ON THE NUMBER OF ORDERED FACTORIZATIONS OF AN INTEGER

Noah Lebowitz-Lockard
Department of Mathematics & Data Science, College of Coastal Georgia,
Brunswick, Georgia
nlebowitzlockard@ccga.edu

Received: 5/9/20, Accepted: 10/15/20, Published: 10/19/20

Abstract
Let $g(n)$ be the number of ordered factorizations of $n$ into parts greater than 1. We establish a new upper bound on the number of numbers in the range of $g$ which do not exceed $x$. This work improves a theorem of Klazar and Luca and closely follows a proof of Balasubramanian and Srivastav.

1. Introduction
Let $f(n)$ and $g(n)$ be the number of unordered and ordered factorizations of the integer $n$ into parts greater than 1. These functions were first studied by Oppenheim and Kalmár, respectively [9, 5], who showed that
\[
\sum_{n \leq x} f(n) \sim \frac{1}{2\sqrt{\pi}} \frac{x \exp(2\sqrt{\log x})}{(\log x)^{3/4}},
\]
\[
\sum_{n \leq x} g(n) \sim -\frac{1}{\rho \zeta'(\rho)} x^\rho,
\]
where $\zeta$ refers to the Riemann zeta function and $s = \rho \approx 1.73$ is the unique solution to the equation $\zeta(s) = 2$ in $(1, \infty)$.

Define $\mathcal{F}(x)$ and $\mathcal{G}(x)$ to be the sets of $n \leq x$ which lie in the ranges of $f$ and $g$, i.e.,
\[
\mathcal{F}(x) = f(\mathbb{Z}_+) \cap [1, x],
\]
\[
\mathcal{G}(x) = g(\mathbb{Z}_+) \cap [1, x].
\]
Multiple people have found upper bounds for $\#\mathcal{F}(x)$. Canfield, Erdős, and Pomerance [3] stated (without proof) that $\#\mathcal{F}(x) = x^{o(1)}$. Luca, Mukhopadhyay, and Srinivas [8] later showed that
\[
\#\mathcal{F}(x) = \exp\left(O\left(\frac{\log x \log_3 x}{\log_2 x}\right)\right),
\]
where \( \log_k x \) refers to the \( k \)th fold iterate of the logarithm. Soon afterward, Balasubramanian and Luca \[1\] proved that
\[
\#\mathcal{F}(x) \leq \exp(9(\log x)^{2/3})
\]
for all \( x \geq 1 \). More recently, Balasubramanian and Srivastav \[2\] showed that
\[
\#\mathcal{F}(x) \leq \exp \left( (1 + o(1)) \frac{\pi}{2} \sqrt{\frac{\log x}{\log_2 x}} \right).
\]
Through a slight modification of their proof, the author \[7, \text{Section 8}\] reduced the constant in the exponent, obtaining
\[
\#\mathcal{F}(x) \leq \exp \left( (1 + o(1)) \frac{\pi}{2} \sqrt{\frac{\log x}{\log_2 x}} \right).
\]
In addition, Balasubramanian and Srivastav conjecture that a bound of this type is optimal in the sense that there exists a positive constant \( C \) such that
\[
\#\mathcal{F}(x) \geq \exp \left( (C + o(1)) \sqrt{\frac{\log x}{\log_2 x}} \right).
\]
As for \( \#\mathcal{G}(x) \), Klazar and Luca \[6, \text{Proposition 5.7}\] proved that
\[
\#\mathcal{G}(x) \leq \exp \left( (1 + o(1)) \frac{\pi}{2} \sqrt{\frac{2}{\log 2 \log x}} \right).
\]
The proofs of the last two upper bounds on \( \#\mathcal{F}(x) \) rely entirely on a lower bound for \( f(n) \). Because \( g(n) \geq f(n) \) for all \( n \), this observation serves as a simple proof that
\[
\#\mathcal{G}(x) \leq \exp \left( (1 + o(1)) \frac{\pi}{2} \sqrt{\frac{\log x}{\log_2 x}} \right).
\]
Using a method similar to that of \[2\], we obtain a better upper bound for \( \#\mathcal{G}(x) \).

**Theorem 1.** We have
\[
\#\mathcal{G}(x) \leq \exp \left( (1 + o(1)) \frac{2\pi}{\sqrt{3}} \sqrt{\frac{\log x}{\log_2 x}} \right).
\]

2. Preliminary Results

Let \( n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \). For notational convenience, we let \( \mathbf{a} \) be the vector \((\alpha_1, \ldots, \alpha_r)\).

