# THE UNION OF TWO ARITHMETIC PROGRESSIONS WITH THE SAME COMMON DIFFERENCE IS NOT SUM-DOMINANT

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**Abstract**

Given a finite set $A \subseteq \mathbb{R}$, define the sum set $A + A = \{a_i + a_j \mid a_i, a_j \in A\}$ and the difference set $A - A = \{a_i - a_j \mid a_i, a_j \in A\}$. The set $A$ is said to be sum-dominant if $|A + A| > |A - A|$. In the literature, sum-dominant sets are also called more-sum-than-difference (MSTD) sets. We prove the following results:

1. The union of two arithmetic progressions with the same common difference is not sum-dominant. This result partially proves a conjecture proposed by the author in a previous paper; that is, the union of any two arbitrary arithmetic progressions is not sum-dominant.

2. Hegarty proved that a sum-dominant set of integers must have at least 8 elements with computers' help. The author of the current paper provided a human-verifiable proof that a sum-dominant set of integers must have at least 7 elements. A natural question is about the largest cardinality of sum-dominant subsets of an arithmetic progression. Fix $n \geq 16$. Let $N$ be the cardinality of the largest sum-dominant subset(s) of $\{0, 1, \ldots, n-1\}$. Then $n - 7 \leq N \leq n - 4$; that is, from an arithmetic progression of length $n \geq 16$, we need to discard at least 4 and at most 7 elements (in a clever way) to have the largest sum-dominant set(s).

3. Let $R \in \mathbb{N}$ have the property that for all $r \geq R$, $\{1, 2, \ldots, r\}$ can be partitioned into 3 sum-dominant subsets, while $\{1, 2, \ldots, R-1\}$ cannot. Then $24 \leq R \leq 145$. This result answers a question by the author et al. in another paper on whether we can find a stricter upper bound for $R$.

## 1. Introduction

### 1.1. Background and Main Results

Given a finite set $A \subseteq \mathbb{R}$, define $A + A = \{a_i + a_j \,|\, a_i, a_j \in A\}$ and $A - A = \{a_i - a_j \,|\, a_i, a_j \in A\}$. The set $A$ is said to be

- *sum-dominant*, if $|A + A| > |A - A|$;

- *balanced*, if $|A+A| = |A-A|$; and
- *difference-dominant*, if $|A+A| < |A-A|$.

Sum-dominant sets are also called more-sum-than-difference (MSTD) sets. Because addition is commutative, while subtraction is not, sum-dominant sets are very rare. However, it was first proved by Martin and O'Bryant [13] that as $n \to \infty$, the proportion of sum-dominant subsets of $\{0, 1, 2, \ldots, n-1\}$ is bounded below by a positive constant (about $2 \cdot 10^{-7}$), which was later improved by Zhao [25] to about $4 \cdot 10^{-4}$. However, these works used the probabilistic method and did not give explicit constructions of sum-dominant sets. Later, Miller et al. [15] constructed a family of density $\Theta(1/n^4)$[1] and Zhao [24] gave a family of density $\Theta(1/n)$. The last few years have seen an explosion of papers exploring properties of sum-dominant sets: see [7, 10, 12, 18, 20, 21, 22] for history and overview, [8, 14, 15, 18, 24] for explicit constructions, [5, 9, 13, 25] for positive lower bounds for the percentage of sum-dominant sets, [11, 16] for generalized sum-dominant sets, and [1, 4, 6, 25] for extensions to other settings.

We know that an arithmetic progression is not a sum-dominant set (see Corollary 2.2 below). It is natural to ask whether numbers from the union of several arithmetic progressions produce a sum-dominant set. Our first result is that the union of two arithmetic progressions with the same common difference is not sum-dominant.

