LATTICE CONFIGURATIONS DETERMINING FEW DISTANCES

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Abstract

We begin by revisiting a paper of Erdős and Fishburn, which posed the following question: given $k \in \mathbb{N}$, what is the maximum number of points in a plane that determine at most k distinct distances, and can such optimal configurations be classified? We rigorously verify claims made in remarks in that paper, including the fact that the vertices of a regular polygon, with or without an additional point at the center, cannot form an optimal configuration for any $k \geq 7$. Further, we investigate configurations in both triangular and rectangular lattices studied by Erdős and Fishburn. We collect a large amount of data related to these and other configurations, some of which correct errors in the original paper, and we use that data and additional analysis to provide explanations and make conjectures.

1. Introduction

In 1996, Erdős and Fishburn [2] posed the following question: given $k \in \mathbb{N}$, what is the maximum number $g(k)$ of points in a plane that determine at most k distinct distances, and can such optimal configurations be classified?

Here and throughout we say that a set $E \subseteq \mathbb{R}^2$ *determines* a distance $\lambda > 0$ if there exist two points $P, Q \in E$ satisfying $\|P - Q\| = \lambda$, where $\|\cdot\|$ is the standard Euclidean norm. In other words, E determines at most k distances if the cardinality of $\{\|P - Q\| : P, Q \in E, P \neq Q\}$ is at most k. By convention and for the remainder of this paper, 0 is not counted as a distance determined by a set of points.

The question of Erdős and Fishburn can be thought of as a precise, small picture, inverse formulation of the famous Erdős distinct distance problem, introduced by Erdős [1] 50 years earlier, which concerns the minimum number $f(n)$ of distinct distances determined by n points in a plane. In that original paper, Erdős proved via an elementary counting argument that $f(n) = \Omega(\sqrt{n})$, and he conjectured that the correct order of growth is $n/\sqrt{\log n}$ (we use log to denote the natural logarithm), as attained by a square subset of the integer lattice. After decades of incremental progress, the question of the asymptotic behavior of $f(n)$ was effectively resolved in a celebrated result of Guth and Katz [3], who established that $f(n) = \Omega(n/\log n)$. However, the problem of precisely determining $g(k)$ for many k, and classifying optimal configurations, which we refer to as the *Erdős-Fishburn problem*, is still very much open for business. To aid our exploration, we introduce the following definitions.

Definition 1. We use *configuration* to refer to finite subsets of $\mathbb{R}^2$ modulo *similarity*, meaning equivalence via scaling and distance-preserving transformation. For $n \in \mathbb{N}$, we let R_n denote the configuration given by the set of vertices of a regular n-gon, and we let R_n^+ denote R_n with an additional point at the center of the unique circle containing the vertices. For $k \in \mathbb{N}$, we say that a configuration is *k-optimal* if it determines at most k distinct distances and contains $g(k)$ points. We say that a configuration is *EF-optimal* if it is k-optimal for some $k \in \mathbb{N}$.

We start by observing that $g(1) = 3$, and the only 1-optimal set is R_3. To see this, we fix $U, V \in \mathbb{R}^2$, which after translation and rotation we can assume are $U = (-1, 0)$, $V = (1, 0)$. We note that in order to add any additional points without determining an additional distance, those points must lie on the circles of radius 2 centered at U and V, respectively. These two circles intersect at two points, $Q = (\sqrt{3}, 0)$ and $R = (-\sqrt{3}, 0)$, and since these two points are more than distance 2 apart, we can only add one of them while maintaining a single distance.

For $k = 2$, we see that $g(2) \geq 5$ by considering R_5, the vertices of a regular pentagon, but showing equality is nontrivial. More generally, by rotational symmetry, we see that R_n determines $\lfloor n/2 \rfloor$ distinct distances, and hence $g(k) \geq 2k + 1$ by considering R_{2k+1}. Starting with $k = 3$, we start to see a new player enter the picture, as both R_7 and R_6^+ are 7-point configurations determining 3 distances.

Further, R_6^+ is particularly compelling, as it is a hexagonal array of points that lie in a lattice forming equilateral triangles. Foreshadowing future discussion, we introduce the following lattices.

Definition 2. We let $L_\Delta = \left\{\left(a + \frac{1}{2}b, \frac{\sqrt{3}}{2}b\right) : a, b \in \mathbb{Z}\right\}$, which we refer to as the *triangular lattice*, and we let $L_\Box = \{(a,b) : a,b \in \mathbb{Z}\}$ denote the usual integer lattice, which we refer to as the *rectangular lattice*.

We now give a synopsis of fully resolved cases, established by Erdős and Fishburn [2] for $1 \le k \le 4$, by Shinahara [7] for $k = 5$, and by Wei [8] for $k = 6$. The problem remains open for $k \ge 7$.

