#A90 INTEGERS 20 (2020)

ON THE NUMBER OF ORDERED FACTORIZATIONS OF AN INTEGER

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Received: 5/9/20, Accepted: 10/15/20, Published: 10/19/20

Abstract

Let $g(n)$ be the number of ordered factorizations of n into parts greater than 1. We establish a new upper bound on the number of numbers in the range of g which do not exceed x. This work improves a theorem of Klazar and Luca and closely follows a proof of Balasubramanian and Srivastav.

1. Introduction

Let $f(n)$ and $g(n)$ be the number of unordered and ordered factorizations of the integer n into parts greater than 1. These functions were first studied by Oppenheim and Kalmár, respectively [9, 5], who showed that

$$\sum_{n \leq x} f(n) \sim \frac{1}{2\sqrt{\pi}} \frac{x \exp(2\sqrt{\log x})}{(\log x)^{3/4}},$$

$$\sum_{n \leq x} g(n) \sim -\frac{1}{\rho \zeta'(\rho)} x^{\rho},$$

where ζ refers to the Riemann zeta function and $s = \rho \approx 1.73$ is the unique solution to the equation $\zeta(s) = 2$ in $(1, \infty)$.

Define $\mathcal{F}(x)$ and $\mathcal{G}(x)$ to be the sets of $n \leq x$ which lie in the ranges of f and g, i.e.,

$$\mathcal{F}(x) = f(\mathbb{Z}_+) \cap [1, x],$$

$$\mathcal{G}(x) = g(\mathbb{Z}_+) \cap [1, x].$$

Multiple people have found upper bounds for $\#\mathcal{F}(x)$. Canfield, Erdős, and Pomerance [3] stated (without proof) that $\#\mathcal{F}(x) = x^{o(1)}$. Luca, Mukhopadhyay, and Srinivas [8] later showed that

$$\#\mathcal{F}(x) = \exp\left(O\left(\frac{\log x \log_3 x}{\log_2 x}\right)\right),$$

where $\log_k x$ refers to the kth fold iterate of the logarithm. Soon afterward, Balasubramanian and Luca [1] proved that

$$\#\mathcal{F}(x) \leq \exp(9(\log x)^{2/3})$$

for all $x \geq 1$. More recently, Balasubramanian and Srivastav [2] showed that

$$\#\mathcal{F}(x) \leq \exp\left((1+o(1))2\pi\sqrt{\frac{2}{3}}\sqrt{\frac{\log x}{\log_2 x}}\right).$$

Through a slight modification of their proof, the author [7, Section 8] reduced the constant in the exponent, obtaining

$$\#\mathcal{F}(x) \leq \exp\left((1+o(1))\pi\sqrt{2}\sqrt{\frac{\log x}{\log_2 x}}\right).$$

In addition, Balasubramanian and Srivastav conjecture that a bound of this type is optimal in the sense that there exists a positive constant C such that

$$\#\mathcal{F}(x) \geq \exp\left((C+o(1))\sqrt{\frac{\log x}{\log_2 x}}\right).$$

As for $\#\mathcal{G}(x)$, Klazar and Luca [6, Proposition 5.7] proved that

$$\#\mathcal{G}(x) \leq \exp\left((1+o(1))\pi\sqrt{\frac{2}{3\log 2}}\sqrt{\log x}\right).$$

The proofs of the last two upper bounds on $\#\mathcal{F}(x)$ rely entirely on a lower bound for $f(n)$. Because $g(n) \geq f(n)$ for all n, this observation serves as a simple proof that

$$\#\mathcal{G}(x) \leq \exp\left((1+o(1))\pi\sqrt{2}\sqrt{\frac{\log x}{\log_2 x}}\right).$$

Using a method similar to that of [2], we obtain a better upper bound for $\#\mathcal{G}(x)$.

Theorem 1. *We have*

$$\#\mathcal{G}(x) \leq \exp\left((1+o(1))\frac{2\pi}{\sqrt{3}}\sqrt{\frac{\log x}{\log_2 x}}\right).$$

2. Preliminary Results

Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. For notational convenience, we let $\boldsymbol{\alpha}$ be the vector $(\alpha_1, \ldots, \alpha_r)$. In order to obtain their bound on $\#\mathcal{F}(x)$, Balasubramanian and Srivastav proved the following result.