**Theorem 1.1.** *The union of two arithmetic progressions $P_1$ and $P_2$ with the same common difference is not sum-dominant.*

This result partially proves the conjecture by the author of the current paper [2] that the union of any two arbitrary arithmetic progressions is not sum-dominant. Note that $\{0, 2\} \cup \{3, 7, 11, \ldots, 4k-1\} \cup \{4k, 4k+2\}$ for $k \geq 5$ is sum-dominant [18], and the set is the union of three arithmetic progressions. Hence, [2, Conjecture 17] is the most we can do.

Our next result concerns the cardinality of a sum-dominant set. Hegarty [8] proved that a sum-dominant set of integers must have at least 8 elements with the help of computers. The author of the current paper provided a human-understandable proof that a sum-dominant set of integers must have at least 7 elements [2, 3]. Another natural question is about the largest cardinality of a sum-dominant set of integers. It is well-known that a sum-dominant set can be arbitrarily large, so we put a restriction on the size of the set to have the following result

**Theorem 1.2.** *Fix $n \geq 16$. Let $N$ be the cardinality of the largest sum-dominant subset(s) of $\{0, 1, \ldots, n-1\}$. Then $n - 7 \leq N \leq n - 4$.*

The theorem implies that from an arithmetic progression of length at least 16, we need to discard at least 4 elements and not more than 7 elements (in a clever

[1] A more refined analysis improves the bound to $\Theta(1/n^2)$ [11].

way) to have the largest sum-dominant set(s). A corollary is that if we want to search for all sum-dominant subsets of $\{0, 1, \dots, n-1\}$, we only need to look for subsets of size between 8 and $n-4$.

**Conjecture 1.3.** Fix $n \geq 16$. Let $N$ be the cardinality of the largest sum-dominant subset(s) of $\{0, 1, \dots, n-1\}$. Then $N = n - 7$.

We run a computer program to find that the conjecture holds for all $16 \leq n \leq 34$. For $n \leq 14$, $N$ does not exist. For $n = 15$, $N = 9$, corresponding to the set $\{0, 1, 2, 4, 5, 9, 12, 13, 14\}$; that is, we discard 6 elements.

Our final result is related to the partition of an arithmetic progression into sum-dominant subsets. Asada et al. proved that as $r \to \infty$, the proportion of 2-decompositions of $\{1, 2, \dots, r\}$ into sum-dominant subsets is bounded below by a positive constant [1]. Continuing the work, the author of the current paper with Luntzlara, Miller, and Shao proved that it is possible to partition $\{1, 2 \dots, r\}$ (for $r$ sufficiently large) into $k \geq 3$ sum-dominant subsets [4]. By defining $R := R(k)$ to be the smallest integer such that for all $r \geq R$, $\{1, 2, \dots, r\}$ can be $k$-decomposed into MSTD subsets, while $\{1, 2, \dots, R-1\}$ cannot, the authors established rough lower and upper bounds for $R$. However, the upper bound when $k = 3$ is very loose because it depends on a sum-dominant subset $A$ with $|A + A| - |A - A| \geq 10|A|$. In particular, [4, Theorem 1.4] gives an upper bound equal to $4 \min\{\max A : |A + A| - |A - A| \geq 10|A|\} + 24$. Due to [8, Theorem 1] and [19, Theorem 22], we know that $|A| \geq 10$, which implies that $|A + A| \geq 10|A| + |A - A| \geq 12|A| - 1 \geq 119$. Because $2 \max A - 1 \geq |A + A|$, we have that $\max A \geq 60$, which gives an upper bound of $R$ of at least 264. Our next theorem gives an upper bound of 145.

**Theorem 1.4.** *Let $R \in \mathbb{N}$ have the property that for $r \geq R$, $\{1, 2, \dots, r\}$ can be partitioned into 3 sum-dominant subsets, while $\{1, 2, \dots, R-1\}$ cannot. Then $24 \leq R \leq 145$.*

This theorem answers a question raised by the author of the current paper et al. about whether we can find a more efficient way to decompose $\{1, 2, \dots, r\}$ into 3 sum-dominant sets. We find a smaller upper bound by a new way of partitioning $\{1, 2, \dots, n\}$ into 3 sum-dominant subsets. Our construction is similar to that of Miller et al. [15] and utilizes the fact that their construction allows a long run of missing elements. The long run of missing elements is where we can insert a fixed sum-dominant set.