Theorem 1. *The following are completely resolved cases for the Erdős-Fishburn problem:*

(i) $g(1) = 3$*, and the only* 1*-optimal configuration is* R_3*;*

(ii) $g(2) = 5$*, and the only* 2*-optimal configuration is* R_5*;*

(iii) $g(3) = 7$*, and the only* 3*-optimal configurations are* R_7 *and* R_6^+*;*

(iv) $g(4) = 9$*, and there are four* 4*-optimal configurations:* R_9*, two subsets of* L_Δ*, and one additional;*

(v) $g(5) = 12$*, and the only* 5*-optimal configuration is a subset of* L_Δ*;*

(vi) $g(6) = 13$*, and there are three* 6*-optimal configurations:* R_{13}*,* R_{12}^+*, and a subset of* L_Δ*.*

We see that $k = 5$ is the first case in which R_{2k+1} is not a k-optimal configuration, which is quickly followed by the case $k = 6$ in which R_{13} and R_{12}^+ are 6-optimal. However, the aforementioned fact that n points can be arranged within a square subset of $L_\Box$ in order to determine $O(n/\sqrt{\log n})$ distinct distances ensures that $g(k) = \Omega(k\sqrt{\log k})$. Therefore, there exists $k_0 \in \mathbb{N}$ such that neither R_{2k}^+ nor R_{2k+1} is k-optimal for all $k \ge k_0$ (note that R_{2k} is never k-optimal because R_{2k+1} has more points with the same number of distinct distances).

Following some illuminating constructions, Erdős and Fishburn [2] indicate in a remark that one can take $k_0 = 7$, but the remaining details are left unverified. We also observe that at least one subset of L_Δ is k-optimal for $3 \le k \le 6$. The appeal of L_Δ for the purposes of minimizing distinct distances is intuitive, based on the fact that the lattice forms equilateral triangles. The following conjecture of Erdős and Fishburn makes precise the belief that L_Δ is the correct place to look for EF-optimal configurations.

Conjecture 1 (Conjecture 1, [2])**.** There exists at least one k-optimal configuration in L_Δ for all $k \ge 3$, and all k-optimal configurations are represented by subsets of L_Δ for all $k \ge 7$.

A mechanism by which Erdős and Fishburn gather further information and provide evidence for Conjecture 1 is the presentation of data on the number of distances determined by particular subsets of L_Δ and $L_\Box$. Specifically, they focus on hexagonal arrays in L_Δ and square arrays in $L_\Box$, particularly on cases when these arrays have approximately the same number of points. They observe that, in these cases, the hexagonal arrays of L_Δ have about 26% fewer distances than their square counterparts.

2. Results and Outline

In Section 3, by filling in gaps and constructing new examples, we rigorously verify the following result mentioned in the introduction ($k_0 = 7$), which is included in a remark without proof in [2].

Theorem 2. *$g(k) = 2k + 1$ for $k \in \{1, 2, 3, 4, 6\}$, while $g(k) > 2k + 1$ otherwise.*

Since R_n determines $\lfloor n/2 \rfloor$ distinct distances and R_n^+ determines $n/2$ distinct distances if n is divisible by 6 and $\lfloor n/2 \rfloor + 1$ distinct distances otherwise, Theorem 2 resolves the question of when the vertices of a regular polygon, with or without an additional point in the center, is an EF-optimal configuration.

Corollary 1. *R_n is an EF-optimal configuration if and only if $n \in \{3, 5, 7, 9, 13\}$, and R_n^+ is an EF-optimal configuration if and only if $n \in \{6, 12\}$.*

In an effort to verify and expand upon the aforementioned data on lattice configurations provided in [2], we made the surprising discovery that the data tables contain numerous, albeit relatively small, errors in the number of distinct distances determined by said configurations. After repeatedly and rigorously checking our code, we carried out some calculations by hand, the most readily doable of which concerned a 5×5 square configuration in $L_\Box$, in other words $\{0, 1, 2, 3, 4\}^2$. The distances determined by this configuration are $\sqrt{n}$ for $n = 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 32$, for a total of 14 distinct distances, while the data table in [2] indicates only 13 distinct distances.

Further, we sought additional insight as to the signifcance of the 26% figure observed by Erdős and Fishburn when comparing the number of distances in comparably sized configurations in L_Δ and $L_\Box$. In particular, we considered density and number theoretic properties, and investigated whether the hexagonal and square arrays were the best choices to compare the two lattices.

The following provides an outline of Section 4:

(i) We provide corrected and expanded data concerning the number of points and distances determined by hexagonal arrays in L_Δ and square arrays in $L_\Box$.

(ii) We provide a heuristic explanation, through density and number theoretic considerations, for why an optimal configuration in L_Δ should be about 27.6% better than an optimal configuration in $L_\square$ for the purposes of minimizing distinct distances. This is close to the 26% observed in [2], although that observation was influenced by the errors in the data. We see that our heuristic is in fact rigorous in the case of intersections of the respective lattices with large disks centered at the origin.

(iii) We provide new data indicating that, even amongst subsets of L_Δ and $L_\square$, respectively, the hexagonal and square arrays are not optimal with regard to minimizing the number of distinct distances for a fixed number of points. We see that these configurations are routinely "beaten" by the aforementioned lattice disks. We provide a large amount of numerical data for all four types of configurations, focusing on instances where all four types contain approximately the same number of points.

(iv) Using considerations from items (ii) and (iii), as well as Conjecture 1, we make a detailed conjecture related to the original Erdős distinct distance problem.

