IDENTITIES AND INEQUALITIES FOR SUMS INVOLVING BINOMIAL COEFFICIENTS

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Abstract

We present identities and inequalities for sums involving the partial sums of binomial coefficients. One of our results states that

$$T_n = \sum_{k=0}^{n} \binom{n}{k} k \sum_{j=0}^{k} \binom{n}{j} = n2^{2n-2} + \frac{n}{2}\binom{2n}{n} \quad (n = 0, 1, 2, \dots),$$

$$\sqrt{T_{n-1}T_{n+1}} < T_n < \frac{T_{n-1} + T_{n+1}}{2} \quad (n = 1, 2, 3, \dots)$$

and

$$T_m + T_n \le T_{m+n} \quad (m, n = 0, 1, 2, \dots)$$

with equality if and only if $m = 0$ or $n = 0$.

1. Introduction and Statement of Main Results

In 1994, Calkin [2] presented an interesting identity of sums of powers of the partial sums of binomial coefficients:

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^3 = n2^{3n-1} + 2^{3n} - 3n2^{n-2}\binom{2n}{n}. \tag{1}$$

This result attracted the attention of several mathematicians who discovered new proofs as well as various extensions and variants of (1). We refer to [3], [5], [13], [16] and the references cited therein.

Our work is inspired by three remarkable papers published by Hirschhorn [6], Wang and Zhang [12] and Zhang [14]. Among others, these authors offered identities for the following sums which are related to Calkin's sum given in (1):

$$A_n = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j}, \quad A_n^* = \sum_{k=0}^{n}(-1)^k\sum_{j=0}^{k}\binom{n}{j},$$
$$B_n = \sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2, \quad B_n^* = \sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2,$$
$$C_n = \sum_{k=0}^{n}k\sum_{j=0}^{k}\binom{n}{j}, \quad D_n = \sum_{k=0}^{n}k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2.$$

They proved that for $n \geq 1$ we have

$$A_n = (n+2)2^{n-1}, \quad A_n^* = (-1)^n 2^{n-1}, \tag{2}$$

$$B_n = (n+2)2^{2n-1} - \frac{n}{2}\binom{2n}{n}, \tag{3}$$

$$B_n^* = 2^{2n-1}, \quad \text{if } n \text{ is even}, \tag{4}$$

$$B_n^* = -2^{2n-1} + (-1)^{(n+1)/2}\binom{n-1}{(n-1)/2}, \quad \text{if } n \text{ is odd}, \tag{5}$$

$$C_n = n(3n+5)2^{n-3}, \tag{6}$$

$$D_n = n(3n+5)2^{2n-3} - \frac{(n-1)n}{4}\binom{2n}{n}. \tag{7}$$

With regard to these identities it is natural to ask for similar results for the alternating sums

$$C_n^* = \sum_{k=0}^{n}(-1)^k k\sum_{j=0}^{k}\binom{n}{j} \quad \text{and} \quad D_n^* = \sum_{k=0}^{n}(-1)^k k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2.$$

We have $C_1^* = -2$ and for $n \geq 2$,

$$\begin{aligned} C_n^* &= \sum_{j=0}^{n}\binom{n}{j}\sum_{k=j}^{n}(-1)^k k = \sum_{j=0}^{n}\binom{n}{j}\left[(-1)^n\left(\frac{n}{2}+\frac{1}{4}\right) + (-1)^j\left(\frac{j}{2}-\frac{1}{4}\right)\right] \\ &= (-1)^n(2n+1)2^{n-2}. \end{aligned} \tag{8}$$

The problem to find a corresponding identity for D_n^* is a bit more difficult. Our first theorem presents the solution.

Theorem 1. *For all natural numbers n we have*

$$D_n^* = (2n+1)4^{n-1} + (-1)^{(n+2)/2}(n+1)\binom{n-2}{(n-2)/2}, \quad \textit{if } n \textit{ is even}, \tag{9}$$

and

$$D_1^* = -4, \quad D_n^* = -(2n+1)4^{n-1} + (-1)^{(n+1)/2}\frac{n-1}{2}\binom{n-1}{(n-1)/2}, \quad \textit{if } n \geq 3 \textit{ is odd}. \tag{10}$$

In what follows, we study the four sums which we obtain from A_n, A_n^*, C_n and C_n^* by inserting the factor $\binom{n}{k}$ behind the first summation sign, that is,

$$S_n = \sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{n}{j}, \quad S_n^* = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{k}\binom{n}{j},$$
$$T_n = \sum_{k=0}^{n}\binom{n}{k}k\sum_{j=0}^{k}\binom{n}{j}, \quad T_n^* = \sum_{k=0}^{n}(-1)^k\binom{n}{k}k\sum_{j=0}^{k}\binom{n}{j}.$$

We show that each of the four sums can be expressed in terms of the central binomial coefficient $\binom{2N}{N}$. The next result offers counterparts of the two identities given in (2).