Theorem 2 ([2, Proposition 2.7])**.** *Let $z = z(\boldsymbol{\alpha})$ be the unique positive solution to the equation*

$$z = \prod_{i=1}^{r} \left(1 + \frac{\alpha_i}{z}\right),$$

and $N = \lfloor z \rfloor$. We have

$$f(n) \geq \frac{e^{N-2}}{2N^{3/2}} \prod_{i=1}^{r} \frac{1}{2\sqrt{2N}} \left(1 + \frac{N}{\alpha_i}\right)^{\alpha_i + (1/2)}.$$

In addition, if $f(n) \leq x$, then

$$r \leq (2 + o(1)) \frac{\log x}{\log_2 x}.$$

Deléglise, Hernane, and Nicolas provide a similar lower bound for $g(n)$. Let $\Omega(n)$ be the number of (not necessarily distinct) prime factors of n.

Theorem 3 ([4, Eqs. (3.1), (3.26)])**.** *Let $c = c(\boldsymbol{\alpha})$ be the unique solution to the equation*

$$\prod_{i=1}^{r} \left(1 + \frac{\alpha_i}{c}\right) = 2.$$

Then,

$$g(n) \gg \sqrt{\Omega(n)} \prod_{i=1}^{r} \frac{1}{e\sqrt{\alpha_i}} \left(1 + \frac{c}{\alpha_i}\right)^{\alpha_i}.$$

We also write three lemmas for future use.

Lemma 1. *If $g(n) \leq x$, then*

$$r \leq (1 + o(1)) \frac{\log x}{\log_2 x}.$$

Proof. In order to maximize r, we assume n is squarefree. If $n = p_1 \cdots p_r$, then $g(n) \geq r!$ because we can express n as a product of primes in exactly $r!$ ways. Because $r! \leq x$, we have

$$r \leq (1 + o(1)) \frac{\log x}{\log_2 x}. \qquad \square$$

From [2, Eq. (2.9)], the following result holds for all n satisfying $f(n) \leq x$. Because $f(n) \leq g(n)$, it holds when $g(n) \leq x$ as well.

Lemma 2. *If $g(n) \leq x$, then $\alpha_i \leq (\log x)^2$ for all i.*

Using this result, we bound the sum of $\log \alpha_i$.

Lemma 3. *If $g(n) \leq x$, then*

$$\sum_{i=1}^{r} \log \alpha_i = o(\log x).$$

Proof. Fix a large number M which we determine more precisely later. Let $\mathcal{S}_1$ be the set of $i \leq r$ satisfying $\alpha_i > M$ and $\mathcal{S}_2$ the set of all other i. For all $n \in \mathcal{S}_1$, we have

$$g(n) \geq g(p_1^M \cdots p_{\#\mathcal{S}_1}^M).$$

In this case,

$$\prod_{i=1}^{\#\mathcal{S}_1} \left(1 + \frac{M}{c}\right) = 2,$$

which implies that

$$c = \frac{M}{2^{1/\#\mathcal{S}_1} - 1}.$$

By Theorem 3,

$$x \geq g(n) \gg \prod_{i=1}^{\#\mathcal{S}_1} \frac{1}{e\sqrt{M}} \left(1 + \frac{1}{2^{1/\#\mathcal{S}_1} - 1}\right)^M = \exp((1 + o(1))M(\#\mathcal{S}_1)\log(\#\mathcal{S}_1)).$$

Fix $\epsilon > 0$. Letting $M = (\log x)^\epsilon$ gives us

$$\#\mathcal{S}_1 = o((\log x)^{1-\epsilon}).$$

We bound our desired sum on $\#\mathcal{S}_1$. By the previous lemma, we have $\alpha_i \leq (\log x)^2$ for all i. Therefore,

$$\sum_{i \in \mathcal{S}_1} \log \alpha_i \ll \sum_{i \in \mathcal{S}_1} \log_2 x = o((\log x)^{1-\epsilon} \log_2 x) = o(\log x).$$

Consider $\mathcal{S}_2$. By definition, $\alpha_i \leq (\log x)^\epsilon$ for all $i \in \mathcal{S}_2$. We have

$$\sum_{i \in \mathcal{S}_2} \log \alpha_i \leq \sum_{i \in \mathcal{S}_2} \epsilon \log_2 x \leq \sum_{i=1}^{r} \epsilon \log_2 x = \epsilon r \log_2 x.$$

By Lemma 1, this quantity is at most $(1 + o(1))\epsilon \log x$. Letting ϵ go to 0 gives us our desired result. $\square$

The final result follows naturally from the asymptotic formula for the partition function.

