HYPER b-ARY EXPANSIONS AND STERN POLYNOMIALS

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Abstract

We study a recently introduced base b polynomial analog of Stern's diatomic sequence, which generalizes Stern polynomials of Klavžar, Dilcher, Ericksen, Mansour, Stolarsky, and others. We lift some basic properties of base 2 Stern polynomials to arbitrary base, and introduce a matrix characterization of Stern polynomials. By specializing, we recover some new number theoretic results about hyper b-ary partitions, which count partitions of n into powers of b.

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1. Introduction

The aim of this note is to study the properties of arbitrary base polynomial analogs of the Stern sequence, recently introduced and studied in [7, 8]. These are intimately connected to both the theory of automatic sequences and binary partitions. The classical *Stern diatomic sequence* is defined by the two recurrences

$$\begin{aligned} s(2n) &= s(n), \\ s(2n+1) &= s(n) + s(n+1), \end{aligned}$$

with the initial conditions $s(0) = 0, s(1) = 1$. The Stern sequence is closely linked to the theory of automatic sequences [1]; informally, an automatic sequence can be represented by a finite automaton, to which we feed in the digits of n in sequential order. These sequences exhibit interesting aspects of both order and disorder, making them ideal for modeling semi-chaotic physical systems such as quasicrystals.

For our purposes, the most important property of the Stern sequence is its combinatorial interpretation: $s(n+1)$ is equal to the number of *hyperbinary expansions* of n, which is the number of ways to write n as the sum of powers of 2, with each part used at most twice (also known as a base 2 overexpansion of n). While this may seem like a contrived definition, it is one of the simplest possible binary partition functions. While we typically consider partition functions built from sets of integers with nonzero asymptotic density, binary partitions are restricted partitions built from the parts $\{1, 2, 4, 8, 16, \ldots\}$. When each part can be used at most once, this is simply the base 2 expansion of n. When we place no restriction on the number of parts, we obtain the binary partition function popularized by Churchhouse [2]. In this paper, we consider Stern numbers, which allow each part to be used at most twice. Any progress on the Stern case will provide intuition for treating more complicated binary partition functions, which qualitatively behave very differently from more commonly studied partition functions.

We can then consider a base b analog of the Stern sequence, $s_b(n)$, such that the $(n+1)^{\text{th}}$ term equals the number of hyper b-ary expansions (also known as base b overexpansions) of n. These are partitions from the set $\{1, b, b^2, b^3, \ldots\}$, such that each part can be used at most b times. These are completely determined by the recurrence

$$\begin{aligned} &s_b(b(n-1)+j+1) = s_b(n), \ \ 1 \le j \le b-1, \\ &s_b(bn+1) = s_b(n) + s_b(n+1), \\ &s_b(1) = 1, \\ &s_b(0) = 0. \end{aligned}$$

The goal of this paper is to consider an even more general case, the *base b Stern polynomials*, which are polynomials in the b variables $\{z_1, \ldots, z_b\}$, They are indexed by the tuple of positive integers $T := (t_1, \ldots, t_b)$, and are completely determined by the following set of b recurrences and two initial conditions:

$$w_T(b(n-1)+j+1|z_1 \ldots, z_b) = z_j w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}), \;\; 1 \le j \le b-1, \tag{1}$$
$$w_T(bn+1|z_1 \ldots, z_b) = z_b w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}) + w_T(n+1|z_1^{t_1} \ldots, z_b^{t_b}) \tag{2}$$
$$w_T(1|z_1 \ldots, z_b) = 1, \tag{3}$$
$$w_T(0|z_1 \ldots, z_b) = 0. \tag{4}$$

These were recently introduced by Dilcher and Ericksen [7]. When $z_1 = \cdots = z_b = 1$, we have the reduction $w_T(n+1|1, \ldots, 1) = s_b(n)$. In the case $b = 2$, with certain forms of t_i and z_i, we recover polynomial analogues of the Stern sequence introduced by Klavžar et. al [10], Mansour [11], and Dilcher and Stolarsky [9]. These are the most general Stern polynomials studied to date.

In particular, we introduce a generating product for the base b Stern polynomials and prove an identity for a finite version of this product. We then introduce a representation for Stern polynomials in terms of products of 2×2 matrices, following results for automatic sequences due to Allouche and Shallit [1]. Finally, we characterize the behavior of w_T at indices where $s_b(n)$ is maximized, and generalize some recent continued fractions of Dilcher and Ericksen [5]. There is some overlap with the extremely recent results of [8], but we also prove some further new results.