Theorem 2. *For all nonnegative integers n we have*

$$S_n = 2^{2n-1} + \frac{1}{2}\binom{2n}{n} \tag{11}$$

and

$$S_0^* = 1, \quad S_n^* = \frac{(-1)^{n/2}}{2}\binom{n}{n/2}, \quad \textit{if } n \geq 2 \textit{ is even}, \tag{12}$$

$$S_n^* = (-1)^{(n+1)/2}\binom{n-1}{(n-1)/2}, \quad \textit{if } n \textit{ is odd}. \tag{13}$$

The following theorem provides counterparts of (6) and (8).

Theorem 3. *For all nonnegative integers n we have*

$$T_n = n2^{2n-2} + \frac{n}{2}\binom{2n}{n} \tag{14}$$

and

$$T_0^* = 0, \quad T_n^* = (-1)^{(n+2)/2}2\binom{n-2}{(n-2)/2}, \quad \textit{if } n \geq 2 \textit{ is even}, \tag{15}$$

$$T_1^* = -2, \quad T_n^* = (-1)^{(n+1)/2}n\binom{n-1}{(n-1)/2}, \quad \textit{if } n \geq 3 \textit{ is odd}. \tag{16}$$

From (3), (7), (11) and (14) we conclude that the sums B_n, D_n, S_n and T_n are connected by the identities

$$S_n - \frac{1}{n}T_n = \frac{B_n + T_n}{3n+4} = \frac{2D_n - (n-1)B_n}{n^2+3n+4} = 4^{n-1} \quad (n = 1, 2, 3, \dots).$$

Identities for sums and their alternating relatives can be used to obtain identities for sums, where the summation index runs over only even and over only odd integers, respectively. For example, from (11), (12) and (13) we get

$$\sum_{\substack{k=0\\k \text{ even}}}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{n}{j} = \frac{S_n + S_n^*}{2} = 4^{n-1} + \frac{1}{4}\binom{2n}{n} + \sigma_n \quad (n = 1, 2, 3, \dots),$$

where

$$\sigma_n = \frac{1}{4}(-1)^{n/2}\binom{n}{n/2}, \quad \text{if } n \text{ is even}, \quad \text{and}$$
$$\sigma_n = \frac{1}{2}(-1)^{(n+1)/2}\binom{n-1}{(n-1)/2}, \quad \text{if } n \text{ is odd}.$$

We apply (11) and (14) to prove that the sequences $(S_n)_{n\geq 0}$ and $(T_n)_{n\geq 0}$ satisfy certain convexity/concavity and superadditive properties.

Theorem 4. *The sequence $(S_n)_{n\geq 0}$ is strictly log-convex, that is, we have*

$$S_n < \sqrt{S_{n-1}S_{n+1}} \quad (n = 1, 2, 3, \dots). \tag{17}$$

Moreover, $(\log S_n)_{n\geq 0}$ is superadditive, that is,

$$S_m S_n \leq S_{m+n} \quad (m, n = 0, 1, 2, \dots). \tag{18}$$

Equality holds in (18) if and only if $m = 0$ or $n = 0$.

We show next that T_n separates the geometric and arithmetic means of T_{n-1} and T_{n+1}.

Theorem 5. *The sequence $(T_n)_{n\geq 0}$ is strictly log-concave, strictly convex and superadditive, that is, we have*

$$\sqrt{T_{n-1}T_{n+1}} < T_n < \frac{T_{n-1} + T_{n+1}}{2} \quad (n = 1, 2, 3, \dots) \tag{19}$$

and

$$T_m + T_n \leq T_{m+n} \quad (m, n = 0, 1, 2, \dots). \tag{20}$$

Equality holds in (20) if and only if $m = 0$ or $n = 0$.