### 1.2. Notation

We introduce some notation. Let $A$ and $B$ be sets. We write $A \to B$ to mean the introduction of elements in $A$ to $B$. We also use a different notation to write a set, which was first introduced by Spohn [23]. Given a set $S = \{m_1, m_2, \dots, m_n\}$, we arrange its elements in increasing order and find the differences between two consecutive numbers to form a sequence. Suppose that $m_1 < m_2 < \cdots < m_n$, then our sequence is $m_2 - m_1, m_3 - m_2, m_4 - m_3, \ldots, m_n - m_{n-1}$, and we represent $S = (m_1 \,|\, m_2 - m_1, m_3 - m_2, m_4 - m_3, \ldots, m_n - m_{n-1}) = (m_1 \,|\, a_1, \ldots, a_{n-1})$, where $a_i = m_{i+1} - m_i$. Any positive difference in $S - S$ must be equal to a sum $a_i + \cdots + a_j$ for some $1 \le i \le j \le n-1$. Take $S = \{3, 2, 15, 10, 9\}$, for example. We arrange the elements in increasing order to have 2, 3, 9, 10, 15, form a sequence by looking at the difference between two consecutive numbers: 1, 6, 1, 5, and write $S = (2 \,|\, 1, 6, 1, 5)$. All information about a set is preserved in this notation.

An arithmetic progression is a sequence of the form $(a, a+d, a+2d, a+3d, \ldots, a+kd)$ for any arbitrary numbers $a$, $k$, and the common difference $d$. Because sum-dominance is preserved under affine transformations, we can safely assume that our arithmetic progressions contain nonnegative numbers with 1 being the common difference. To emphasize, all arithmetic progressions we consider will have nonnegative numbers and have the same common difference, which is 1.

## 2. Important Results

We use the definition of a symmetric set given by Nathanson [17]: a set $A$ is symmetric if there exists a number $a$ such that $a - A = A$. If so, we say that the set $A$ is symmetric about $a$. The following proposition was proved by Nathanson [17].

**Proposition 2.1.** *A symmetric set is balanced.*

*Proof.* Let $A$ be a symmetric set about $a$. We have $|A + A| = |A + (a - A)| = |a + (A - A)| = |A - A|$. Hence, $A$ is balanced. □

Though symmetric sets are not sum-dominant, adding a few numbers into these sets (in a clever way) can produce sum-dominant sets. Examples of such a technique were provided by Hegarty [8] and Nathanson [18].

**Corollary 2.2.** *An arithmetic progression is not sum-dominant.*

Note that a set of numbers from an arithmetic progression is symmetric about the sum of the maximum and the minimum of the arithmetic progression. For example, the set $E = \{3, 5, 7, 9, 11\}$ is symmetric about 14. The following lemma is proved by Macdonald and Street [14].

**Lemma 2.3.** *Given a finite set $A = (0 \,|\, a_1, a_2, \ldots, a_n)$, the following claims hold.*

*(1) If $a_i \in \{1, 2\}$ for all $i$, then $A$ is not sum-dominant.*

*(2) If $a_i \in \{1, k\}$ and the first and last times that 1 occurs as a difference, it occurs in a block of at least $k-1$ consecutive differences, then $A$ is not sum-dominant.*

The following lemma is trivial but very useful in our proof of Theorem 1.1.