2. Product Representation

Following from the interpretation of $s(n)$ as counting hyperbinary expansions, we have the generating product

$$\sum_{n=1}^{\infty} s(n)t^n = t \prod_{i=0}^{\infty} \left(1 + t^{2^i} + t^{2^{i+1}}\right).$$

Interpreted as a product generating partitions, each term in the product, $1 + t^{2^i} + t^{2^{i+1}}$, corresponds to the choice of whether to pick no occurrences of the part 2^i, one occurrence of this part, or two. This bijectively generates all hyperbinary expansions of n. We can find an analog of this product for the base b Stern polynomials, the proof of which we reproduce to make this paper relatively self-contained.

Theorem 1 ([7, 8]). *The base b Stern polynomials have the generating product*

$$\sum_{n=1}^{\infty} w_T(n|z_1 \ldots, z_b)t^n = t \prod_{i=0}^{\infty} \left(1 + \sum_{j=1}^{b} z_j^{t_j^i} t^{j \cdot b^i}\right). \tag{5}$$

Proof. We consider the simpler case of the hyperternary polynomials ($b = 3$), defined by

$$\begin{aligned}
w_T(3n-1|x,y,z) &= xw_T(n|x^u,y^v,z^w)\\
w_T(3n|x,y,z) &= yw_T(n|x^u,y^v,z^w)\\
w_T(3n+1|x,y,z) &= zw_T(n|x^u,y^v,z^w) + w_T(n+1|x^u,y^v,z^w),
\end{aligned}$$

with $w_T(0) = 0$ and $w_T(1) = 1$. Now consider the generating function

$$F(t|x,y,z) = \sum_{n=1}^{\infty} w_T(n|x,y,z)t^{n-1}.$$

We can form a functional equation for F as follows:

$$\begin{aligned}
F(t|x,y,z) &= \sum_{n=1}^{\infty} w_T(3n-1|x,y,z)t^{3n-2} + \sum_{n=1}^{\infty} w_T(3n|x,y,z)t^{3n-1}\\
&\quad + \sum_{n=1}^{\infty} w_T(3n+1|x,y,z)t^{3n}\\
&= \sum_{n=1}^{\infty} xw_T(n|x^u,y^v,z^w)t^{3n-2} + \sum_{n=1}^{\infty} yw_T(n|x^u,y^v,z^w)t^{3n-1}\\
&+ \sum_{n=1}^{\infty} zw_T(n|x^u,y^v,z^w)t^{3n} + \sum_{n=1}^{\infty} w_T(n+1|x^u,y^v,z^w)t^{3n}\\
&= xtF(t^3|x^u,y^v,z^w) + yt^2F(t^3|x^u,y^v,z^w) + zt^3F(t^3|x^u,y^v,z^w)\\
&\quad + F(t^3|x^u,y^v,z^w)\\
&= (1 + xt + yt^2 + zt^3)F(t^3|x^u,y^v,z^w).
\end{aligned}$$

We then iterate this functional equation while noting that for $|t|$ sufficiently small, $F(t^n|x,y,z) \to 1$ as $n \to \infty$. This is a subtle fact; if each $|z_i| < 1$ then by the triangle inequality we have $|w_T(n+1|z_1,\ldots,z_b)| < |w_T(n+1|1,\ldots,1)|$, which counts the number of hyper b-ary expansions of n. We then have a (difficult) characterization of the maximal order (see [3] and [4]),

$$\limsup_{n\to\infty} \frac{s_b(n)}{n^{\log_b \phi}} = \frac{\phi^{\log_b(b^2-1)}}{\sqrt{5}},$$

where $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$ is the golden ratio. This shows that $s_b(n) = O(n^{\log_b \phi})$ grows polynomially, so that the generating function $F(t^n|x,y,z)$ will converge to its constant term of 1 for sufficiently small t. Altogether, this gives us the infinite product representation

$$F(t|x,y,z) = \prod_{i=0}^{\infty}\left(1 + x^{u^i}t^{3^i} + y^{v^i}t^{2\cdot 3^i} + z^{w^i}t^{3\cdot 3^i}\right).$$

n	$\omega_T(n\|x,y,z)$	n	$\omega_T(n\|x,y,z)$	n	$\omega_T(n\|x,y,z)$
1	1	10	$x^{u^2}+y^vz+z^w$	19	$x^{u^2}y^vz+x^{u^2}z^w+y^{v^2}$
2	x	11	$x^{1+u^2}+xz^w$	20	$x^{1+u^2}z^w+xy^{v^2}$
3	y	12	$x^{u^2}y+yz^w$	21	$x^{u^2}yz^w+y^{1+v^2}$
4	x^u+z	13	$x^{u+u^2}+x^{u^2}z+z^{1+w}$	22	$x^{u^2}z^{1+w}+x^uy^{v^2}+y^{v^2}z$
5	x^{1+u}	14	x^{1+u+u^2}	23	$x^{1+u}y^{v^2}$
6	x^uy	15	$x^{u+u^2}y$	24	$x^uy^{1+v^2}$
7	x^uz+y^v	16	$x^{u+u^2}z+x^{u^2}y^v$	25	$x^uy^{v^2}z+y^{v+v^2}$
8	xy^v	17	$x^{1+u^2}y^v$	26	xy^{v+v^2}
9	y^{1+v}	18	$x^{u^2}y^{1+v}$	27	y^{1+v+v^2}

