tag{34}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(4n+3)\,q^n \equiv 10\frac{f_1^3 f_5 f_{10}}{f_2} + 12\frac{f_2^4 f_5^2}{f_1^2}. \tag{35}$$

Using (6) and (7) in (34), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(8n+1)\,q^n \equiv 2\frac{f_2^2}{f_1^2} + 12q\frac{f_1 f_2^2 f_5 f_{20}}{f_4} \tag{36}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(8n+5)\,q^n \equiv 4\frac{f_1^2 f_4 f_{10}^3}{f_2 f_{20}} + 8f_2^7. \tag{37}$$

Substituting (5) in (37), we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(16n+5)\,q^n \equiv 4\frac{f_4 f_5^3}{f_2 f_{10}} + 8f_1^7 \tag{38}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(16n+13)\,q^n \equiv 8f_2^7 f_5. \tag{39}$$

Congruence (39) is the $\beta = 0$ case of (18). Suppose that Congruence (18) is true for $\beta \geq 0$ and using (13) in (18), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16\cdot 5^{2\beta+1}n + \frac{46\cdot 5^{2\beta+1}+1}{3}\right) q^n \equiv 8q^2 f_1 f_{10}^7. \tag{40}$$

Utilizing (13) in (40) and then collecting the coefficients of q^{5n+3} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16 \cdot 5^{2\beta+2}n + \frac{38 \cdot 5^{2\beta+2}+1}{3}\right) q^n \equiv 8f_2^7 f_5, \tag{41}$$

which implies that Congruence (18) is true for $\beta+1$. Hence, by induction, Congruence (18) holds for all integers $\beta \geq 0$.

Using (13) in (18) and then extracting the coefficients of q^{5n+4} from both sides of the resultant equation, we get (19).

From Congruence (19) along with (13), we obtain (20).

Substituting (8) and (12) in (35), we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(8n+3)\, q^n \equiv 10\frac{f_1 f_2 f_5^3}{f_{10}} + 12f_2^4 \tag{42}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(8n+7)\, q^n \equiv 12\frac{f_2^3 f_5 f_{10}}{f_1} + 14f_5^4. \tag{43}$$

Substituting (6) and (8) in (43), we have

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(16n+7)\, q^n \equiv 12\frac{f_1 f_4 f_5 f_{10}^2}{f_{20}} + 14\frac{f_{10}^2}{f_5^2} \tag{44}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(16n+15)\, q^n \equiv 12\frac{f_2^3 f_5^2 f_{20}}{f_4 f_{10}} + 8q^2 f_{10}^7. \tag{45}$$

Employing (5) in (45), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+15)\, q^n \equiv 12\frac{f_1^3 f_{20}}{f_2 f_{10}} + 8qf_5^7 \tag{46}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+31)\, q^n \equiv 8q^2 f_1 f_{10}^7. \tag{47}$$

Congruence (47) is the $\beta = 0$ case of (21). Suppose that Congruence (21) is true for $\beta \geq 0$, we have

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32 \cdot 5^{2\beta}n + \frac{92 \cdot 5^{2\beta}+1}{3}\right) q^n \equiv 8q^2 f_1 f_{10}^7. \tag{48}$$

Employing (13) in (48) and then collecting the coefficients of q^{5n+3} from both sides, we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32\cdot 5^{2\beta+1}n+\frac{76\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 8f_2^7 f_5. \tag{49}$$

Again, using (13) in (49) and then comparing the coefficients of q^{5n+4} on both sides, we have

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32\cdot 5^{2\beta+2}n+\frac{92\cdot 5^{2\beta+2}+1}{3}\right)q^n \equiv 8q^2 f_1 f_{10}^7, \tag{50}$$

which implies that Congruence (21) is true for $\beta+1$. Hence, by mathematical induction, Congruence (21) holds for all integers $\beta \geq 0$.

Employing (13) in (21) and then collecting the coefficients of q^{5n+3} from both sides of the resultant equation, we obtain (22).

Employing (13) in (21) and then comparing the coefficients of q^{5n+i} for $i=0,1$ on both sides of the resultant equation, we get (23).