**Lemma 2.4.** *Let $P_1$ and $P_2$ be arithmetic progressions with common difference* 1*. The following claims hold.*

*(1) Given an arithmetic progression $P_1$, $\{\max P_1 + 1\} \to P_1$ gives* 2 *new sums.*

*(2) Given arithmetic progressions $P_1$ and $P_2$, $\{\max P_1 + 1\} \to (P_1 \cup P_2)$ gives at most* 3 *new sums.*

*Proof.* We first prove item 1. Without loss of generality, assume $P_1 = \{0, 1, \ldots, n\}$ for some $n \geq 0$. Denote $Q_1 = P_1 \cup \{n+1\}$. Then $P_1 + P_1 = \{0, 1, \ldots, 2n\}$ and $Q_1 + Q_1 = \{0, 1, \ldots, 2n+2\}$. Clearly, $|Q_1 + Q_1| - |P_1 + P_1| = 2$.

We proceed to prove item 2. New sums come from the interactions of $\max P_1 + 1$ with $P_1$, with $P_2$, and with itself. By item 1, the interactions of $\max P_1 + 1$ with $P_1$ and itself give at most 2 new sums. We consider the interactions of $\max P_1 + 1$ with $P_2$. We have $(\max P_1 + 1 + P_2) \backslash (\max P_1 + P_2) = \{\max P_1 + \max P_2 + 1\}$. Therefore, the interactions of $\{\max P_1 + 1\}$ with $P_2$ gives at most 1 new sum. In total, we have at most 3 new sums, as desired. □

## 3. Proof of Theorem 1.1

Because sum-dominance is preserved under affine transformations, without loss of generality, assume that $0 = \min P_1 \leq \min P_2$ and $|P_1| \geq |P_2|$. Let $m_i$ and $M_i$ denote $\min P_i$ and $\max P_i$, respectively. Finally, we only consider $P_1 \cap P_2 = \emptyset$ because if $P_1 \cap P_2 \neq \emptyset$, $P_1 \cup P_2$ is an arithmetic progression[2], which does not form a sum-dominant set by Corollary 2.2. Our proof considers $P_1$ as the original set and sees how $P_2 \to P_1$ changes the number of sums and differences.

### 3.1. Part I. $\max \mathbf{P_1} < \min \mathbf{P_2}$

Let $k = \min P_2 - \max P_1$. If $k = 1$, $P_1 \cup P_2$ is an arithmetic progression, not a sum-dominant set. We consider two cases corresponding to $k < 1$ and $k > 1$.

`Case I.1:` $k < 1$. We consider $P_2 \to P_1$. The set of new positive and distinct differences includes $k < k+1 < \cdots < k + |P_1| + |P_2| - 2$. Hence, the number of new differences is at least $2(|P_1| + |P_2| - 1)$. Now, we count the number of new sums. Consider $m_2 \to P_1$. We have exactly $|P_1| + 1$ new sums. Due to Lemma 2.4, $m_2 + j \to P_1 \cup \{m_2, \ldots, m_2 + j - 1\}$ gives at most 3 new sums for all $j \geq 1$. Therefore, $P_2 \to P_1$ gives at most $(|P_1| + 1) + 3(|P_2| - 1) = |P_1| + 3|P_2| - 2$ new sums.

[2] Recall that $P_1$ and $P_2$ have the same common difference.

Because $|P_1| \geq |P_2|$, we have $2(|P_1|+|P_2|-1) \;\geq\; |P_1|+3|P_2|-2$, and so, we do not have a sum-dominant set.

`Case I.2:` $k > 1$. If $k$ is not an integer, then with the same reasoning as Case I.1, we are done. If $k$ is an integer, we consider two following subcases.

`Subcase I.2.1:` $k > \max P_1$. Then $m_2 \to P_1$ gives $|P_1|$ new positive differences

$$m_2 - \max P_1 \;<\; m_2 - \max P_1 + 1 \;<\; \cdots \;<\; m_2$$

while at most $|P_1|+1$ new sums. Due to Lemma 2.4, $m_2+j \to P_1 \cup \{m_2, \ldots, m_2+j-1\}$ gives at most 3 new sums and at least 2 new differences $\pm(m_2+j)$ for all $j \geq 1$. Therefore, $P_2 \to P_1$ gives at most $|P_1|+1+3(|P_2|-1)$ new sums while at least $2|P_1|+2(|P_2|-1)$ new differences. Because $|P_1| \geq |P_2|$, the number of new differences is not smaller than the number of new sums, and so, $P_1 \cup P_2$ is not sum-dominant.