From (17)—(20) we conclude that the following double-inequalities are valid for all $n \geq 1$:

$$S_n^2 < S_{n-1}S_{n+1} \leq S_{2n} \quad \text{and} \quad 2T_n < T_{n-1} + T_{n+1} \leq T_{2n}.$$

The sign of equality is valid if and only if $n = 1$.

In the next section, we establish the identities presented in Theorems 1, 2 and 3. In Section 3, we collect a few lemmas which we need to prove the inequalities (17)—(20). The proofs of Theorems 4 and 5 are given in Section 4. Throughout, we maintain the notations introduced in this section.

2. Proofs of Theorems 1, 2, and 3

In order to prove the identities for D_n^*, S_n, S_n^* and T_n we use an elementary unified approach, whereas the proof of the identity for T_n^* makes use of techniques from Complex Analysis. More precisely, we apply an integral formula given by Hautus and Klarner [4] in 1971. We remark that the Hautus–Klarner method can also be applied to establish the other identities.

To prove Theorem 1 we need the identities (15) and (16) stated in Theorem 3. Therefore, first we establish Theorems 2 and 3.

Proofs of (11), (12), (13) *and* (14). Let $k \geq 0$ and $n \geq 0$ be integers. We define

$$X_{k,n} = \sum_{j=0}^{k} \binom{n}{j}, \quad Y_n = \sum_{k=0}^{n} \delta_k X_{k,n}^2,$$

where $(\delta_k)_{k \geq 0}$ are complex numbers. Using

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}$$

we find

$$X_{k,n+1} = X_{k-1,n} + X_{k,n}.$$

Moreover, we have

$$X_{n+1.n} = X_{n,n} = 2^n \quad \text{and} \quad X_{k-1,n} = X_{k,n} - \binom{n}{k}.$$

Applying these relations we obtain

$$Y_{n+1} = \sum_{k=0}^{n+1} \delta_k X_{k,n+1}^2 = \sum_{k=0}^{n+1} \delta_k (X_{k-1,n} + X_{k,n})^2$$

$$
\begin{aligned}
&= \sum_{k=0}^{n}(\delta_k + \delta_{k+1})X_{k,n}^2 + 2^{2n}\delta_{n+1} + 2\sum_{k=0}^{n+1}\delta_k X_{k-1,n}X_{k,n} \\
&= \sum_{k=0}^{n}(\delta_k + \delta_{k+1})X_{k,n}^2 + 2^{2n}\delta_{n+1} + 2\sum_{k=0}^{n+1}\delta_k \left(X_{k,n} - \binom{n}{k} \right) X_{k,n} \\
&= \sum_{k=0}^{n}(3\delta_k + \delta_{k+1})X_{k,n}^2 - 2\sum_{k=0}^{n}\binom{n}{k}\delta_k X_{k,n} + 3 \cdot 2^{2n}\delta_{n+1} \\
&= 3Y_n + \sum_{k=0}^{n}\delta_{k+1}X_{k,n}^2 - 2\sum_{k=0}^{n}\binom{n}{k}\delta_k X_{k,n} + 3 \cdot 2^{2n}\delta_{n+1}.
\end{aligned}
$$

This leads to

$$
\sum_{k=0}^{n}\binom{n}{k}\delta_k X_{k,n} = \frac{3}{2}Y_n - \frac{1}{2}Y_{n+1} + 3 \cdot 2^{2n-1}\delta_{n+1} + \frac{1}{2}\sum_{k=0}^{n}\delta_{k+1}X_{k,n}^2. \tag{21}
$$

We consider three cases.

Case 1. $\delta_k = 1$.

Then,

$$
\sum_{k=0}^{n}\binom{n}{k}\delta_k X_{k,n} = S_n \quad \text{and} \quad Y_n = B_n. \tag{22}
$$

Applying (21) and (22) gives

$$
S_n = 2B_n - \frac{1}{2}B_{n+1} + 3 \cdot 2^{2n-1}. \tag{23}
$$

From (3) and (23) we conclude that (11) is valid.

Case 2. $\delta_k = (-1)^k$.

We have

$$
\sum_{k=0}^{n}\binom{n}{k}\delta_k X_{k,n} = S_n^* \quad \text{and} \quad Y_n = B_n^*, \tag{24}
$$

so that (21) and (24) imply

$$
S_n^* = B_n^* - \frac{1}{2}B_{n+1}^* + (-1)^{n+1}3 \cdot 2^{2n-1}. \tag{25}
$$

We apply (4), (5) and (25). This yields (12) and (13).