Employing (10) in (42), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(16n+3)\,q^n \equiv 6f_1^4 + 4q\frac{f_1 f_2 f_{10}^3}{f_5} \tag{51}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(16n+11)\,q^n \equiv 6\frac{f_1^3 f_5 f_{10}}{f_2} + 12q f_{20}^2. \tag{52}$$

Utilizing (12) in (52), we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+11)\,q^n \equiv 6\frac{f_1 f_2 f_5^3}{f_{10}} + 12q\frac{f_1^2 f_{20}^2}{f_5^2} \tag{53}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+27)\,q^n \equiv 12\frac{f_2^3 f_5 f_{10}}{f_1} + 14f_5^4. \tag{54}$$

Employing (6) and (8) in (54), we have

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+27)\,q^n \equiv 12\frac{f_1 f_4 f_5 f_{10}^2}{f_{20}} + 14\frac{f_{10}^2}{f_5^2} \tag{55}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+59)\,q^n \equiv 12\frac{f_2^3 f_5^2 f_{20}}{f_4 f_{10}} + 8q^2 f_{10}^7. \tag{56}$$

Using (5) in (56), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(128n+59)\,q^n \equiv 12\frac{f_1^3 f_{20}}{f_2 f_{10}} + 8q f_5^7 \tag{57}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(128n+123)\,q^n \equiv 8q^2 f_1 f_{10}^7. \tag{58}$$

Equation (58) is the $\beta = 0$ case of (24). The rest of the proofs of the identities (24)-(26) are similar to the proofs of the identities (21)-(23). So, we omit the details.

Using (7) and (10) in (53) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+43)\,q^n \equiv 10\frac{f_1^3 f_5 f_{10}}{f_2} + 12\frac{f_2^4 f_5^2}{f_1^2}. \tag{59}$$

In view of Congruences (35) and (59), we see that

$$\overline{a}_{4,5}(64n+43) \equiv \overline{a}_{4,5}(4n+3). \tag{60}$$

By induction on α, we arrive at (27).

Employing (6) and (7) in (51), we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+3)\,q^n \equiv 6\frac{f_2^2}{f_1^2} + 12q\frac{f_1 f_2^2 f_5 f_{20}}{f_4} \tag{61}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+19)\,q^n \equiv 8f_2^7 + 4\frac{f_1^2 f_4 f_{10}^3}{f_2 f_{20}}. \tag{62}$$

Utilizing (5) in (62), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+19)\,q^n \equiv 8f_1^7 + 4\frac{f_4 f_5^3}{f_2 f_{10}} \tag{63}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+51)\,q^n \equiv 8f_2^7 f_5. \tag{64}$$

Congruence (64) is the $\beta = 0$ case of (28). The rest of the proofs of the identities (28)-(30) are similar to the proofs of the identities (18)-(20). So, we omit the details. $\square$

Theorem 2. *Let* $v_5 \in \{62, 158\}, v_6 \in \{166, 214\}, v_7 \in \{82, 178\}, v_8 \in \{332, 428\}$, $v_9 \in \{124, 316\}$, $v_{10} \in \{164, 356\}$, $v_{11} \in \{1328, 1712\}$, $v_{12} \in \{496, 1264\}$, $v_{13} \in \{656, 1424\}$, $v_{14} \in \{248, 632\}$, $v_{15} \in \{664, 856\}$ *and* $v_{16} \in \{328, 712\}$. *Then for all* $n \geq 0$ *and* $\beta \geq 0$, *we have, modulo* 8,

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16\cdot 5^{2\beta}n + \frac{14\cdot 5^{2\beta}+1}{3}\right) q^n \equiv 4f_2 f_5, \tag{65}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16\cdot 5^{2\beta+1}n+\frac{22\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 4f_1f_{10}, \tag{66}$$

$$\overline{a}_{4,5}\left(16\cdot 5^{2\beta+1}n+\frac{v_5\cdot 5^{2\beta}+1}{3}\right) \equiv 0, \tag{67}$$

$$\overline{a}_{4,5}\left(16\cdot 5^{2\beta+2}n+\frac{v_6\cdot 5^{2\beta+1}+1}{3}\right) \equiv 0, \tag{68}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16\cdot 5^{2\beta}n+\frac{26\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 4f_1^{13}+4f_1^3f_{10}, \tag{69}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16\cdot 5^{2\beta+1}n+\frac{34\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 4f_2f_5^3+4q^2f_5^{13}, \tag{70}$$