`Subcase I.2.2:` $k \leq \max P_1$. If $|P_2| \geq k$, we are done due to item 2 of Lemma 2.3. So, we consider $|P_2| \leq k-1$. Consider $m_2 \to P_1$. It is easily checked that $m_2 \to P_1$ gives $2k$ new differences and $k+1$ new sums. Due to Lemma 2.4, $m_2+j \to P_1 \cup \{m_2, \ldots, m_2+j-1\}$ gives at most 3 new sums and 2 new differences $\pm(m_2+j)$ for all $j \geq 1$. The total number of new sums is at most $k+1+3(|P_2|-1)$, while the number of new differences is at least $2k+2(|P_2|-1)$. We have

$$2k+2(|P_2|-1)-(k+1+3(|P_2|-1)) \;=\; k-|P_2| \;\geq\; 1.$$

Hence, $P_1 \cup P_2$ is difference-dominant.

### 3.2. Part II. $\max \mathbf{P_1} > \min \mathbf{P_2}$

If $m_2 - 1/2 \in \mathbb{Z}$, we consider $2(P_1 \cup P_2)$. Because the difference between any two consecutive numbers in increasing order is either 1 or 2, by item 1 of Lemma 2.3, we do not have a sum-dominant set. Hence, we assume that $m_2 - 1/2 \notin \mathbb{Z}$. Suppose that $n < m_2 < n+1$ for some $n \in P_1$. The following are new and pairwise distinct positive differences from $m_2 \to P_1$

$$\begin{aligned} &m_2 - n \;<\; m_2-(n-1) \;<\; \cdots \;<\; m_2 - 0, \\ &n+1-m_2 \;<\; n+2-m_2 \;<\; \cdots \;<\; \max P_1 - m_2. \end{aligned}$$

Hence, we have at least $2|P_1|$ new differences. On the other hand, $m_2 \to P_1$ gives at most $|P_1|+1$ new sums. Due to Lemma 2.4, $m_2+j \to P_1 \cup \{m_2, \ldots, m_2+j-1\}$ gives at most 3 new sums and at least 2 new differences $\pm(m_2+j)$ for all $j \geq 1$. Hence, the total number of new sums as a result of $P_2 \to P_1$ is at most

$$|P_1|+1+3(|P_2|-1) \;=\; |P_1|+3|P_2|-2,$$

while the number of new differences is at least

$$2|P_1|+2(|P_2|-1) \;=\; 2|P_1|+2|P_2|-2.$$

Because $|P_1| \geq |P_2|$, we have $|P_1|+3|P_2|-2 \ \leq \ 2|P_1|+2|P_2|-2$. Therefore, $P_1 \cup P_2$ is not sum-dominant and our proof of Theorem 1.1 is complete.

## 4. Proof of Theorem 1.2

**Lemma 4.1.** *For $m \geq 9$, the set*

$$\begin{aligned} K &= \{0,1,\ldots,m+7\}\backslash\{3,5,6,m+1,m+2,m+3,m+5\} \\ &= \{0,1,2,4\} \cup \{7,8,\ldots,m\} \cup \{m+4,m+6,m+7\} \end{aligned}$$

*is sum-dominant. Note that $K$ is a sum-dominant subset of $\{0,1,\ldots,m+7\}$ after we discard 7 numbers from the arithmetic progression.*

*Proof.* Observe that $K-K = \{\pm 0, \pm 1, \ldots, \pm(m+7)\}\backslash\{\pm(m+1)\}$, while $K+K = \{0,1,\ldots,2m+14\}\backslash\{2m+9\}$. Hence, $|K+K|-|K-K| = 1$. □

We now prove Theorem 1.2. Fix $n \geq 16$. Let $N$ be the cardinality of the largest sum-dominant subset(s) of $\{0,1,\ldots,n-1\}$. Lemma 4.1 proves the lower bound for $N$ in Theorem 1.2; that is, $N \geq n-7$. We proceed to show that $N \leq n-4$.