Case 3. $\delta_k = k$.

Then,

$$
\sum_{k=0}^{n}\binom{n}{k}\delta_k X_{k,n} = T_n \quad \text{and} \quad Y_n = D_n. \tag{26}
$$

Using (21) and (26) gives

$$
T_n = 2D_n - \frac{1}{2}D_{n+1} + \frac{1}{2}B_n + 3(n+1)2^{2n-1}. \tag{27}
$$

Finally, we apply (13), (17) and (27). This leads to (14).

Proofs of (15) *and* (16). We define

$$T^*_{m,n} = \sum_{k=0}^{m}(-1)^k\binom{m}{k}k\sum_{j=0}^{k}\binom{n}{j} = \sum_{0\le j\le k}(-1)^k k\binom{m}{k}\binom{n}{j}$$

and

$$F(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} T^*_{m,n}x^m y^n.$$

In what follows, we assume that x, y, z are sufficiently small real numbers. Let $a = x/(x-1)$ and $b = ay/(1-y)$. Using

$$\sum_{n=j}^{\infty}\binom{n}{j}t^n = \frac{t^j}{(1-t)^{j+1}}$$

and

$$\sum_{n=j}^{\infty} nt^n = \frac{1}{1-t}\Big(j+\frac{t}{1-t}\Big)t^j \quad (|t|<1;\ j=0,1,2,\dots)$$

we obtain

$$\begin{aligned}
F(x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{0\le j\le k}(-1)^k k\binom{m}{k}\binom{n}{j}x^m y^n\\
&= \sum_{0\le j\le k}\sum_{m\ge k} x^m(-1)^k k\binom{m}{k}\sum_{n\ge j}\binom{n}{j}y^n\\
&= \sum_{0\le j\le k}\sum_{m\ge k} x^m(-1)^k k\binom{m}{k}\frac{y^j}{(1-y)^{j+1}}\\
&= \sum_{0\le j\le k}(-1)^k k\frac{x^k}{(1-x)^{k+1}}\frac{y^j}{(1-y)^{j+1}}\\
&= \frac{1}{(1-x)(1-y)(1-a)}\sum_{j\ge 0}\Big(j+\frac{a}{1-a}\Big)b^j\\
&= \frac{1}{(1-x)(1-y)(1-a)}\Big(\frac{1}{1-b}\frac{b}{1-b}+\frac{a}{1-a}\frac{1}{1-b}\Big)\\
&= \frac{x(x-1)(2xy-x+1)}{(2xy-x-y+1)^2}.
\end{aligned}$$

Let

$$f_z(s) = \frac{1}{s}F(s,z/s) = -\frac{(s-1)(s-2z-1)s^2}{\big(s^2-(2z+1)s+z\big)^2}.$$

The function f_z has double poles at

$$s_1 = z + \frac{1}{2} + \frac{1}{2}\sqrt{4z^2+1} \quad \text{and} \quad s_2 = z + \frac{1}{2} - \frac{1}{2}\sqrt{4z^2+1}.$$

A theorem of Hautus and Klarner [4] states that

$$\frac{1}{2\pi i}\int_C f_z(s)ds = \sum_{n=0}^{\infty} T^*_{n,n} z^n, \tag{28}$$

and from the residue theorem we obtain

$$\frac{1}{2\pi i}\int_C f_z(s)ds = \mathrm{Res}_{s=s_2} f_z(s). \tag{29}$$

Here, C is a positively oriented circle with center s_2 and sufficiently small radius. We set $c_n = 4^n\binom{-3/2}{n}$. Then,

$$\begin{aligned} \mathrm{Res}_{s=s_2} f_z(s) &= \frac{z(8z^3+2z-1)}{(4z^2+1)^{3/2}} - z \\ &= 8\sum_{n=0}^{\infty} c_n z^{2n+4} + 2\sum_{n=0}^{\infty} c_n z^{2n+2} - \sum_{n=0}^{\infty} c_n z^{2n+1} - z \\ &= -2z + 2\sum_{n=1}^{\infty}(c_{n-1}+4c_{n-2})z^{2n} - \sum_{n=1}^{\infty} c_n z^{2n+1}. \end{aligned} \tag{30}$$

We have $T^*_n = T^*_{n,n}$, so that (28), (29) and (30) yield

$$T^*_0 = 0, \quad T^*_{2n} = 2(c_{n-1}+4c_{n-2}) = (-1)^{n-1}2\binom{2n-2}{n-1} \quad (n = 1, 2, 3, \dots)$$

and

$$T^*_1 = -2, \quad T^*_{2n+1} = -c_n = (-1)^{n-1}(2n+1)\binom{2n}{n} \quad (n = 1, 2, 3, \dots).$$

This leads to (15) and (16).