$$\overline{a}_{4,5}\left(16\cdot 5^{2\beta+2}n+\frac{v_7\cdot 5^{2\beta+1}+1}{3}\right) \equiv 0, \tag{71}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32\cdot 5^{2\beta}n+\frac{44\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 4f_1f_{10}, \tag{72}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32\cdot 5^{2\beta+1}n+\frac{28\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 4f_2f_5, \tag{73}$$

$$\overline{a}_{4,5}\left(32\cdot 5^{2\beta+1}n+\frac{v_8\cdot 5^{2\beta}+1}{3}\right) \equiv 0, \tag{74}$$

$$\overline{a}_{4,5}\left(32\cdot 5^{2\beta+2}n+\frac{v_9\cdot 5^{2\beta+1}+1}{3}\right) \equiv 0, \tag{75}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32\cdot 5^{2\beta}n+\frac{68\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 4f_2f_5^3+4q^2f_5^{13}, \tag{76}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32\cdot 5^{2\beta+1}n+\frac{52\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 4f_1^3f_{10}+4f_1^{13}, \tag{77}$$

$$\overline{a}_{4,5}\left(32\cdot 5^{2\beta+1}n+\frac{v_{10}\cdot 5^{2\beta}+1}{3}\right) \equiv 0, \tag{78}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(128\cdot 5^{2\beta}n+\frac{176\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 4f_1f_{10}, \tag{79}$$

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(128\cdot 5^{2\beta+1}n+\frac{112\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 4f_2f_5, \tag{80}$$

$$\overline{a}_{4,5}\left(128\cdot 5^{2\beta+1}n+\frac{v_{11}\cdot 5^{2\beta}+1}{3}\right) \equiv 0, \tag{81}$$

$$\overline{a}_{4,5}\left(128\cdot 5^{2\beta+2}n+\frac{v_{12}\cdot 5^{2\beta+1}+1}{3}\right)\equiv 0, \tag{82}$$

$$\sum_{n=0}^{\infty}\overline{a}_{4,5}\left(128\cdot 5^{2\beta}n+\frac{272\cdot 5^{2\beta}+1}{3}\right)q^n\equiv 4f_2f_5^3+4q^2f_5^{13}, \tag{83}$$

$$\sum_{n=0}^{\infty}\overline{a}_{4,5}\left(128\cdot 5^{2\beta+1}n+\frac{208\cdot 5^{2\beta+1}+1}{3}\right)q^n\equiv 4f_1^3f_{10}+4f_1^{13}, \tag{84}$$

$$\overline{a}_{4,5}\left(128\cdot 5^{2\beta+1}n+\frac{v_{13}\cdot 5^{2\beta}+1}{3}\right)\equiv 0, \tag{85}$$

$$\sum_{n=0}^{\infty}\overline{a}_{4,5}\left(64\cdot 5^{2\beta}n+\frac{56\cdot 5^{2\beta}+1}{3}\right)q^n\equiv 4f_2f_5, \tag{86}$$

$$\sum_{n=0}^{\infty}\overline{a}_{4,5}\left(64\cdot 5^{2\beta+1}n+\frac{88\cdot 5^{2\beta+1}+1}{3}\right)q^n\equiv 4f_1f_{10}, \tag{87}$$

$$\overline{a}_{4,5}\left(64\cdot 5^{2\beta+1}n+\frac{v_{14}\cdot 5^{2\beta}+1}{3}\right)\equiv 0, \tag{88}$$

$$\overline{a}_{4,5}\left(64\cdot 5^{2\beta+2}n+\frac{v_{15}\cdot 5^{2\beta+1}+1}{3}\right)\equiv 0, \tag{89}$$

$$\sum_{n=0}^{\infty}\overline{a}_{4,5}\left(64\cdot 5^{2\beta}n+\frac{104\cdot 5^{2\beta}+1}{3}\right)q^n\equiv 4f_1^{13}+4f_1^3f_{10}, \tag{90}$$

$$\sum_{n=0}^{\infty}\overline{a}_{4,5}\left(64\cdot 5^{2\beta+1}n+\frac{136\cdot 5^{2\beta+1}+1}{3}\right)q^n\equiv 4f_2f_5^3+4q^2f_5^{13}, \tag{91}$$

$$\overline{a}_{4,5}\left(64\cdot 5^{2\beta+2}n+\frac{v_{16}\cdot 5^{2\beta+1}+1}{3}\right)\equiv 0. \tag{92}$$