In order to obtain their bound on \( \#\mathcal{F}(x) \), Balasubramanian and Srivastav proved the following result.
Theorem 2 ([2, Proposition 2.7]). Let \( z = z(\alpha) \) be the unique positive solution to the equation

\[
z = \prod_{i=1}^{r} \left( 1 + \frac{\alpha_i}{z} \right),
\]

and \( N = \lfloor z \rfloor \). We have

\[
f(n) \geq e^{N-2} \prod_{i=1}^{r} \frac{1}{2N^{3/2}} \left( 1 + \frac{N}{\alpha_i} \right)^{\alpha_i + (1/2)}.
\]

In addition, if \( f(n) \leq x \), then

\[
r \leq (2 + o(1)) \frac{\log x}{\log_2 x}.
\]

Delégile, Hernane, and Nicolas provide a similar lower bound for \( g(n) \). Let \( \Omega(n) \) be the number of (not necessarily distinct) prime factors of \( n \).

Theorem 3 ([4, Eqs. (3.1), (3.26)]). Let \( c = c(\alpha) \) be the unique solution to the equation

\[
\prod_{i=1}^{r} \left( 1 + \frac{\alpha_i}{c} \right) = 2.
\]

Then,

\[
g(n) \gg \sqrt{\Omega(n)} \prod_{i=1}^{r} \frac{1}{e^{\sqrt{\alpha_i}}} \left( 1 + \frac{c}{\alpha_i} \right)^{\alpha_i}.
\]

We also write three lemmas for future use.

Lemma 1. If \( g(n) \leq x \), then

\[
r \leq (1 + o(1)) \frac{\log x}{\log_2 x}.
\]

Proof. In order to maximize \( r \), we assume \( n \) is squarefree. If \( n = p_1 \cdots p_r \), then \( g(n) \geq r! \) because we can express \( n \) as a product of primes in exactly \( r! \) ways. Because \( r! \leq x \), we have \( r \leq (1 + o(1)) \frac{\log x}{\log_2 x} \).

From [2, Eq. (2.9)], the following result holds for all \( n \) satisfying \( f(n) \leq x \). Because \( f(n) \leq g(n) \), it holds when \( g(n) \leq x \) as well.

Lemma 2. If \( g(n) \leq x \), then \( \alpha_i \leq (\log x)^2 \) for all \( i \).

Using this result, we bound the sum of \( \log \alpha_i \).
Lemma 3. If $g(n) \leq x$, then
\[ \sum_{i=1}^{r} \log \alpha_i = o(\log x). \]

Proof. Fix a large number $M$ which we determine more precisely later. Let $S_1$ be the set of $i \leq r$ satisfying $\alpha_i > M$ and $S_2$ the set of all other $i$. For all $n \in S_1$, we have
\[ g(n) \geq g(p_1^M \cdots p_{\#S_1}^M). \]

In this case,
\[ \prod_{i=1}^{\#S_1} \left(1 + \frac{M}{e}\right) = 2, \]

which implies that
\[ e = \frac{M}{2^{\#S_1} - 1}. \]

By Theorem 3,
\[ x \geq g(n) \gg \prod_{i=1}^{\#S_1} \frac{1}{\sqrt[2M]{1 + \frac{1}{2^{\#S_1} - 1}}} = \exp((1 + o(1))M(\#S_1)\log(\#S_1)). \]

Fix $\epsilon > 0$. Letting $M = (\log x)^{\epsilon}$ gives us
\[ \#S_1 = o((\log x)^{1-\epsilon}). \]

We bound our desired sum on $\#S_1$. By the previous lemma, we have $\alpha_i \leq (\log x)^2$ for all $i$. Therefore,
\[ \sum_{i \in S_1} \log \alpha_i \ll \sum_{i \in S_1} \log x = o((\log x)^{1-\epsilon} \log x) = o(\log x). \]

Consider $S_2$. By definition, $\alpha_i \leq (\log x)^{\epsilon}$ for all $i \in S_2$. We have
\[ \sum_{i \in S_2} \log \alpha_i \leq \epsilon \sum_{i \in S_2} \log x \leq \sum_{i=1}^{r} \epsilon \log x = \epsilon r \log x. \]

By Lemma 1, this quantity is at most $(1 + o(1))\epsilon \log x$. Letting $\epsilon$ go to 0 gives us our desired result.