Lemma 4 ([2, Lemma 2.8])**.** *For all $y \geq 1$, the number of unordered tuples $(n_1, \ldots, n_k)$ of positive integers satisfying $n_1 + \cdots + n_k \leq y$ is at most*

$$\exp\left((1 + o(1))\pi\sqrt{\frac{2y}{3}}\right).$$

3. The Proof

Given the results from the previous section, we obtain our desired upper bound for $\#\mathcal{G}(x)$, which we rewrite here.

Theorem 1. We have

$$\#\mathcal{G}(x) \leq \exp\left((1+o(1))\frac{2\pi}{\sqrt{3}}\sqrt{\frac{\log x}{\log_2 x}}\right).$$

Proof. Suppose $n \leq x$. By Theorem 3,

$$\sqrt{\Omega(n)}\prod_{i=1}^{r}\frac{1}{e\sqrt{\alpha_i}}\left(1+\frac{c}{\alpha_i}\right)^{\alpha_i} \ll g(n) \leq x.$$

If $g(n)$ is sufficiently large, we have

$$\prod_{i=1}^{r}\frac{1}{e\sqrt{\alpha_i}}\left(1+\frac{c}{\alpha_i}\right)^{\alpha_i} < x.$$

Taking logarithms and rearranging terms gives us

$$\sum_{i=1}^{r}\alpha_i \log\left(1+\frac{c}{\alpha_i}\right) < \log x + r + \frac{1}{2}\sum_{i=1}^{r}\log \alpha_i.$$

Lemmas 1 and 3 imply that

$$\sum_{i=1}^{r}\alpha_i \log\left(1+\frac{c}{\alpha_i}\right) \leq (1+o(1))\log x.$$

Let $\mathcal{S}_1$ be the set of all $i \leq r$ satisfying $\alpha_i \leq Ac$ with

$$A = \frac{(\log_2 x)^2}{(\log x)^{1/2}}$$

and $\mathcal{S}_2$ the set of all other $i \leq r$. If $i \in \mathcal{S}_1$, then

$$\log\left(1+\frac{c}{\alpha_i}\right) \geq \log\left(1+\frac{1}{A}\right) \sim \frac{1}{2}\log_2 x.$$

Therefore,

$$\sum_{i=1}^{r}\alpha_i \log\left(1+\frac{c}{\alpha_i}\right) \geq \sum_{i\in\mathcal{S}_1}\alpha_i \log\left(1+\frac{c}{\alpha_i}\right) \geq (1+o(1))\frac{1}{2}\log_2 x\sum_{i\in\mathcal{S}_1}\alpha_i,$$

which implies that

$$\sum_{i\in\mathcal{S}_1}\alpha_i \leq (1+o(1))\frac{2\log x}{\log_2 x}.$$

By Lemma 4, the number of possible sets $\mathcal{S}_1$ is at most

$$\exp\left((1+o(1))\frac{2\pi}{\sqrt{3}}\sqrt{\frac{\log x}{\log_2 x}}\right).$$

We bound the number of possible sets $\mathcal{S}_2$ using an approach similar to the proof of [2, Lemma 2.10]. We have

$$2 = \prod_{i=1}^{r}\left(1+\frac{\alpha_i}{c}\right) \geq \prod_{i\in\mathcal{S}_2}\left(1+\frac{\alpha_i}{c}\right) \geq (1+A)^{\#\mathcal{S}_2},$$

which implies that

$$(\#\mathcal{S}_2)\log(1+A) < \log 2.$$

Because $A = o(1)$, we have $\log(1+A) > A/2$ for x sufficiently large. Hence,

$$\#\mathcal{S}_2 < \frac{2\log 2}{A} = O\left(\frac{(\log x)^{1/2}}{(\log_2 x)^2}\right).$$

By Lemma 2, $\alpha_i \leq (\log x)^2$ for all i. Therefore, the number of possible sets $\mathcal{S}_2$ is at most

$$((\log x)^2)^{\#\mathcal{S}_2} = \exp\left(O\left(\frac{\sqrt{\log x}}{\log_2 x}\right)\right) = \exp\left(o\left(\sqrt{\frac{\log x}{\log_2 x}}\right)\right).$$

Multiplying our bounds for $\mathcal{S}_1$ and $\mathcal{S}_2$ completes the proof. □

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