Proof. From Equation (38), we have, modulo 8,

$$\sum_{n=0}^{\infty}\overline{a}_{4,5}\left(16n+5\right)q^n\equiv 4f_2f_5, \tag{93}$$

which is the $\beta=0$ case of (65). Suppose that Congruence (65) is true for $\beta\geq 0$ and utilizing (13) in (65), we find that

$$\sum_{n=0}^{\infty}\overline{a}_{4,5}\left(16\cdot 5^{2\beta+1}n+\frac{22\cdot 5^{2\beta+1}+1}{3}\right)q^n\equiv 4f_1f_{10}. \tag{94}$$

Substituting (13) in (94) and then collecting the coefficients of q^{5n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16\cdot 5^{2\beta+2}n+\frac{14\cdot 5^{2\beta+2}+1}{3}\right)q^n \equiv 4f_2f_5, \tag{95}$$

which implies that Congruence (65) is true for $\beta+1$. So, by induction, Congruence (65) holds for all integers $\beta\geq 0$.

From Congruence (65) along with (13), we obtain (66) and (67).

From Equation (66) along with (13), we arrive at (68).

Equation (36) becomes

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(8n+1\right)q^n \equiv 2f_1^2+4qf_1f_5^3f_{10}. \tag{96}$$

Using (5) and (10) in (96), we have

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16n+1\right)q^n \equiv 2\frac{f_1f_4}{f_2^2}+4qf_5^5 \tag{97}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16n+9\right)q^n \equiv 4f_1^{13}+4f_1^3f_{10}. \tag{98}$$

Equation (98) is the $\beta=0$ case of (69). Suppose that Congruence (69) is true for $\beta\geq 0$. Employing (13) in (69), we find that

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16\cdot 5^{2\beta+1}n+\frac{34\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 4f_2f_5^3+4q^2f_5^{13}. \tag{99}$$

Utilizing (13) in (99) and then comparing the coefficients of q^{5n+2} on both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(16\cdot 5^{2\beta+2}n+\frac{26\cdot 5^{2\beta+2}+1}{3}\right)q^n \equiv 4f_1^{13}+4f_1^3f_{10}, \tag{100}$$

which implies that Congruence (69) is true for $\beta+1$. So, by induction, Congruence (69) holds for all integers $\beta\geq 0$.

Utilizing (13) in (69) and then extracting the terms involving q^{5n+3} from both sides of the resultant equation, we obtain (70).

From Equation (70) along with (13), we arrive at (71).

Equation (46) reduce to

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32n+15\right)q^n \equiv 4f_1f_{10}, \tag{101}$$

which is the $\beta = 0$ case of (72). Suppose that Congruence (72) is true for $\beta \geq 0$ and utilizing (13) in (72), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32 \cdot 5^{2\beta+1}n + \frac{28 \cdot 5^{2\beta+1}+1}{3}\right) q^n \equiv 4f_2 f_5. \tag{102}$$

Employing (13) in (102) and then extracting the coefficients of q^{5n+2} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32 \cdot 5^{2\beta+2}n + \frac{44 \cdot 5^{2\beta+2}+1}{3}\right) q^n \equiv 4f_1 f_{10}, \tag{103}$$

which implies that Congruence (72) is true for $\beta + 1$. Hence, by mathematical induction, Congruence (72) holds for all integers $\beta \geq 0$.

Using (13) in (72), we obtain (73) and (74).

Employing (13) in (73) and then comparing the coefficients of q^{5n+i} for $i = 1, 3$ on both sides of the resultant equation, we get (75).