Table 1: Small n hyperternary polynomials [7, Table 2]

The arbitrary base analog is proved exactly analogously. We omit the proof because it offers no new insight; the hyperternary case is much more obvious and notationally clearer. □

Remark 1. Note that basically any polynomial generalization of hyperternary expansions begins with the infinite product and inserts variables somewhere inside the product, so that this product representation is excellent motivation for how to define a polynomial version of the base b Stern sequence. This phenomenon is apparent for $b = 2$; despite having very different recursive definitions, any form of base 2 Stern polynomial that has been introduced thus far has a compact generating product.

Using this infinite product as a starting point, we can derive several properties of the base b Stern polynomials which mirror the base 2 case. The advantage of this representation is that it gives us some intuition for exactly which theorems are possible to generalize, which isn't very apparent from the recursive definition.

A first nice consequence of this infinite product is that we can recover the main theorem of [7], essentially by inspection. We can check the following result with Table 1 of small hyperternary polynomials. For what follows, let $\mathbb{H}_{b,n}$ denote the set of hyper b-ary expansions of $n \geq 1$.

Theorem 2 ([7, Theorem 11]). *For any integer $n \geq 1$ we have*

$$w_T(n+1|z_1,\ldots,z_b) = \sum_{h\in\mathbb{H}_{b,n}} z_1^{p_{h,1}(t_1)}\cdots z_b^{p_{h,b}(t_b)},$$

where for each h in $\mathbb{H}_{b,n}$, the exponents $p_{h,1}(t_1),\ldots,p_{h,b}(t_b)$ are polynomials in $t_1,\ldots,t_b$, respectively, with only 0 and 1 as coefficients. Furthermore, if for $1 \leq j \leq b$ we write

$$p_{h,j}(t_j) = t_j^{\tau_j(1)} + t_j^{\tau_j(2)} + \cdots + t_j^{\tau_j(\nu_j)},\ 0 \leq \tau_j(1) < \cdots < \tau_j(\nu_j),\ \nu_j \geq 0, \tag{6}$$

then the powers that are used exactly j times in the hyper b-ary representation of n are

$$b^{\tau_j(1)}, b^{\tau_j(2)}, \ldots, b^{\tau_j(\nu_j)}. \tag{7}$$

If $\nu_j = 0$ in (6), we set $p_{h,j}(t_j) = 0$ by convention, and accordingly (7) is the empty set.

Proof. Consider the infinite product (5). Consider a hyper b-ary expansion of n of the form $n = a_0 + a_1 b + a_2 b^2 + \cdots$, where each $0 \le a_i \le b$. Upon expanding the product this corresponds to a monomial of the form

$$\prod_i z_{a_i}^{t_{a_i}^i} t^{a_i b^i} = t^{\sum_i a_i b^i} \prod_i z_{a_i}^{t_{a_i}^i} = t^n \prod_i z_{a_i}^{t_{a_i}^i}$$

in the resulting infinite sum. Furthermore, every possible way to expand the infinite product leads to a monomial of this form which corresponds in a bijective fashion to a hyper b-ary expansion of n. The coefficient of t^n sums over a polynomial in $\{z_j^{t_j}\}$, where each monomial corresponds to a hyper b-ary expansion of n in this bijective fashion. Furthermore, given such a monomial term we can obtain the corresponding hyperbinary expansion by the procedure given above, in the statement of this theorem. Adjusting the indexing by 1, since the $(n+1)$th Stern polynomial corresponds to hyper b-ary expansions of n, completes the proof. $\square$

We can consider partial products to give a base b polynomial analog of the classic result

$$\sum_{n=2^N+1}^{2^{N+1}} s(n) = 3^N,$$

which was then generalized to base 2 Stern polynomials [5, Proposition 5.2] as (in our notation)

$$\prod_{i=0}^{N-1} \left(2 + z^{t^i}\right) = \sum_{n=2^N+1}^{2^{N+1}} w_T(n|1, z). \tag{8}$$

Our direct proof is combinatorial and quite different from that of [5], which inductively depends on recursive properties of the Stern polynomials. The fact that we have $z_1 = 1$ is in fact essential for this identity; inspecting small hyperternary polynomials indicates that there is no closed form analog in terms of $\{z_1, \ldots, z_b\}$.