3. Proof of Theorem 2

For each integer $s \geq 2$, let H_s denote the hexagonal array in L_Δ with s points on each side. For clarity, we note that Figure 1 below depicts H_4. For the following two lemmas, we take the convention that the leftmost vertex of H_s lies at the origin.

Lemma 1. *Every distance that occurs in H_s occurs between the origin and a point of H_s in the closed upper half-plane ($y \geq 0$).*

Proof. Fix $P, Q \in H_s$.

`Case 1:` Suppose P is one of the six vertices of H_s. If P is not the origin, H_s can be rotated by an integer multiple of $60°$ to take P to the origin. If the image of Q under this rotation lies in the lower half-plane, we can reflect H_s about the x-axis to take Q to the upper half-plane. We note that, due to its symmetry, H_s is invariant under this transformation, call it ϕ. Further, since ϕ is distance-preserving, we have $\|\phi(Q)\| = \|P - Q\|$.

`Case 2:` Suppose P lies on the boundary of H_s but is not one of the vertices. Since there are s points on each side of the boundary, P is at most distance $\lfloor \frac{s-1}{2} \rfloor$ away from the nearest vertex.

We call this minimum distance d_1, and we call the distance from P to the opposite vertex on the same side e_1, hence $d_1 + e_1 = s - 1$. As for Q, it lies on some edge of points parallel to the side of the boundary containing P. Along this parallel edge, Q has two distances to the boundary of H_s, one in each direction, so we let d_2 denote the distance in the same direction as d_1, and we let e_2 denote the distance in the same direction as e_1. We note that the parallel edge containing Q is at least as long as the side-length on the boundary, so $d_2 + e_2 \geq s - 1 = d_1 + e_1$. Therefore, either $d_2 \geq d_1$ or $e_2 \geq e_1$.

If $d_2 \geq d_1$, we can translate P by d_1 units to a vertex, with Q remaining inside H_s. Similarly, if $e_2 \geq e_1$, we can translate P by e_1 units to the other vertex, with Q remaining inside H_s. In either case, we have reduced to Case 1. The translation procedure is illustrated in Figure 1 below.

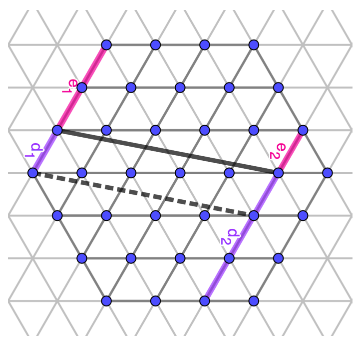


Figure 1: The translation procedure described in Case 2 of the proof of Lemma 1.

`Case 3:` If neither P nor Q lie on the boundary of H_s, then P and Q can both be translated left a unit at a time, preserving their distance, until one reaches the boundary, thus reducing to previous cases. $\square$

As with the conclusion of Theorem 2, the following facts about H_s were included in remarks in [2], and we rigorously verify them here.

Lemma 2. *H_s contains $3s^2 - 3s + 1$ points and determines at most $s^2 - 1$ distances.*

Proof. For the first part of the lemma, we see that we can decompose H_s into a disjoint union of $\tilde{H}_j$, a boundary hexagon with j points on each side, for $1 \leq j \leq s$ (including $\tilde{H}_1$, which is a single point). By the inclusion-exclusion principle, the number of points in $\tilde{H}_j$ is $6j - 6 = 6(j-1)$, with the exception of $|\tilde{H}_1| = 1$. Therefore,

$$|H_s| = 1 + \sum_{j=2}^{s} 6(j-1) = 1 + \sum_{j=1}^{s-1} 6j = 1 + 6s(s-1)/2 = 3s^2 - 3s + 1.$$

The cardinality $|H_s|$ is known as the s-th *central hexagonal number*. For the second part of the lemma, we have from Lemma 2 that we only need to consider distances that occur from the origin to points in H_s in the closed upper half-plane. Recall that points in L_Δ take the form $P = \langle a, b\rangle = a(1,0) + b(1/2, \sqrt{3}/2)$ for $a, b \in \mathbb{Z}$, in which case the distance from P to the origin is $\sqrt{a^2+ab+b^2}$. We note that if the pair $\langle a, b\rangle$ occurs in H_s with $b \geq a \geq 0$, then the pair $\langle b, a\rangle$ also occurs. Since the expression a^2+ab+b^2 is symmetric in a and b, we can assume when counting distances from the origin to the upper half-plane in H_s that $a \geq b$.

We see that in order to exhaust the upper half of H_s, we can first consider the pairs $\langle a, b\rangle$ for $0 \leq a \leq s-1$ and $0 \leq b \leq s-1$. What remains is a triangle of points on the right-hand side, for which the allowable range of b decreases as a increases. Therefore, the total number of distances in H_s is at most

$$\sum_{j=2}^{s} j + \sum_{i=1}^{s-1}(s-i) = \sum_{j=2}^{s} j + \sum_{i=1}^{s-1} i = \frac{s(s+1)}{2} - 1 + \frac{(s-1)s}{2} = s^2 - 1.$$

$\square$

In particular, for any $k \in \mathbb{N}$, we can let $s = \lfloor\sqrt{k+1}\rfloor$, and Lemma 2 tells us that H_s contains $3s^2-3s+1$ points and determines at most $s^2-1 \leq k$ distances, hence $g(k) \geq 3s^2-3s+1$. We have therefore established the following corollary.

