A GRAPH THEORETIC FORMULA FOR THE NUMBER OF PRIMES $\pi(n)$

R. Jacobs
Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, Virginia

C. E. Larson[1]
Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, Virginia
clarson@vcu.edu

Received: 5/5/20, Accepted: 9/23/20, Published: 10/9/20

Abstract

Let $\mathrm{PR}[n]$ be the graph whose vertices are $2, 3, \ldots, n$ with vertex v adjacent to vertex w if and only if $\gcd(v, w) > 1$. It is shown that $\pi(n)$, the the number of primes no more than n, equals the Lovász number of this graph. This result suggests new avenues for graph-theoretic investigations of number-theoretic problems.

1. The Result

In *Written on the Wall* (or WoW), Fajtlowicz's notes on conjectures of his program GRAFFITI [2, 3], Fajtlowicz defined the graphs $\mathrm{RP}[S]$ and $\mathrm{PR}[S]$ whose vertices are a set S of integers [1]. For $\mathrm{RP}[S]$, two distinct vertices are adjacent if and only if they are relatively prime; while in $\mathrm{PR}[S]$, two distinct vertices are adjacent if and only if they have a non-trivial common factor (and are thus the complements of the $RP[S]$ graphs). Let $\mathrm{PR}[n]$ be the graph where $S = \{2, 3, \ldots, n\}$. WoW is indexed by conjecture numbers, often with useful commentary of its author and correspondents. These graphs are defined in WoW #434 (1988). Study of these graphs may yield new insights into number theoretic questions. Among other things WoW records Staton's proof of the interesting fact that *every graph is an induced subgraph of* $\mathrm{PR}[n]$, *for some positive integer* n (WoW #446).

An independent set in a graph is a set of vertices which are pair-wise nonadjacent. The independence number $\alpha = \alpha(G)$ of a graph G is the cardinality of a maximum independent set. Fajtlowicz observed, and it is easy to see, that the primes are a

[1]Research supported by the Simons Foundation Mathematics and Physical Sciences–Collaboration Grants for Mathematicians Award (426267)

maximum independent set in PR[n] and thus the independence number of PR[n] is the number of primes up to n, denoted $\pi(n)$. So $\alpha(\mathrm{PR}[n]) = \pi(n)$. The Prime Number Theorem gives an asymptotic formula for $\pi(n)$ and the Riemann Hypothesis is equivalent to a conjecture about the error term in a formula for $\pi(n)$. Thus the study of the independence number of the PR[n] graphs may yield new insights into $\pi(n)$. GRAFFITI is well-known for its conjectures for the independence number of a graph—many of which were proved. While following up on Fajtlowicz's idea and also investigating the class of graphs where α equals Lovász's theta function ϑ, we discovered the following new formula for $\pi(n)$.

Theorem 1.1. Fon every integer $n \geq 2$, $\pi(n) = \vartheta(\mathrm{PR}[n])$.

This may be of interest for a few reasons: Lovász's theta function is widely studied, has a number of equivalent definitions [4], is efficiently computable, and there is an efficient algorithm for recognizing graphs where the independence number α equals Lovász's theta function ϑ.

An *orthonormal representation* of a graph G is an assignment of a unit vector $\hat{x}_v$ to each vertex v in the vertex set $V(G)$ having the property that vectors assigned to non-adjacent vertices are orthogonal. Lovász's theta function $\vartheta = \vartheta(G)$ of a graph G is defined to be the minimum over all feasible orthonormal representations $\{\hat{x}_v : v \in V(G)\}$ (or simply $\{\hat{x}_v\}$) and all unit vectors $\hat{c}$:

$$\vartheta = \min_{\hat{c},\{\hat{x}_v\}} \max_{v \in V(G)} \frac{1}{(\hat{c} \cdot \hat{x}_v)^2}.$$

Proof. Let n be a positive integer. Since Lovász proved that for any graph G, $\vartheta(G) \geq \alpha(G)$ [5] and $\alpha(\mathrm{PR}[n]) = \pi(n)$, it is enough to show that there is in fact a feasible orthonormal representation of PR[n] and unit vector $\hat{c}$ that realizes this lower bound; that is, such that:

$$\pi(n) = \min_{\hat{c},\{\hat{x}_v\}} \max_{v \in V(G)} \frac{1}{(\hat{c} \cdot \hat{x}_v)^2}.$$

Suppose there are exactly k primes no more than n: $p_1, p_2, \ldots, p_k$. For each vertex v with l_v distinct prime factors we define $\hat{x}_v$ to be a k-component vector where the i^{th} component is 0 unless p_i is a factor of v in which case the i^{th} component is $\frac{1}{\sqrt{l_v}}$. It is easy to see that if v and w are relatively prime, and thus non-adjacent, then $\hat{x}_v \cdot \hat{x}_w = 0$. Then let $\hat{c}$ be the vector whose components are all $\frac{1}{\sqrt{k}}$. Then $\max_{v \in V(G)} \frac{1}{(\hat{c} \cdot \hat{x}_v)^2} = \max_{v \in V(G)} \frac{1}{(l_v \cdot \frac{1}{\sqrt{k}} \cdot \frac{1}{\sqrt{l_v}})^2} = \max_{v \in V(G)} \frac{k}{l_v}$ occurs for a vertex w when $l_w = 1$, for instance when w is prime, and in this case equals k, the number of primes no more than n. □

Since perfect graphs have the property that $\alpha = \vartheta$, it might be thought that the PR[n] graphs are perfect. They are for $n < 35$. PR[35] has an odd hole: $5 \cdot 7, 7 \cdot 3, 3 \cdot 11, 11 \cdot 2, 2 \cdot 5$.

References

[1] S. Fajtlowicz, Written on the wall: conjectures of Graffiti, http://math.uh.edu/~siemion, 1986–2004.

[2] S. Fajtlowicz, On conjectures of Graffiti, in *Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986)* **72** (1988), 113–118.

[3] S. Fajtlowicz, On conjectures of Graffiti, V, *Graph Theory, Combinatorics, and Algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ., 367–376, 1995.

[4] D. E. Knuth, The sandwich theorem. *Electron. J. Combin.* **1** (1994), #R1, 49pp.

[5] L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory* **25**(1) (1979), 1–7.