If $N = n$, we have the arithmetic progression $\{0,1,\ldots,n-1\}$, which is not sum-dominant.

If $N = n-1$, we do not have a sum-dominant set due to item 1 of Lemma 2.3.

If $N = n-2$, we have two cases. If the two missing numbers are not next to each other, we do not have a sum-dominant set due to item 1 of Lemma 2.3. If the two missing numbers are next to each other, we do not have a sum-dominant set due to Theorem 1.1.

If $N = n-3$, we have three cases.

1. Case 4.1: If the three missing numbers are consecutive, then we do not have a sum-dominant set due to Theorem 1.1.

2. Case 4.2: If no two numbers are next to each other, then we do not have a sum-dominant set due to item 1 of Lemma 2.3.

3. Case 4.3: Two numbers are next to each other, while the other is not next to any of these numbers. Let the two numbers that are next to each other be $k$ and $k+1$ for some $k \geq 0$. Without loss of generality, assume that the third number is $k+p$ such that $k+p > k+2$.

   (a) If $k = 0$, we have the set $\{2,3,\ldots,n-1\}\backslash\{k+p\}$, which is not sum-dominant due to item 1 of Lemma 2.3.

   (b) If $k+p = n-1$, we are back to the case $N = n-2$.

(c) Suppose that $k > 0$ and $k + p < n - 1$. We have all differences in $\{0, 1, \ldots, n-1\} \backslash \{k, k+1, k+p\}$ by looking at $\{0, 1, \ldots, n-1\} \backslash \{k, k+1, k+p\} - 0$. If we do not have any missing differences, then we are done.

If $k = 1$, because $n \geq 16$ and we miss only 3 numbers, it must be that we have three consecutive numbers in our set. So, we have differences of 1 and 2, and so, $k$ and $k+1$ are in the difference set. Hence, we miss at most 2 differences, which are $\pm(k+p)$. However, we also miss at least 2 sums, which are 1 and 2. Therefore, we do not have a sum-dominant set.

If $k = 2$, then 1 is in our set. We have $k+p$ by looking at $(k+p+1)-1$ and $k+1$ by looking at $(k+2)-1$. Because $n \geq 16$ and we miss only 3 numbers, it must be that we have three consecutive numbers in our set. So, we have a difference of 2, and so, $k$ is in the difference set. We are done.

If $k > 2$, then 1 and 2 are in our set. We have $k + p$ by looking at $(k + p + 1) - 1$, $k + 1$ by looking at $(k + 2) - 1$, and $k$ by looking at $(k + 2) - 2$. We are done.

## 5. Proof of Theorem 1.4

We will use the construction discussed in [4, Theorem 1.1] to partition $\{1, 2, \ldots, r\}$ into 3 sum-dominant subsets. Following the construction, we fix $n = k = 20$ and set

$$\begin{aligned} L_1 &= \{1, 2, 3, 4, 8, 9, 11, 13, 14, 15, 20\}, \\ R_1 &= \{21, 26, 27, 28, 31, 33, 37, 38, 39, 40\}, \\ L_2 &= \{5, 6, 7, 10, 12, 16, 17, 18, 19\}, \\ R_2 &= \{22, 23, 24, 25, 29, 30, 32, 34, 35, 36\}. \end{aligned}$$