Proofs of (9) *and* (10). From (21) with $\delta_k = (-1)^k k$ we obtain

$$T^*_n = -\frac{1}{2}B^*_n + D^*_n - \frac{1}{2}D^*_{n+1} + (-1)^{n+1}3(n+1)2^{2n-1}. \tag{31}$$

Since

$$X_{k,n+1} = 2X_{k,n} - \binom{n}{k},$$

we get

$$\begin{aligned} D^*_{n+1} &= \sum_{k=0}^{n}(-1)^k k\left(4X^2_{k,n} - 4\binom{n}{k}X_{k,n} + \binom{n}{k}^2\right) + (-1)^{n+1}(n+1)2^{2n+2} \\ &= 4D^*_n - 4T^*_n + R_n + (-1)^{n+1}(n+1)2^{2n+2}, \end{aligned} \tag{32}$$

where

$$R_n = \sum_{k=0}^{n} (-1)^k k \binom{n}{k}^2.$$

The sum R_n can be expressed in terms of the classical Legendre polynomials P_n:

$$R_n = (-1)^n 2^{n-1} \bigl(nP_n(0) + P_n'(0)\bigr);$$

see [8, section 1.2.7]. We obtain

$$R_n = (-1)^{n/2} \frac{n}{2} \binom{n}{n/2}, \quad \text{if } n \text{ is even}, \tag{33}$$

$$R_n = (-1)^{(n+1)/2} n \binom{n-1}{(n-1)/2}, \quad \text{if } n \text{ is odd}. \tag{34}$$

Next, we replace in (31) D_{n+1}^* by the expression given in (32) and find

$$D_n^* = T_n^* - \frac{1}{2} B_n^* - \frac{1}{2} R_n + (-1)^n (n+1) 2^{2n-1}. \tag{35}$$

Finally, we use (4), (5), (15), (16), (33) and (34). Then we conclude from (35) that (9) and (10) are valid.

3. Lemmas

Our first lemma offers a known property of log-convex functions; see [11, p. 19].

Lemma 1. *If f is log-convex and g is strictly log-convex, then $f+g$ is strictly log-convex.*

Remark. If f and g are twice differentiable, then Lemma 1 can be proved by using the formula

$$\bigl(\log(f+g)\bigr)'' = \frac{f(\log f)''}{f+g} + \frac{g(\log g)''}{f+g} + \frac{(fg' - f'g)^2}{fg(f+g)^2}.$$

The following elegant functional inequality for convex functions is due to Petrović [10]; see also [9, section 1.4.7].

Lemma 2. *Let f be strictly convex on $[0,\infty)$. Then, for $x, y \geq 0$,*

$$f(x) + f(y) \leq f(0) + f(x+y).$$

Equality holds if and only if $x = 0$ or $y = 0$.

The next lemma collects a few basic properties of the logarithmic derivative of Euler's gamma function, $\psi = \Gamma'/\Gamma$; see [1, chapter 6].

Lemma 3. *Let $x > 0$. Then,*

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \quad \psi(2x) = \frac{1}{2}\psi(x) + \frac{1}{2}\psi(x+1/2) + \log 2,$$

$$(-1)^{n+1}\psi^{(n)}(x) = \int_0^\infty e^{-xt}\frac{t^n}{1-e^{-t}}dt \quad (n = 1, 2, 3, \dots).$$

4. Proofs of Theorems 4 and 5

Proof of Theorem 4. Let

$$u(x) = 2^{2x-1} \quad \text{and} \quad v(x) = \frac{\Gamma(2x+1)}{2\Gamma(x+1)^2}. \tag{36}$$