Equation (44) reduces to

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(16n+7)\, q^n \equiv 4f_1^3 f_2 f_5 + 6f_5^2. \tag{104}$$

Substituting (5) and (12) in (104), we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+7)\, q^n \equiv 4f_1^5 + 6\frac{f_5 f_{20}}{f_{10}^2} \tag{105}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+23)\, q^n \equiv 4f_2 f_5^3 + 4q^2 f_5^{13}. \tag{106}$$

Equation (106) is the $\beta = 0$ case of (76). Suppose that Congruence (76) is true for $\beta \geq 0$ and employing (13) in (76), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32 \cdot 5^{2\beta+1}n + \frac{52 \cdot 5^{2\beta+1}+1}{3}\right) q^n \equiv 4f_1^3 f_{10} + 4f_1^{13}. \tag{107}$$

Substituting (13) in (107) and then collecting the coefficients of q^{5n+3} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\left(32 \cdot 5^{2\beta+2}n + \frac{68 \cdot 5^{2\beta+2}+1}{3}\right) q^n \equiv 4f_2 f_5^3 + 4q^2 f_5^{13}, \tag{108}$$

which implies that Congruence (76) is true for $\beta + 1$. Hence, by induction, Congruence (76) holds for all integer $\beta \geq 0$.

Employing (13) in (76), we obtain (77) and (78).

Equation (57) becomes

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(128n+59)\,q^n \equiv 4f_1f_{10}, \tag{109}$$

which is the $\beta = 0$ case of (79). The rest of the proofs of the identities (79)-(82) are similar to the proofs of the identities (72)-(75). So, we omit the details.

Equation (55) reduces to

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+27)\,q^n \equiv 4f_1^3f_2f_5 + 6f_5^2. \tag{110}$$

Utilizing (5) and (12) in (110), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(128n+27)\,q^n \equiv 4f_1^5 + 6\frac{f_5f_{20}}{f_{10}^2} \tag{111}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(128n+91)\,q^n \equiv 4f_2f_5^3 + 4q^2f_5^{13}. \tag{112}$$

Equation (112) is the $\beta = 0$ case of (83). The rest of the proofs of the identities (83)-(85) are similar to the proofs of the identities (76)-(78). So, we omit the details.

Equation (63) becomes

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+19)\,q^n \equiv 4f_2f_5, \tag{113}$$

which is the $\beta = 0$ case of (86). The rest of the proofs of the identities (86)-(89) are similar to the proofs of the identities (65)-(68). So, we omit the details.

From Equation (61), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+3)\,q^n \equiv 6f_1^2 + 4qf_1f_5^3f_{10}. \tag{114}$$

Using (5) and (10) in (114), we have

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+3)\,q^n \equiv 6\frac{f_1f_4}{f_2^2} + 4qf_5^5 \tag{115}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+35)\,q^n \equiv 4f_1^{13} + 4f_1^3f_{10}. \tag{116}$$

Congruence (116) is the $\beta = 0$ case of (90). The rest of the proofs of the identities (90)-(92) are similar to the proofs of the identities (69)-(71). So, we omit the details. $\square$

Theorem 3. *For all $n \geq 0$, we have, modulo 4,*

$$\overline{a}_{4,5}\,(16n+1) \equiv \begin{cases} 2 & \text{if } n \text{ is a pentagonal number,} \\ 0 & \text{otherwise,} \end{cases} \tag{117}$$

$$\overline{a}_{4,5}\,(32(5n+i)+7) \equiv 0, \quad \text{where } i = 1,2,3,4, \tag{118}$$

$$\overline{a}_{4,5}\,(160n+7) \equiv \begin{cases} 2 & \text{if } n \text{ is a pentagonal number,} \\ 0 & \text{otherwise,} \end{cases} \tag{119}$$

$$\overline{a}_{4,5}\,(128(5n+i)+27) \equiv 0, \quad \text{where } i = 1,2,3,4, \tag{120}$$

$$\overline{a}_{4,5}\,(640n+27) \equiv \begin{cases} 2 & \text{if } n \text{ is a pentagonal number,} \\ 0 & \text{otherwise,} \end{cases} \tag{121}$$

$$\overline{a}_{4,5}\,(64n+3) \equiv \begin{cases} 2 & \text{if } n \text{ is a pentagonal number,} \\ 0 & \text{otherwise.} \end{cases} \tag{122}$$

Proof. From Equation (97), we have, modulo 4,

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(16n+1)\, q^n \equiv 2f_1. \tag{123}$$

Result (117) follows from Equation (123).