Note that in [4, Theorem 1.1], $A_1 = L_1 \cup R_1$ and $A_2 = L_2 \cup R_2$. Pick $m \geq 21$. Set

$$\begin{aligned} R_1' &= R_1 + m + 84, \\ R_2' &= R_2 + m + 84, \\ O_{11} &= \{24\} \cup \{25, 27, 29, \ldots, 61\} \cup \{62\}, \\ O_{12} &= \{63 + m\} \cup \{64 + m, 66 + m, 68 + m, \ldots, 100 + m\} \cup \{101 + m\}, \\ O_{21} &= \{21, 22, 23\} \cup \{26, 28, 30, \ldots, 60\} \cup \{63, 64, 65\}, \\ O_{22} &= \{60 + m, 61 + m, 62 + m\} \cup \{65 + m, 67 + m, \ldots, 99 + m\} \\ &\cup \ \{102 + m, 103 + m, 104 + m\}. \end{aligned}$$

Let $M_1 \subseteq \{66, 67, \ldots, 59+m\} \backslash \{66, 68, 69, 70, 73, 77, 78, 80\}$ such that within $M_1$, there exists a sequence of pairs of consecutive elements, where consecutive pairs are not more than 39 apart and the sequence starts with a pair in $\{66, 67, \ldots, 101\}$ and ends with a pair in $\{24 + m, 25 + m, \ldots, 59 + m\}$. Let $M_2 \subseteq \{66, 67, \ldots, 59 + m\}$ such that within $M_2$, there exists a sequence of triplets of consecutive elements, where consecutive triplets are not more than 40 apart and the sequence starts with a triplet in $\{66, 67, \ldots, 105\}$ and ends with a triplet in $\{20+m, 21+m, \ldots, 59+m\}$. Also, $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = \{66, 67, \ldots, 59+m\} \backslash \{66, 68, 69, 70, 73, 77, 78, 80\}$. By [4, Theorem 1.1], we know that

$$
\begin{aligned}
A_1' &= L_1 \cup O_{11} \cup M_1 \cup O_{12} \cup R_1' \\
A_2' &= L_2 \cup O_{21} \cup M_2 \cup O_{22} \cup R_2'
\end{aligned}
$$

are both sum-dominant and along with $S = \{66, 68, 69, 70, 73, 77, 78, 80\}$ partition $\{1, 124+m\}$. Note that $S$ is sum-dominant because it is a translation of the smallest sum-dominant set $S' = \{0, 2, 3, 4, 7, 11, 12, 14\}$.

**Example 5.1.** Let $m = 21$. Set

$$
\begin{aligned}
M_1 &= \{71, 72\} \\
M_2 &= \{67, 74, 75, 76, 79\}.
\end{aligned}
$$

We partition $\{1, 145\}$ into the following three sum-dominant sets

$$
\begin{aligned}
A_1' &= L_1 \cup O_{11} \cup M_1 \cup O_{12} \cup R_1' \\
&= \{1, 2, 3, 4, 8, 9, 11, 13, 14, 15, 20, 24\} \cup \{25, 27, 29, \ldots, 61\} \\
&\cup \{62, 71, 72, 84\} \cup \{85, 87, \ldots, 121\} \\
&\cup \{122, 126, 131, 132, 133, 136, 138, 142, 143, 144, 145\} \\
&\text{with } |A_1' + A_1'| - |A_1' - A_1'| = 2, \\
A_2' &= L_2 \cup O_{21} \cup M_2 \cup O_{22} \cup R_2' \\
&= \{5, 6, 7, 10, 12, 16, 17, 18, 19, 21, 22, 23\} \cup \{26, 28, 30, \ldots, 60\} \\
&\cup \{63, 64, 65, 67, 74, 75, 76, 79, 81, 82, 83\} \cup \{86, 88, \ldots, 120\} \cup \{123, 124, 125\} \\
&\cup \{127, 128, 129, 130, 134, 135, 137, 139, 140, 141\} \\
&\text{with } |A_2' + A_2'| - |A_2' - A_2'| = 2, \\
S &= \{66, 68, 69, 70, 73, 77, 78, 80\} \text{ with } |S + S| - |S - S| = 1.
\end{aligned}
$$

Example 5.1 proves the upper bound of 145 for $R$ in our Theorem 1.4.

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