Corollary 2. *$g(k) \geq 3(\lfloor\sqrt{k+1}\rfloor)^2 - 3(\lfloor\sqrt{k+1}\rfloor) + 1$ for all $k \in \mathbb{N}$.*

Some basic algebra now reduces the proof of Theorem 2 to a manageable number of remaining cases.

Corollary 3. *$g(k) > 2k+1$ for all $k \geq 63$.*

Proof. Fix $k \in \mathbb{N}$, and let $u = \sqrt{k+1}$. Since $\lfloor u\rfloor > u-1$, we have by Corollary 2 that

$$g(k) > 3(u-1)^2 - 3(u-1) + 1 = 3u^2 - 9u + 7.$$

Further, we see that $3u^2-9u+7 \geq 2u^2-1 = 2k+1$ provided $u^2-9u+8 = (u-8)(u-1) \geq 0$, which holds for $u \geq 8$, or in other words $k \geq 63$. $\square$

Proof of Theorem 2. Based on numerical data provided in [2] and re-verified, H_s for $s \in \{3,4,5,6,7,8\}$ yield (k,n) pairs, where k is the number of distinct distances and n is the number of points, of

$$(8,19), (15,37), (23,61), (34,91), (46,127), (59,169).$$

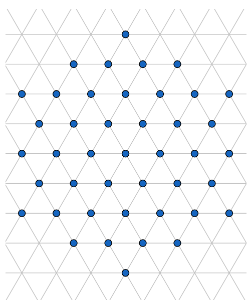
Further, the same paper displays arrays yielding pairs

$$(7, 16), (9, 21), (10, 25), (11, 27), (13, 31).$$

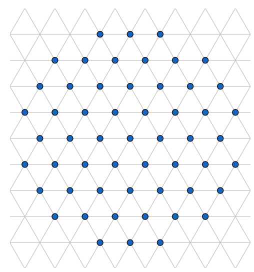
In particular, we know that $g(k) > 2k+1$ for $7 \leq k \leq 17$, $23 \leq k \leq 29$, $34 \leq k \leq 44$, and $46 \leq k \leq 83$. Further, we have by Corollary 3 that $g(k) > 2k+1$ for all $k \geq 63$. This leaves only the following list of exceptions:

$$k \in \{18, 19, 20, 21, 22, 30, 31, 32, 33, 45\}.$$

The following four configurations, the number of distinct distances in which were verified both by hand and by computer, account for the remaining exceptions, and the theorem follows. □

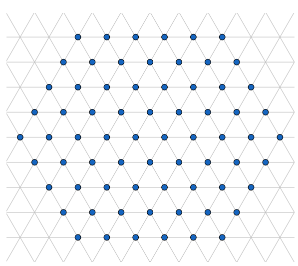


(a) $k = 18$, $n = 43$

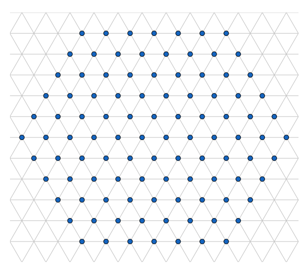


(b) $k = 21$, $n = 55$

Figure 2: These two configurations show $g(k) > 2k + 1$ for $k \in \{18, 19, 20, 21, 22\}$.



(a) $k = 29$, $n = 70$



(b) $k = 40$, $n = 102$

Figure 3: These two configurations show $g(k) > 2k + 1$ for $k \in \{30, 31, 32, 33, 45\}$.

4. Numerical Data for Lattice Configurations

4.1. Erdős-Fishburn Data Revisited

We begin by presenting corrected and slightly expanded versions of the data tables from [2] containing the number of points and distinct distances determined by hexagonal arrays in $L_\triangle$ and square arrays in $L_\square$.

$H_s \subseteq L_\triangle$					
n	k	s	n	k	s
7	3	2	469	150	13
19	8	3	547	**172** (173)	14
37	15	4	631	**196** (197)	15
61	23	5	721	**222** (223)	16
91	34	6	817	**249** (250)	17
127	46	7	919	**277** (280)	18
169	59	8	1027	**308** (312)	19
217	74	9	1141	**339** (345)	20
271	90	10	1261	**372** (382)	21
331	109	11	1387	405	22
397	129	12	1519	440	23