Theorem 3. *Let $\ell_N := \frac{b^{N+1}-1}{b-1}$. Then we have the finite product*

$$\prod_{i=0}^{N-1} \left(2 + \sum_{j=2}^{b} z_j^{t_j^i}\right) = \sum_{n=\ell_{N-1}+2}^{\ell_N+1} w_T(n|1, z_2, z_3 \ldots, z_b).$$

Proof. At the outset, fix $b \geq 2$ and $N \geq 1$. For $n \geq 1$, define

$$\mathbb{H}_{b,n} := \{(a_0, \ldots, a_n) \big| \sum_{i=0}^{n} a_i b^i = n, 0 \leq a_i \leq b\},$$

and

$$\mathcal{H}_{b,n}(m) := \{(a_0, \ldots, a_n) \big| \sum_{i=0}^{n} a_i b^i = n, 0 \leq a_i \leq b, a_{\log_b m} = 1\}.$$

Intuitively, $\mathbb{H}_{b,n}$ is the set of hyper b-ary expansions of n, and $\mathcal{H}_{b,n}(m)$ is the set of hyper b-ary expansions of n containing the part m with multiplicity one. The use of $a_{\log_b m}$ in the definition is well-defined, since by definition hyper b-ary expansions only count decompositions into powers of b, so the logarithm of such a part will always be a non-negative integer.

Initially, note that we have a bijection,

$$\bigcup_{n=0}^{\ell_N - 1} \mathcal{H}_{b,n}\left(b^N\right) \to \bigcup_{n=0}^{\ell_{N-1}-1} \mathbb{H}_{b,n}, \tag{9}$$

which preserves parts of multiplicity at least 2.

This proceeds in the obvious fashion: consider every nonempty hyper b-ary expansion on the left–hand side and subtract b^N. Note that since by assumption every element in $\mathcal{H}_{b,n}(b^N)$ contains the part b^N, we have that $\mathcal{H}_{b,n}(b^N)$ is empty for $n < b^N$. Removing b^N thus gives a hyper b-ary expansion of $n - b^N$, which transforms the upper limit on n as

$$\ell_N - 1 - b^N = \frac{b^{N+1} - 1}{b - 1} - b^N - 1 = \frac{b^N - 1}{b - 1} - 1 = \ell_{N-1} - 1.$$

Furthermore, any hyper b-ary expansion on the right–hand side cannot contain the part b^N since

$$\ell_{N-1} - 1 = \frac{b^N - 1}{b - 1} - 1 = \frac{b^N}{b - 1} - \frac{b}{b - 1} < b^N.$$

Thus, removing b^N removes a part of multiplicity 1, and the operation preserves parts of multiplicity at least 2. For the reverse mapping, appending b^N to a hyper b-ary expansion of $n, 0 \leq n \leq \ell_{N-1} - 1$, takes a part of multiplicity 0 (since nothing on the right–hand side can contain b^N) to multiplicity 1, and hence also preserves parts of multiplicity at least 2.

Secondly, we can expand the finite product

$$\prod_{i=0}^{N-1} \left(1 + \sum_{j=1}^{b} z_j^{t_j^i} t^{j \cdot b^i}\right) = \sum_{(a_0, \ldots, a_n) \in \mathcal{S}} t^{\sum_{i=0}^{n} a_i b^i} \prod_{i=0}^{n} z_{a_i}^{t_{a_i}^i}, \tag{10}$$

where $(a_0, \ldots, a_n)$ encodes the hyper b-ary expansion $\sum_{i=0}^{n} a_i b^i, 0 \le a_i \le b$, we have the set of expansions

$$\mathcal{S} := \bigcup_{n=0}^{\ell_N - 1} \mathbb{H}_{b,n} \setminus \bigcup_{n=0}^{\ell_N - 1} \mathcal{H}_{b,n}\left(b^N\right),$$

and we set $z_0 := 1$.

Note that when we expand the left–hand side as a polynomial in t, the largest power of t will be $b + b^2 + \cdots + b^N = \ell_N - 1 < 2b^N$, so consider the set of all hyper b-ary expansions of integers less than or equal to $\ell_N - 1$. Since the left–hand product generates all hyper b-ary expansions with largest part b^{N-1}, the only elements of $\cup_{n=0}^{\ell_N - 1} \mathbb{H}_{b,n}$ which are omitted in the sum are those which contain the part b^N exactly once (since $\ell_N - 1 < 2b^N$), matching the definition of $\mathcal{S}$. Finally, notice that we neglect to track the occurrence of parts which are not used, which is why we have $z_0 = 1$.