The final result follows naturally from the asymptotic formula for the partition function.

Lemma 4 ([2, Lemma 2.8]). For all $y \geq 1$, the number of unordered tuples $(n_1, \ldots, n_k)$ of positive integers satisfying $n_1 + \cdots + n_k \leq y$ is at most
\[ \exp\left((1 + o(1))\pi \sqrt{\frac{2y}{3}}\right). \]
3. The Proof

Given the results from the previous section, we obtain our desired upper bound for $\#G(x)$, which we rewrite here.

**Theorem 1.** We have

$$\#G(x) \leq \exp \left( (1 + o(1)) \frac{2\pi}{\sqrt{3}} \sqrt[3]{\frac{\log x}{\log_2 x}} \right).$$

**Proof.** Suppose $n \leq x$. By Theorem 3,

$$\sqrt{\Omega(n)} \prod_{i=1}^{r} \frac{1}{e^{\sqrt{\alpha_i}}} \left( 1 + \frac{c}{\alpha_i} \right)^{\alpha_i} \ll g(n) \leq x.$$

If $g(n)$ is sufficiently large, we have

$$\prod_{i=1}^{r} \frac{1}{e^{\sqrt{\alpha_i}}} \left( 1 + \frac{c}{\alpha_i} \right)^{\alpha_i} < x.$$

Taking logarithms and rearranging terms gives us

$$\sum_{i=1}^{r} \alpha_i \log \left( 1 + \frac{c}{\alpha_i} \right) < \log x + r + \frac{1}{2} \sum_{i=1}^{r} \log \alpha_i.$$

Lemmas 1 and 3 imply that

$$\sum_{i=1}^{r} \alpha_i \log \left( 1 + \frac{c}{\alpha_i} \right) \leq (1 + o(1)) \log x.$$

Let $S_1$ be the set of all $i \leq r$ satisfying $\alpha_i \leq A c$ with

$$A = \frac{(\log_2 x)^2}{(\log x)^{1/2}}$$

and $S_2$ the set of all other $i \leq r$. If $i \in S_1$, then

$$\log \left( 1 + \frac{c}{\alpha_i} \right) \geq \log \left( 1 + \frac{1}{A} \right) \sim \frac{1}{2} \log_2 x.$$

Therefore,

$$\sum_{i=1}^{r} \alpha_i \log \left( 1 + \frac{c}{\alpha_i} \right) \geq \sum_{i \in S_1} \alpha_i \log \left( 1 + \frac{c}{\alpha_i} \right) \geq (1 + o(1)) \frac{1}{2} \log_2 x \sum_{i \in S_1} \alpha_i,$$

which implies that

$$\sum_{i \in S_1} \alpha_i \leq (1 + o(1)) \frac{2 \log x}{\log_2 x}.$$
By Lemma 4, the number of possible sets $S_1$ is at most
\[
\exp \left( (1 + o(1)) \frac{2\pi}{\sqrt{3}} \sqrt[3]{\frac{\log x}{\log_2 x}} \right).
\]

We bound the number of possible sets $S_2$ using an approach similar to the proof of [2, Lemma 2.10]. We have
\[
2 = \prod_{i=1}^{r} \left( 1 + \frac{\alpha_i}{c} \right) \geq \prod_{i \in S_2} \left( 1 + \frac{\alpha_i}{c} \right) \geq (1 + A)^{|S_2|},
\]
which implies that
\[
(|S_2|) \log (1 + A) < \log 2.
\]
Because $A = o(1)$, we have $\log(1 + A) > A/2$ for $x$ sufficiently large. Hence,
\[
|S_2| < \frac{2 \log 2}{A} = O \left( \frac{(\log x)^{1/2}}{\log_2 x} \right).
\]
By Lemma 2, $\alpha_i \leq (\log x)^2$ for all $i$. Therefore, the number of possible sets $S_2$ is at most
\[
((\log x)^2)^{|S_2|} = \exp \left( O \left( \frac{\sqrt{\log x}}{\log_2 x} \right) \right) = \exp \left( o \left( \sqrt{\frac{\log x}{\log_2 x}} \right) \right).
\]
Multiplying our bounds for $S_1$ and $S_2$ completes the proof. 

References