$s \times s$ square array in $L_\square$					
n	k	s	n	k	s
4	2	2	441	**197** (194)	21
9	5	3	484	**215** (212)	22
16	9	4	529	**234** (228)	23
25	**14** (13)	5	576	**251** (248)	24
36	19	6	625	**272** (268)	25
49	**26** (25)	7	676	**293** (288)	26
64	**33**(32)	8	729	**314** (309)	27
81	**41**(40)	9	784	**336** (331)	28
100	**50**(49)	10	841	**359** (352)	29
121	**60** (58)	11	900	**381** (377)	30
144	**70** (69)	12	961	**407** (400)	31
169	**82** (80)	13	1024	**430** (425)	32
196	**93** (91)	14	1089	**456** (451)	33
225	**105** (104)	15	1156	**483** (474)	34
256	**119** (118)	16	1225	**507** (501)	35
289	**134** (130)	17	1296	535	36
324	**147** (146)	18	1369	566	37
361	**164** (160)	19	1444	594	38
400	**179** (177)	20	1521	623	39

Table 1: Number of points, n, and distinct distances, k, determined by hexagonal and square arrays in $L_\triangle$ and $L_\square$, respectively. Corrected data is in bold, previously reported data from [2] is in parentheses. Data for $s = 22, 23$ for $L_\triangle$ and $36 \leq s \leq 39$ for $L_\square$ is new.

Data was collected using brute force searches with nested for-loops in Java. Specifically, we used that distances in the $s \times s$ square array in $L_\square$ take the form $\sqrt{a^2+b^2}$ for $0 \leq b \leq a \leq s-1$, while by Lemma 1, distances in H_s take the form $\sqrt{(a+\frac{1}{2}b)^2+(\frac{\sqrt{3}}{2}b)^2} = \sqrt{a^2+ab+b^2}$ for the pairs $\langle a, b\rangle$ indicated in the proof of Lemma 2. In focusing on instances where the two arrays have approximately the same number of points, Erdős and Fishburn indicate that H_s determines about 26% fewer distances. However, with the corrected table, using the respective n-values $(169, 169)$, $(1027, 1024)$, and the newly collected pair $(1519, 1521)$, we find savings between 28% and 29.4% in the number of distinct distances determined by the hexagonal arrays in L_Δ versus the square arrays in $L_\square$.

4.2. A Heuristic for L_Δ versus $L_\square$

To gain a better understanding of the savings in distinct distances in L_Δ as compared to $L_\square$, we start with two questions:

(a) How much "less dense" than L_Δ is $L_\square$? In other words, if a nice region contains n_1 points of L_Δ and n_2 points of $L_\square$, what would we expect n_2/n_1 to be?

(b) How much "sparser" are the distances determined by L_Δ than the distances determined by $L_\square$? In other words, if k_1 distances determined by L_Δ lie in some interval $(0, r]$, while k_2 distances determined by $L_\square$ lie in $(0, r]$, what would we expect k_1/k_2 to be?

Multiplying these two expectations together yields an expectation for

$$\frac{(n_2/k_2)}{(n_1/k_1)}$$

which compares the "efficiency" of $L_\square$ to that of L_Δ with regard to maximizing the number of "points per distinct distance". Extending this predicted ratio beyond cases where the region determining n_1 and n_2 are the same, and in particular to cases where $n_1 \approx n_2$, yields a prediction for k_1/k_2 in such cases.

Of the two, Question (a) is the more straightforward, and is answered by considering the *covolumes* of the lattices. Specifically, the number of points of $L_\square$ lying inside any nice region can be well-approximated by drawing a 1×1 square centered at each point, hence n_2 is very close to the area of the region. For L_Δ, we can instead draw parallelograms spanned by the vectors $(1, 0)$ and $(1/2, \sqrt{3}/2)$ centered at each point, which have area $\sqrt{3}/2$, hence n_1 is approximately the area of the region divided by $\sqrt{3}/2$. Therefore, we predict that n_2/n_1 is approximately $\sqrt{3}/2$.

Question (b) is in fact a question of number theory, specifically binary quadratic forms, as distances in $(0, r]$ determined by $L_\square$ are in correspondence with integers

$1 \leq n \leq r^2$ that can be represented as $n = a^2 + b^2$ for $a, b \in \mathbb{Z}$. Meanwhile, as discussed above, distances in $(0, r]$ determined by L_Δ are in correspondence with integers $1 \leq n \leq r^2$ that can be represented as $n = a^2 + ab + b^2$ for $a, b \in \mathbb{Z}$.

Remark 1. While our framing here is somewhat purpose-built for the Erdős-Fishburn problem, the interested reader should note that these inquiries are closely related to a conjecture of Schmutz Schaller [6], roughly stating that the triangular lattice determines the fewest distances of all lattices, resolved partially by Moree and te Riele [5] and later fully by Moree and Osburn [4].

It is known (see for example [4]) that the number of integers $1 \leq n \leq N$ representable as $n = a^2 + b^2$ is approximately $cN/\sqrt{\log N}$, where

$$c = \left(\frac{1}{2} \cdot \prod_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{p^2}{p^2 - 1} \right)^{1/2} \approx 0.764223654$$

is known as the Landau-Ramanujan constant. Similarly (again see [4]), the number of integers $1 \leq n \leq N$ representable as $a^2 + ab + b^2$ is approximately $c'N/\sqrt{\log N}$, where

$$c' = \left(\frac{1}{2\sqrt{3}} \prod_{p \equiv 2 (\mathrm{mod}\ 3)} \frac{p^2}{p^2 - 1} \right)^{1/2} \approx 0.638909405, \tag{1}$$

and therefore we predict $k_1/k_2 \approx c'/c \approx 0.83602$.