Now, set $z_1 = t = 1$ in (10) to obtain

$$\prod_{i=0}^{N-1} \left(2 + \sum_{j=2}^{b} z_j^{t_j^i}\right) = \sum_{(a_0, \ldots, a_n) \in \mathcal{S}} t^{\sum_{i=0}^{n} a_i b^i} \prod_{\substack{i=0 \\ a_i \ge 2}}^{n} z_{a_i}^{t_{a_i}^i}, \tag{11}$$

where the restriction $a_i \ge 2$ means that we only track parts of multiplicity at least 2. Then we apply (9), transforming $\mathcal{S}$ as

$$\bigcup_{n=0}^{\ell_N - 1} \mathbb{H}_{b,n} \setminus \bigcup_{n=0}^{\ell_N - 1} \mathcal{H}_{b,n}\left(b^N\right) \to \bigcup_{n=0}^{\ell_N - 1} \mathbb{H}_{b,n} \setminus \bigcup_{n=0}^{\ell_{N-1} - 1} \mathbb{H}_{b,n} \to \bigcup_{n=\ell_{N-1}}^{\ell_N - 1} \mathbb{H}_{b,n},$$

while noting that we can then rewrite the right–hand side of (11) in terms of Stern polynomials as

$$\sum_{n=\ell_{N-1}+1}^{\ell_N} w_T(n|1, z_2, z_3 \ldots, z_b).$$

Finally, in order to correctly reduce to (8) in the $b = 2$ case, we note that since for any N we have $\ell_N + 1 = 2 + b(1 + b + \cdots + b^{N-1}) \equiv 2 \pmod{b}$, by iterating recurrence (1) we have

$$\begin{aligned} w_T\left(\ell_N + 1|1, z_2, \ldots, z_b\right) &= 1 \cdot w_T\left(1 + \frac{\ell_N - 1}{b}\middle|1, z_2^{t_2}, \ldots, z_b^{t_b}\right) \\ &= w_T\left(\ell_{N-1} + 1|1, z_2^{t_2}, \ldots, z_b^{t_b}\right) \\ &= \cdots = w_T\left(\ell_0 + 1|1, z_2^{t_2^N}, \ldots, z_b^{t_b^N}\right) \\ &= 1. \end{aligned}$$

Therefore, we can rewrite the bounds of summation as in the given theorem. $\square$

Even when $z_i = 1$, this identity for the number of hyper b-ary expansions appears to be new:

$$(b+1)^N = \sum_{n=\ell_{N-1}+2}^{\ell_N+1} s_b(n) = \sum_{n=\ell_{N-1}+1}^{\ell_N} s_b(n). \tag{12}$$

3. Matrix Representation

We can also describe an explicit matrix characterization of the Stern polynomials. This is a variant of Allouche and Shallit's famous result [1] about k-regular sequences, a generalization of automatic sequences. There are several equivalent definitions of k-regular sequences, the most basic of which is that the sequence $s(n)$ is k-regular if the *k-kernel* $\{(s(k^e n + r))_{n\geq 0} | e \geq 0 \text{ and } 0 \leq r \leq k^e - 1\}$ generates a finite dimensional vector space over $\mathbb{Q}$.

Theorem 4 ([1])**.** *The integer sequence $\{s(n)\}$ is k-regular if and only if there exists positive integers m and d, matrices $\mathbf{A}_0, \ldots, \mathbf{A}_{b-1} \in \mathbb{Z}^{d\times d}$, and vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d$ such that*

$$s(n) = \mathbf{w}^T \mathbf{A}_{i_0} \cdots \mathbf{A}_{i_s} \mathbf{v},$$

where $i_s \cdots i_0$ is the base b expansion of n.

The base b Stern sequence satisfies this characterization [3] with

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{A}_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{A}_i = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, 2 \leq i \leq b-1.$$

We can derive an analog of this result for Stern polynomials, at the cost of parametrizing the matrices.

Theorem 5. *Let $i_s \cdots i_0$ be the base b expansion of n. Then*

$$w_T(n|z_1 \ldots, z_b) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{A}_{i_0}(0) \cdots \mathbf{A}_{i_s}(s) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with

$$\mathbf{A}_0(d) = \begin{pmatrix} z_{b-1}^{t_{b-1}^d} & 0 \\ z_b^{t_b^d} & 1 \end{pmatrix}, \mathbf{A}_1(d) = \begin{pmatrix} z_b^{t_b^d} & 1 \\ 0 & z_1^{t_1^d} \end{pmatrix}, \mathbf{A}_i(d) = \begin{pmatrix} 0 & z_{i-1}^{t_{i-1}^d} \\ 0 & z_i^{t_i^d} \end{pmatrix}, 2 \leq i \leq b-1.$$

Proof. We proceed by induction on the value of s, and will show the even stronger statement that

$$\mathbf{A}_{i_0}(0) \cdots \mathbf{A}_{i_s}(s) = \begin{pmatrix} * & w_T(n|z_1, \ldots, z_b) \\ * & w_T(n+1|z_1, \ldots, z_b) \end{pmatrix},$$

where asterisks denote quantities that need not be taken into account for the result. Our base case is when n is a single digit, and can easily be verified since we consider $\mathbf{A}_i(0)$, $0 \leq i \leq b-1$.