Equation (105) becomes

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(32n+7)\, q^n \equiv 2f_5. \tag{124}$$

Extracting the coefficients of q^{5n+i} for $i = 1,2,3,4$ from both sides of the above equation, we arrive at (118).

Equation (124) implies

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(160n+7)\, q^n \equiv 2f_1. \tag{125}$$

From Equation (125), we get (119).

Equation (111) becomes

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(128n+27)\, q^n \equiv 2f_5. \tag{126}$$

Collecting the coefficients of q^{5n+i} for $i = 1,2,3,4$ from both sides of the above equation, we arrive at (120).

Equation (126) implies

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(640n+27)\,q^n \equiv 2f_1. \tag{127}$$

From Equation (127), we obtain (121).

Equation (115) reduces to

$$\sum_{n=0}^{\infty} \overline{a}_{4,5}\,(64n+3)\,q^n \equiv 2f_1. \tag{128}$$

Result (122) follows from Equation (128). □

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References

[1] C. Adiga, M. S. Mahadeva Naika, D. Ranganatha and C. Shivashankar, Congruences modulo 8 for $(2, k)$-regular overpartitions for odd $k > 1$, *Arab. J. Math.* **7** (2018), 61-75.

[2] B. C. Berndt, *Ramanujan's Notebooks Part III*, Springer-Verlag, New York, 1991.

[3] N. Calkin, N. Drake, K. James, S. Law, P. Lee, D. Penniston and J. Radder, Divisibility properties of the 5-regular and 13-regular partition functions, *Integers* **8** (2008), #A60.

[4] S. C. Chen, On the number of overpartitions into odd parts, *Discrete Math.* **325** (2014), 32-37.

[5] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), 1623-1635.

[6] S. P. Cui and N. S. S. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. Appl. Math.* **51** (2013), 507-523.

[7] M. D. Hirschhorn and J. A. Sellers, Elementary proofs of parity results for 5-regular partitions, *Bull. Aust. Math. Soc.* **81** (2010), 58-63.

[8] M. D. Hirschhorn, *The Power of q*, Springer International Publishing, Switzerland, 2017.

[9] M. D. Hirschhorn and J. A. Sellers, Arithmetic relations for overpartitions, *J. Combin. Math. Combin. Comput.* **53** (2005), 65-73.

[10] M. D. Hirschhorn and J. A. Sellers, An infinite family of overpartition congruences modulo 12, *Integers* **5** (2005), #A20.

[11] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of overpartitions into odd parts, *Ann. Comb.* **10** (2006), 353-367.

[12] B. Kim, A short note on the overpartition function, *Discrete Math.* **309** (2009), 2528-2532.

[13] J. Lovejoy, Gordons theorem for overpartitions, *J. Combin. Theory A.* **103** (2003), 393-401.

[14] M. S. Mahadeva Naika and Harishkumar T, Congruences for 5-regular partitions with odd parts overlined. (Communicated).

[15] M. S. Mahadeva Naika and B. Hemanthkumar, Arithmetic properties of 5-regular bipartitions, *Int. J. Number Theory* **13 (4)**, (2017), 937-956.

[16] M. S. Mahadeva Naika, B. Hemanthkumar and H. S. Sumanth Bharadwaj, Color partition identities arising from Ramanujan's theta functions, *Acta Math. Vietnam* **41 (4)**, (2016), 633-660.

[17] K. Mahlburg, The overpartition function modulo small powers of 2, *Discrete Math.* **286** (2004), 263-267.

[18] S. Ramanujan, *Collected Papers*, Cambridge University Press, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, RI, 2000.

[19] C. Ray and R. Barman, New congruences for overpartitions into odd parts, *Integers* **18** (2018), #A50.

[20] E. Y. Y. Shen, Arithmetic properties of ℓ-regular overpartitions, *Int. J. Number Theory* **12 (3)**, (2016), 841-852.

[21] H. S. Sumanth Bharadwaj, B. Hemanthkumar and M. S. Mahadeva Naika, On 3 and 9-regular overpartitions modulo powers of 3, *Colloquium Math.* **154 (1)**, (2018), 121-130.