Combining these considerations, we expect that if configurations in L_Δ and $L_\square$ are optimal within their respective lattices, contain approximately the same number of points, and determine k_1 and k_2 distinct distances, respectively, then

$$\frac{k_1}{k_2} \approx \frac{\sqrt{3}}{2} \cdot \frac{c'}{c} \approx 0.72402.$$

This hypothesized 27.6% saving for L_Δ as compared to $L_\square$ is close to and between the savings observed in [2] and the corrected data in Table 1. However, neither the hexagonal arrays in L_Δ nor the square arrays in $L_\square$ are known to be optimal in their respective lattices. The following section both rigorizes our heuristic in a special case, and introduces alternative candidates for optimal lattice configurations.

4.3. Lattice Disks

The heuristic outlined in Section 4.2 makes some leaps. For example, the general geometric forms of optimal configurations in L_Δ and $L_\square$ might be quite different, which would cast some doubt on the precision of the Question (a) analysis.

Further, a lattice configuration does not have to determine *every* distance determined in the full lattice lying in a particular interval, which makes the Question

(b) analysis tenuous as well. However, there is a family of lattice configurations that completely alleviate these concerns, and also exploit rotational symmetry even more so than our previous candidate configurations: intersections of L_Δ and $L_\Box$ with large disks centered at the origin, which we refer to as *lattice disks*.

With this in mind, we fix $n \in \mathbb{N}$. As discussed in Section 4.2, the intersection of a disk of radius r_1 with L_Δ contains about $2\pi r_1^2/\sqrt{3}$ points. Setting this equal to n yields $r_1 = (\sqrt{3}n/(2\pi))^{1/2}$. Similarly, the intersection of a disk of radius r_2 with $L_\Box$ contains about πr_2^2 points, and setting this equal to n yields $r_2 = (n/\pi)^{1/2}$. A very slight overestimate for the number of distinct distances determined by the L_Δ disk is the number of integers $1 \leq j \leq 4r_1^2 = 2\sqrt{3}n/\pi$ representable as $j = a^2 + ab + b^2$. The missing exceptions stem from distances $\sqrt{a^2+ab+b^2} \leq 2r_1$ that cannot be translated to occur between two points of L_Δ within distance r_1 of the origin. Such distances are at least $2r_1 - 1$, so there are fewer than $4r_1 < 3\sqrt{n}$ of them.

Therefore, the number of distances determined by the L_Δ disk is

$$k_1 = c' \frac{2\sqrt{3}n}{\pi\sqrt{\log n}}(1 + o(1)), \tag{2}$$

where c' is as defined in Section 4.2 and the little-oh notation refers to n tending to infinity. Similarly, the number of distances determined by the $L_\Box$ disk is

$$k_2 = c\frac{4n}{\pi\sqrt{\log n}}(1 + o(1)),$$

where c is the Landau-Ramanujan constant, and both lattice disks contain $n + o(n)$ points. In this case we have, via rigorous argument rather than heuristic, that

$$\frac{k_1}{k_2} = \frac{\sqrt{3}}{2} \cdot \frac{c'}{c} + o(1).$$

4.4. Data for Large Lattice Arrays and Disks

The observations from the previous section are particularly interesting if we believe that lattice disks are good candidates for optimal or near-optimal sets with regard to the Erdős-Fishburn problem or the original Erdős distinct distance problem, or even if we believe them to be good candidates for optimal subsets of their respective lattices.

To collect some evidence on this matter, we compare extensive data for the four families of lattice configurations on which we have focused: hexagonal arrays in L_Δ, square arrays in $L_\Box$, L_Δ disks, and $L_\Box$ disks. To enhance our comparisons, we focus on the cases where these four arrays all have approximately the same numbers of points.

As a starting point, we search via computer for values of s such that $n_1 = 3s^2 - 3s + 1$ is close to a perfect square, in the sense that $\sqrt{n_1}$ is within .01 of an integer s_2. Then, we let $n_2 = s_2^2$, and the hexagonal array with s points on a side and the square array with s_2 points on a side contain n_1 and n_2 points, respectively. Then, as calculated in the previous section, we let $r_1 = (\sqrt{3}n_1/(2\pi))^{1/2}$ and we construct the L_Δ disk of radius r_1, which contains $n_3 \approx n_2 \approx n_1$ points. Finally, we let $r_2 = (n_1/\pi)^{1/2}$, construct the $L_\Box$ disk of radius r_2 containing $n_4 \approx n_3 \approx n_2 \approx n_1$ points, and we compute the number of distances k_1, k_2, k_3, k_4, respectively, determined by each configuration.

We compute k_1 and k_2 as outlined in Section 4, and we compute k_3 and k_4 via a brute force calculation of the number of integers $1 \leq j \leq 4r_1^2$ representable as $a^2 + ab + b^2$ and the number of integers $1 \leq j \leq 4r_2^2$ representable as $a^2 + b^2$, respectively. As discussed in Section 4.3, k_3 and k_4 are actually very slight overestimates for the number of distinct distances in the lattice disks, with relative error decaying quickly to 0, and the error actually works in favor of our eventual conclusions and conjectures.