We inductively assume that the theorem holds for all digit strings of length at most $s+1$, and consider any string of length $s+1$, $i_s \cdots i_0$, which represents the integer n. Then consider a digit string of length $s+2$, which we assume to be $i_{s+1} i_s \cdots i_0$, and which represents $bn + i_0$. By shifting the parametrization by 1, we map $z_i \mapsto z_i^{t_i}$, yielding

$$\mathbf{A}_{i_0}(0)\left[\mathbf{A}_{i_1}(1) \cdots \mathbf{A}_{i_{s+1}}(s+1)\right] = \mathbf{A}_{i_0}(0) \begin{pmatrix} * & w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}) \\ * & w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \end{pmatrix}.$$

We then have 3 cases:

1. $i_0 = 0$:

$$\begin{aligned} &\begin{pmatrix} z_{b-1} & 0 \\ z_b & 1 \end{pmatrix} \begin{pmatrix} * & w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}) \\ * & w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \end{pmatrix} \\ &\qquad = \begin{pmatrix} * & z_{b-1} w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}) \\ * & z_b w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}) + w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \end{pmatrix} \\ &\qquad = \begin{pmatrix} * & w_T(bn|z_1, \ldots, z_b) \\ * & w_T(bn+1|z_1, \ldots, z_b) \end{pmatrix}. \end{aligned}$$

2. $i_0 = 1$:

$$\begin{aligned} &\begin{pmatrix} z_b & 1 \\ 0 & z_1 \end{pmatrix} \begin{pmatrix} * & w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}) \\ * & w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \end{pmatrix} \\ &\qquad = \begin{pmatrix} * & z_b w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}) + w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \\ * & z_1 w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \end{pmatrix} \\ &\qquad = \begin{pmatrix} * & w_T(bn+1|z_1, \ldots, z_b) \\ * & w_T(bn+2|z_1, \ldots, z_b) \end{pmatrix}. \end{aligned}$$

3. $2 \leq i_0 \leq b-1$:

$$\begin{aligned} \begin{pmatrix} 0 & z_{i_0-1} \\ 0 & z_{i_0} \end{pmatrix} \begin{pmatrix} * & w_T(n|z_1^{t_1}, \ldots, z_b^{t_b}) \\ * & w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \end{pmatrix} &= \begin{pmatrix} * & z_{i_0-1} w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \\ * & z_{i_0} w_T(n+1|z_1^{t_1}, \ldots, z_b^{t_b}) \end{pmatrix} \\ &= \begin{pmatrix} * & w_T(bn+i_0|z_1, \ldots, z_b) \\ * & w_T(bn+i_0+1|z_1, \ldots, z_b) \end{pmatrix}. \end{aligned}$$

Since in each case, we recover $bn + i_0$ and $bn + i_0 + 1$, we're done. □

For example, consider the hyperternary polynomials again, with $(z_1, z_2, z_3) \mapsto (x, y, z)$ and $(t_1, t_2, t_3) \mapsto (r, s, t)$ for notational clarity. Consider $n = 7$, which is represented by the digit string 21 in base 3. Then

$$\mathbf{A}_1(0)\mathbf{A}_2(1) = \begin{pmatrix} z & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & x^r \\ 0 & y^s \end{pmatrix} = \begin{pmatrix} 0 & x^r z + y^s \\ 0 & xy^s \end{pmatrix} = \begin{pmatrix} 0 & w_T(7|x, y, z) \\ 0 & w_T(8|x, y, z) \end{pmatrix},$$

just as we expect. To the best of our knowledge, only one such matrix characterization of a polynomial analog of the Stern sequence has previously been stated in the literature [12], despite the fact that it is integral to considering the maximal order of the base b Stern sequence [3]. Spiegelhofer used this matrix representation to prove a digit reversal property for base 2 Stern polynomials; however, an arbitrary base analog has been elusive.