Applying Lemma 3 gives for $x > 0$,

$$\begin{aligned} \frac{d^2}{dx^2}\log v(x) &= \psi'(x+1/2) - \psi'(x) + \frac{1}{x^2} \\ &= \int_0^\infty e^{-(x+1/2)t}\frac{t}{1-e^{-t}}dt - \int_0^\infty e^{-xt}\frac{t}{1-e^{-t}}dt + \int_0^\infty e^{-xt}tdt \\ &= \int_0^\infty e^{-xt}\frac{t}{e^{t/2}+1}dt > 0. \end{aligned} \tag{37}$$

Since u is log-convex on $[0, \infty)$, we conclude from Lemma 1 that

$$w(x) = u(x) + v(x) = 2^{2x-1} + \frac{\Gamma(2x+1)}{2\Gamma(x+1)^2}$$

is strictly log-convex on $[0, \infty)$. An application of Jensen's inequality yields

$$\log w(n) < \frac{1}{2}\big(\log w(n-1) + \log w(n+1)\big) \quad (n = 1, 2, 3, \dots)$$

and Lemma 2 gives

$$\log w(m) + \log w(n) \le \log w(0) + \log w(m+n) = \log w(m+n) \quad (m, n = 0, 1, 2, \dots)$$

with equality if and only if $m = 0$ or $n = 0$. From (11) we obtain that for all integers $n \ge 0$ we have $w(n) = S_n$. This implies that (17) and (18) are valid.

Proof of Theorem 5. (i) First, we prove that $(T_n)_{n\ge0}$ is strictly log-concave. Using (14) yields for $n \ge 1$,

$$\frac{T_{n+1}}{T_n} = 4\Big(1 + \frac{1}{n + 1/\chi_n}\Big) \tag{38}$$

with

$$\chi_n = \frac{1}{n}\Big(1 + \frac{4^n}{\binom{2n}{n}}\Big).$$

Let $x > 0$. We define

$$\theta(x) = \frac{4^x \Gamma(x)\Gamma(x+1)}{\Gamma(2x+1)}.$$

Applying Lemma 3 gives

$$\frac{d}{dx}\log\theta(x) = \psi(x) - \psi(x+1/2) < 0.$$

Since

$$\chi_n = \frac{1}{n} + \theta(n), \quad (n = 1, 2, 3, \dots),$$

we conclude that $(\chi_n)_{n\geq 1}$ is strictly decreasing. From (38) we obtain that $(T_{n+1}/T_n)_{n\geq 1}$ is strictly decreasing. Thus, for $n \geq 2$,

$$\frac{T_{n+1}}{T_n} < \frac{T_n}{T_{n-1}}.$$

This implies that the left-hand side of (19) is valid for $n \geq 2$. By direct computation we obtain that this is also true for $n = 1$.

(ii) In order to prove that $(T_n)_{n\geq 0}$ is strictly convex, we define

$$G(x) = xh(x)$$

with

$$h(x) = 2^{2x-2} + v(x),$$

where v is defined in (36). We have

$$G''(x) = 2h'(x) + xh''(x). \tag{39}$$

An application of Lemma 3 and (37) gives for $x \geq 0$,

$$h'(x) = (\log 2)2^{2x-1} + 2v(x)\big[\psi(2x+1) - \psi(x+1)\big] > 0$$

and

$$h''(x) = (\log 2)^2 4^x + v(x)(\log v(x))'' + \frac{1}{v(x)}\big(v'(x)\big)^2 > 0.$$

From (39) we conclude that G is strictly convex on $[0, \infty)$. Using (14) shows that $G(n) = T_n$ for all integers $n \geq 0$, so that an application of Jensen's inequality reveals that the right-hand side of (19) holds.

(iii) We apply Lemma 2 with $f = G$. Since $G(0) = 0$, we obtain for all integers $m, n \geq 0$,

$$T_m + T_n = G(m) + G(n) \leq G(m+n) = T_{m+n}$$

with equality if and only if $m = 0$ or $n = 0$.

5. Concluding Remark

R. Tauraso kindly informed us that Zhang [15] studied identities for two general classes of combinatorial sums, namely,

$$\sum_{k=0}^{n} f_k \sum_{j=0}^{k} \binom{n}{j} g_j \quad \text{and} \quad \sum_{k=0}^{n} f_k \left(\sum_{j=0}^{k} \binom{n}{j} g_j \right)^2,$$

where f_k and g_k $(k = 0, 1, \dots, n)$ are real numbers. The proof of his identities is based upon an elegant application of MacMahon's Omega operator calculus; see [7]. As special cases of his results he obtained the identities (8), (10) and (13).

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