Table 2 displays the results of this data collection:

$H_s \subseteq L_\Delta$			$s \times s$ square in $L_\Box$			L_Δ Disk		$L_\Box$ Disk	
s	n_1	k_1	s_2	n_2	k_2	n_3	k_3	n_4	k_4
23	1519	440	39	1521	623	1519	441	1513	601
34	3367	925	58	3364	1310	3369	920	3360	1251
38	4219	1139	65	4225	1620	4217	1130	4216	1541
49	7057	1844	84	7056	2628	7059	1818	7049	2486
64	12097	3063	110	12100	4378	12094	3008	12083	4116
75	16651	4136	129	16641	5923	16634	4055	16641	5552
79	18487	4572	136	18496	6558	18482	4475	18480	6122
90	24031	5847	155	24025	8397	24036	5725	24010	7836
105	32761	7841	181	32761	11291	32755	7663	32759	10496
120	42841	10115	207	42849	14568	42848	9870	42841	13528
131	51091	11958	226	51076	17246	51097	11661	51096	15986
135	54271	12660	233	54289	18268	54263	12354	54248	16919
146	63511	14707	252	63504	21196	63519	14325	63509	19640
161	77281	17716	278	77284	25597	77289	17253	77268	23658
172	88237	20099	297	88209	29034	88230	19574	88223	26838
176	92401	21007	304	92416	30348	92406	20446	92332	28029
187	104347	23588	323	104329	34095	104352	22949	104340	31468
191	108871	24557	330	108900	35517	108864	23888	108869	32759
202	121807	27333	349	121801	39539	121812	26580	121785	36454
217	140617	31345	375	140625	45371	140619	30474	140616	41797
228	155269	34463	394	155236	49901	155273	33482	155260	45930
232	160777	35627	401	160801	51610	160771	34614	160760	47489
243	176419	38926	420	176400	56379	176421	37815	176391	51880
247	182287	40162	427	182329	58217	182317	39015	182265	53527
258	198919	43663	446	198916	63291	198916	42405	198912	58161

s	n_1	k_1	s_2	n_2	k_2	n_3	k_3	n_4	k_4
273	222769	48642	472	222784	70564	222768	47234	222761	64804
284	241117	52465	491	241081	76114	241114	50935	241093	69888
288	247969	53901	498	248004	78211	247946	52316	247959	71786
299	267307	57926	517	267289	84051	267323	56199	267302	77117
314	294847	63614	543	294849	92358	294851	61715	294821	84691
329	323737	69586	569	323761	101045	323735	67473	323676	92604
340	345781	74110	588	345744	107620	345756	71853	345771	98630
344	353977	75796	595	354025	110084	353981	73485	353961	100859
355	377011	80509	614	376996	116943	376986	78044	376976	107118
370	409591	87167	640	409600	126667	409562	84483	409575	115964
381	434341	92219	659	434281	133986	434343	89356	434308	122650
385	443521	94093	666	443556	136760	443552	91166	443497	125135
396	469261	99288	685	469225	144351	469249	96221	469208	132091
400	478801	101248	692	478864	147198	478792	98093	478776	134661
411	505531	106665	711	505521	155110	505541	103325	505521	141834
426	543151	114254	737	543169	166216	543137	110682	543138	151934
437	571597	119982	756	571536	174558	571633	116212	571602	159538
441	582121	122120	763	582169	177705	582125	118258	582072	162357
452	611557	128035	782	611524	186289	611562	123979	611530	170223
467	652867	136335	808	652864	198432	652878	131989	652825	181216
482	695527	144892	834	695556	210885	695520	140239	695508	192565
493	727669	151302	853	727609	220271	727659	146454	727647	201095
497	739537	153654	860	739600	223722	739555	148736	739542	204224
508	772669	160264	879	772641	233368	772601	155110	772639	212981
523	819019	169514	905	819025	246864	819023	164028	818990	225234
534	853867	176436	924	853776	256919	853874	170719	853843	234419
538	866719	178995	931	866761	260727	866686	173179	866699	237799
549	902557	186081	950	902500	271056	902576	180050	902467	247218
553	915769	188706	957	915849	274937	915768	182569	915747	250711
564	952597	195999	976	952576	285616	952579	189601	952567	260389
579	1003987	206178	1002	1004004	300439	1004013	199414	1003960	273859
590	1042531	213780	1021	1042441	311537	1042535	206744	1042495	283924
594	1056727	216578	1028	1056784	315648	1056726	209449	1056677	287637
605	1096261	224343	1047	1096209	327044	1096288	216960	1096225	297965
609	1110817	227205	1054	1110916	331216	1110835	219713	1110783	301764
620	1151341	235163	1073	1151329	342863	1151324	227412	1151304	312340
635	1207771	246273	1099	1207801	359046	1207777	238116	1207741	327015
646	1250011	254539	1118	1249924	371136	1249983	246085	1250012	337989
650	1265551	257572	1125	1265625	375609	1265572	249021	1265528	342035
661	1308781	266045	1144	1308736	387968	1308792	257188	1308692	353251
676	1368901	277784	1170	1368900	405139	1368882	268534	1368881	368843
691	1430371	289825	1196	1430416	422777	1430355	280107	1430362	384729
702	1476307	298749	1215	1476225	435779	1476291	288738	1476282	396591