4. Behavior at Maximal Indices

The maximal values of $s_b(n)$ have been well characterized [3]. We can study the related values of w_T at these indices, which will provide polynomial dissections of the Fibonacci numbers. Let $F_k = F_{k-1} + F_{k-2}$ denote the kth Fibonacci number, with the initial values $F_0 = 0, F_1 = 1$. Let $(a_1a_2\cdots a_l)_b$ denote the integer $a_1b^{l-1} + a_2b^{l-2} + \cdots a_l$, which is the number read off in base b. We also let $((a)^k b)_b = (\underbrace{a\cdots a}_{k\text{ times}}b)_b$. Note that much of the material of this section generalizes recent results of Dilcher and Ericksen [6], and that by following the methods of that paper we should be able to describe other new finite and infinite continued fractions involving base b Stern polynomials.

Lemma 1 ([4])**.** *Let $k \geq 2$. Then*

$$\max_{b^{k-2}\leq n<b^{k-1}} s_b(n) = F_k.$$

Moreover, if a_k denotes the smallest n in the interval $[b^{k-2}, b^{k-1})$ for which this maximum is attained, then

$$a_k = \frac{b^k-1}{b^2-1} + \left(\frac{1-(-1)^k}{2}\right)\frac{b}{b+1}.$$

We will heavily depend on the base b expansions (only valid for $l \geq 1$) $a_{2l} = ((10)^{l-1}1)_b$ and $a_{2l+1} = ((10)^{l-1}11)_b$. Note also that $a_1 = a_2 = 1$. In base $b = 2$, these are the well known *Jacobsthal numbers* $\frac{1}{3}(2^n - (-1)^n)$.

We first note some simple properties of this sequence of maximal indices. This first result generalizes the simple recurrence $J_n = 2J_{n-1} + (-1)^n$.

Lemma 2. *The base b Jacobsthal numbers satisfy the recurrence*

$$\begin{aligned} a_n &= ba_{n-1} + 1 - \frac{b}{2}(1+(-1)^n),\ n \geq 2, \\ a_1 &= 1. \end{aligned}$$

Proof. Begin with the base b expansions of $a_{2l}, a_{2l+1}, l \geq 1$, and peel off the last digit to show

$$\begin{aligned} a_{2l} &= (1-b) + ba_{2l-1} \\ a_{2l+1} &= 1 + ba_{2l}. \end{aligned}$$

Rewriting the recurrence with a $1+(-1)^n$ indicator to account for the parity of n completes the proof. □

We then study the values of the base b Stern polynomials at these indices. In the $z_i = 1$ limit we will recover Fibonacci numbers, so we are essentially studying some renormalized variant of Fibonacci polynomials.

Theorem 6. *We have the recurrences*

$$\begin{aligned} &w_T(a_{2l+1}|z_1,\ldots,z_b) = z_b w_T(a_{2l}|z_1^{t_1},\ldots,z_b^{t_b}) + z_1^{t_1} w_T(a_{2l-1}|z_1^{t_1^2},\ldots,z_b^{t_b^2}),\ l \geq 1, \\ &w_T(a_{2l+2}|z_1,\ldots,z_b) = w_T(a_{2l+1}|z_1^{t_1},\ldots,z_b^{t_b}) + z_b z_{b-1}^{t_{b-1}} w_T(a_{2l}|z_1^{t_1^2},\ldots,z_b^{t_b^2}),\ l \geq 1, \\ &w_T(a_1|z_1,\ldots,z_b) = w_T(a_2|z_1,\ldots,z_b) = 1. \end{aligned}$$

Proof. For the base cases, since $a_1 = a_2 = 1$ we have $w_T(1|z_1,\ldots,z_b) = 1$. We will now work directly with the digit expansions of a_n and apply the recursive definitions of the Stern polynomials. In particular, assuming $b \geq 3$ and $l \geq 1$, we have

$$\begin{aligned} w_T(a_{2l+1}|z_1,\ldots,z_b) &= w_T(((10)^{l-1}11)_b|z_1,\ldots,z_b) \\ &= z_b w_T(((10)^{l-1}1)_b|z_1^{t_1},\ldots,z_b^{t_b}) + w_T(((10)^{l-2}102)_b|z_1^{t_1},\ldots,z_b^{t_b}) \\ &= z_b w_T(a_{2l}|z_1^{t_1},\ldots,z_b^{t_b}) + z_1^{t_1} w_T(((10)^{l-2}11)_b|z_1^{t_1^2},\ldots,z_b^{t_b^2}). \end{aligned}$$

If $b = 2$ we have $((10)^{l-2}102)_b = ((10)^{l-2}110)_b$ at the second step, but since $z_1 = z_{b-1}$ we obtain the same final recurrence. We perform a similar calculation with $a_{2l+2} = ((10)^l 1)_b$ to obtain the second recurrence. □