s	n_1	k_1	s_2	n_2	k_2	n_3	k_3	n_4	k_4
706	1493191	302062	1222	1493284	440674	1493181	291932	1493127	400964
717	1540117	311203	1241	1540081	454022	1540131	300711	1540106	413066
732	1605277	323826	1267	1605289	472567	1605270	312938	1605233	429846
743	1653919	333283	1286	1653796	486344	1653950	322046	1653912	442387
747	1671787	336742	1293	1671849	491469	1671808	325387	1671718	446972
758	1721419	346375	1312	1721344	505556	1721389	334664	1721383	459723
762	1739647	349927	1319	1739761	510746	1739653	338071	1739625	464399
773	1790269	359715	1338	1790244	525054	1790299	347523	1790221	477381
788	1860469	373283	1364	1860496	544913	1860505	360604	1860478	495385
799	1912807	383420	1383	1912689	559698	1912800	370371	1912756	508782
803	1932019	387140	1390	1932100	565161	1932048	373949	1932003	513712
814	1985347	397407	1409	1985281	580262	1985338	383870	1985309	527326
818	2004919	401149	1416	2005056	585848	2004941	387510	2004882	532335
829	2059237	411672	1435	2059225	601097	2059246	397582	2059183	546210
844	2134477	426140	1461	2134521	622344	2134456	411547	2134454	565400
855	2190511	436926	1480	2190400	638008	2190527	421951	2190443	579686
859	2211067	440867	1487	2211169	643919	2211058	425754	2211025	584894
870	2268091	451826	1506	2268036	659933	2268113	436313	2268056	599420
885	2347021	466988	1532	2347024	682105	2347029	450917	2347002	619478
900	2427301	482326	1558	2427364	704669	2427293	465744	2427243	639850
911	2487031	493838	1577	2486929	721402	2487049	476764	2487014	655022
915	2508931	497986	1584	2509056	727521	2508908	480809	2508918	660584
926	2569651	509611	1603	2569609	744588	2569702	491991	2569595	675978
941	2653621	525623	1629	2653641	768155	2653614	507466	2653571	697226
952	2716057	537570	1648	2715904	785494	2716052	518955	2716009	713014
956	2738941	541948	1655	2739025	792020	2738981	523156	2738918	718819
967	2802367	554074	1674	2802276	809668	2802353	534812	2802306	734811
971	2825611	558517	1681	2825761	816285	2825597	539085	2825575	740685
982	2890027	570769	1700	2890000	834110	2890025	550893	2890023	756949
997	2979037	587712	1726	2979076	858983	2979000	567254	2979027	779382

Table 2: Number of points, n_1, n_2, n_3, n_4, and distinct distances, k_1, k_2, k_3, k_4, determined by hexagonal arrays in $L_\triangle$, square arrays in $L_\square$, $L_\triangle$ intersected with a disk centered at the origin, and $L_\square$ intersected with a disk centered at the origin, respectively.

As our configurations grow, the originally investigated ratio k_1/k_2 decreases well below our previous observations and indicates at least a 31.5% saving for hexagonal arrays in $L_\triangle$ versus square arrays in $L_\square$. We know that the ratio k_3/k_4 must converge to $\frac{\sqrt{3}}{2} \cdot \frac{c'}{c} \approx 0.724$, and our largest data point yields a ratio of about 0.728. The speed of the convergence is somewhat at the mercy of the convergence of the approximations for the density of the images of the respective binary quadratic forms, which is known to be quite slow.

Importantly, we see that for large configurations, k_3 is notably smaller than k_1 (about a 3.5% saving, and climbing), and k_4 is notably smaller than k_2 (over a 9% saving, and climbing). The fact that switching from a square to a disk had a bigger impact in $L_\square$ than that of switching from a hexagon to a disk in $L_\triangle$ can probably be attributed to a greater increase in rotational symmetry. This discrepancy explains why k_1/k_2 falls well below k_3/k_4, and well below our heuristic ratio.

Based on this data, and our intuition regarding the advantages of rotational symmetry over well-structured arrays, we conjecture that, with regard to minimizing distinct distances for a fixed number of points, or equivalently maximizing the number of points for a fixed number of distinct distances, the $L_\triangle$ and $L_\square$ disks, or efficient subsets thereof, are the optimal configurations within their respective lattices. Combining this belief with Conjecture 1 and the formulas (1) and (2), we conclude our discussion with the following detailed conjecture on the original Erdős distinct distance problem.

Conjecture 2. If $f(n)$ is the minimum number of distinct distances determined by n points in a plane, then

$$f(n) = c\frac{n}{\sqrt{\log n}}(1 + o(1)),$$

where

$$c = \frac{1}{\pi}\left(2\sqrt{3}\prod_{p\equiv 2 \bmod 3}\frac{p^2}{p^2-1}\right)^{1/2} \approx 0.704498\ldots$$

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