We particularly note the first two nontrivial cases, $w_T(a_3|z_1,\ldots,z_b) = z_b + z_1^{t_1}$ and $w_T(a_4|z_1,\ldots,z_b) = z_b z_{b-1}^{t_{b-1}} + z_b^{t_b} + z_1^{t_1^2}$. Since we have a three term recurrence, we can now write down a simple "two level" continued fraction involving a ratio of Stern polynomials. If we set $z_i = 1$ the limit of this ratio is $\lim_{k\to\infty} F_k/F_{k-1} = \frac{1}{2}(1+\sqrt{5})$, the golden ratio. Therefore our continued fraction can be regarded as a finite polynomial generalization of the canonical continued fraction

$$\phi := \frac{1}{2}(1+\sqrt{5}) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}.$$

Theorem 7. *For $l \geq 2$, we have the continued fractions*

$$\frac{w_T(a_{2l+1}|z_1,\ldots,z_b)}{w_T(a_{2l}|z_1^{t_1},\ldots,z_b^{t_b})} = z_b + \cfrac{z_1^{t_1}}{1 + \cfrac{z_b^{t_b} z_{b-1}^{t_{b-1}^2}}{\ddots + \cfrac{z_b^{t_b^{2l-3}} z_{b-1}^{t_{b-1}^{2l-2}}}{z_b^{t_b^{2l-2}} + z_1^{t_1^{2l-1}}}}}$$

and

$$\frac{w_T(a_{2l+2}|z_1,\ldots,z_b)}{w_T(a_{2l+1}|z_1^{t_1},\ldots,z_b^{t_b})} = 1 + \cfrac{z_b z_{b-1}^{t_{b-1}}}{z_b^2 + \cfrac{z_1^{t_1^2}}{\ddots + \cfrac{z_b^{t_b^{2l-2}} z_{b-1}^{t_{b-1}^{2l-1}}}{z_b^{t_b^{2l-1}} + z_1^{t_1^{2l}}}}}.$$

Proof. Assume that $l \geq 2$. Begin with the recurrences of Theorem 6 and divide by the middle term. We obtain

$$\frac{w_T(a_{2l+1}|z_1,\ldots,z_b)}{w_T(a_{2l}|z_1^{t_1},\ldots,z_b^{t_b})} = z_b + \frac{z_1^{t_1}}{\left(\dfrac{w_T(a_{2l}|z_1^{t_1},\ldots,z_b^{t_b})}{w_T(a_{2l-1}|z_1^{t_1^2},\ldots,z_b^{t_b^2})}\right)} \tag{13}$$

and

$$\frac{w_T(a_{2l+2}|z_1,\ldots,z_b)}{w_T(a_{2l+1}|z_1^{t_1},\ldots,z_b^{t_b})} = 1 + \frac{z_b z_{b-1}^{t_{b-1}}}{\left(\dfrac{w_T(a_{2l+1}|z_1^{t_1},\ldots,z_b^{t_b})}{w_T(a_{2l}|z_1^{t_1^2},\ldots,z_b^{t_b^2})}\right)}. \tag{14}$$

Substituting (13) into (14) and vice versa gives

$$\frac{w_T(a_{2l+1}|z_1,\ldots,z_b)}{w_T(a_{2l}|z_1^{t_1},\ldots,z_b^{t_b})} = z_b + \cfrac{z_1^{t_1}}{1 + \cfrac{z_b^{t_b} z_{b-1}^{t_{b-1}^2}}{\left(\dfrac{w_T(a_{2l-1}|z_1^{t_1^2},\ldots,z_b^{t_b^2})}{w_T(a_{2l-2}|z_1^{t_1^3},\ldots,z_b^{t_b^3})}\right)}}$$

and

$$\frac{w_T(a_{2l+2}|z_1,\ldots,z_b)}{w_T(a_{2l+1}|z_1^{t_1},\ldots,z_b^{t_b})} = 1 + \cfrac{z_b z_{b-1}^{t_{b-1}}}{z_b^2 + \cfrac{z_1^{t_1^2}}{\left(\dfrac{w_T(a_{2l}|z_1^{t_1^2},\ldots,z_b^{t_b^2})}{w_T(a_{2l-1}|z_1^{t_1^3},\ldots,z_b^{t_b^3})}\right)}}.$$

We required $l \geq 2$ so that we could apply the recurrence of Theorem 6 twice in a row. We can now avoid mixing recurrences, and iterate this construction while noting the base cases $w_T(a_2|z_1,\ldots,z_b) = 1$ and $w_T(a_3|z_1,\ldots,z_b) = z_b + z_1^{t_1}$. □

These give extremely fastly converging approximations for w_T at maximal indices, since $z_i^{t_i^j} \to 0$ quickly for small z_i.

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