ON THE DIVISIBILITY OF ODD PERFECT NUMBERS, QUASIPERFECT NUMBERS AND AMICABLE NUMBERS BY A HIGH POWER OF A PRIME

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Received: 10/26/16, Revised: 9/30/20, Accepted: 10/10/20, Published: 11/2/20

Abstract

We shall give an explicit upper bound for the smallest prime factor of multiperfect numbers of the form $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s} q_1^{\beta_1} \cdots q_t^{\beta_t}$ with $\beta_1, \ldots, \beta_t$ bounded by a given constant. We shall also give similar results for quasiperfect numbers and relatively prime amicable pairs of opposite parity.

1. Introduction

Let $\sigma(N)$ denote the sum of divisors of N for a positive integer N and define $h(N) = \sigma(N)/N$. An integer N is said to be perfect if $h(N) = 2$. It is one of oldest and most famous problems whether there exists any odd perfect number. Moreover, it is also unknown whether there exists any odd integer N with $h(N) = k$ for some integer $k > 1$.

Although it is unknown whether there exists any odd perfect number, it is known that an odd perfect number must satisfy various conditions. Suppose that N is an odd perfect number. Euler has shown that $N = p^{\alpha} q_1^{\beta_1} \cdots q_t^{\beta_t}$, where $p, q_1, \ldots, q_t$ are distinct odd primes with $p \equiv \alpha \equiv 1 \pmod 4$ and $\beta_1, \ldots, \beta_t$ even. Steuerwald [28] proved that we cannot have $\beta_1 = \cdots = \beta_t = 2$. If $\beta_1 = \cdots = \beta_t = \beta$, then it is known that $\beta \neq 4$ (Kanold [18]), $\beta \neq 6$ (Hagis and McDaniel [16]), $\beta \neq 10, 24, 34, 48, 124$ (McDaniel and Hagis [24]), $\beta \neq 12, 16, 22, 28, 36$ (Cohen and Williams [6]). In their paper [24], Hagis and McDaniel conjecture that $\beta_1 = \cdots = \beta_t = \beta$ does not occur. The author [29] proved that there are only finitely many odd perfect numbers for any given β. McDaniel [22] proved that we cannot have $\beta_1 \equiv \cdots \equiv \beta_t \equiv 2 \pmod 6$, i.e., 3 cannot divide all of $\beta_1 + 1, \beta_2 + 1, \ldots, \beta_t + 1$. If m divides all of $\beta_1 + 1, \beta_2 + 1, \ldots, \beta_t + 1$, then it is known that $m \neq 35$ (Hagis and McDaniel [24]) and $m \neq 65$ (Evans and Pearlman [8]) and eventually Fletcher, Nielsen and Ochem [9] showed that $m \neq 5$ as a by-product of their main result, which will be discussed later. In general, if a prime l divides all of $\beta_1+1, \beta_2+1, \ldots, \beta_t+1$, then l^4 must divide N by a result of Kanold [18].

However, if we relax the condition that there exists some integer dividing all of $\beta_1+1, \beta_2+1, \ldots, \beta_t+1$, then the situation becomes quite different. The simplest problem in this direction would be whether there exists an odd perfect number of the form $p^\alpha q_1^{\beta_1} q_2^{\beta_2} \cdots q_t^{\beta_t}$ with $p \equiv \alpha \equiv 1 \pmod 4$ and $\beta_i \leq 4$. This problem has been studied by McDaniel [23] and Cohen [4]. These papers give *lower* bounds for the smallest prime factor of N: the first paper shows it must be at least 101 and the second shows it must be at least 739.

In general, we can make a conjecture that for a fixed finite set $\mathcal{P}$ of integers, a fixed rational number n/d and a fixed integer s, there exist only finitely many odd n/d-perfect numbers $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s} q_1^{\beta_1} \cdots q_t^{\beta_t}$ with $\beta_1+1, \ldots, \beta_t+1$ contained in $\mathcal{P}$.

This conjecture still seems to be far beyond reach, though this conjecture is weaker than the finiteness conjecture of odd n/d-perfect numbers. In the preprint [30], using sieve methods, the author has proved that for a fixed finite set $\mathcal{P}$ of integers, a fixed rational number n/d and a fixed integer s, there exists an effective constant C such that odd n/d-perfect numbers of the form $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s} q_1^{\beta_1} \cdots q_t^{\beta_t}$ with $\beta_1+1, \ldots, \beta_t+1$ contained in $\mathcal{P}$ must have a prime divisor smaller than C. Moreover, the author has proved that, in the case N is perfect and $\beta_i \leq 4$, then C can be taken to be $\exp(4.97401 \times 10^{10})$.

Using the author's method, but with the aid of the large sieve instead of Selberg's sieve used by the author [29], Fletcher, Nielsen and Ochem [9] proved that if $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s} q_1^{\beta_1} \cdots q_t^{\beta_t}$ satisfies $h(N) = n/d$ and for each i, $\beta_i + 1$ has a prime factor which belongs to a finite set $\mathcal{P}$ of primes, then N has a prime divisor small than a effective constant C, depending only on n, s and $\mathcal{P}$. Moreover, they proved that the smallest prime factor of an odd perfect number N satisfying the above condition with $\mathcal{P} = \{3, 5\}$ lies between 10^8 and 10^{1000}, improving results in [4] and [30]. This implies that neither $\mathcal{P} = \{3\}$ nor $\mathcal{P} = \{5\}$ can occur since a prime l must divide N if $\mathcal{P} = \{l\}$ by the result of Kanold [18] mentioned above.

However, they did not give an explicit value for their effective C in other cases. In this paper, the author would like to give an explicit upper bound for C in general cases.

Theorem 1. *Let $\mathcal{P}$ be a finite but nonempty set of primes and $n, d, \beta_1, \ldots, \beta_t$ be positive integers such that for each $i = 1, \ldots, t$, $\beta_i + 1$ is divisible by at least one prime in the set $\mathcal{P}$ and let P denote the product $\prod_{p \in \mathcal{P}} p$. Define $\Omega_{\mathcal{P}}(x)$ to be the number of prime factors of x that belong to $\mathcal{P}$, counting multiplicity and let $s_0 = s + \omega(n) + \Omega_{\mathcal{P}}(n)$. Furthermore, let $L(\epsilon, n)$ be the real number x such that*

$\Omega(n) = \epsilon x/(\log^2 x)$,

$$x_1 = x_1(l) = x_1(s_0; l, P) = \max\{\exp P, \exp(100.7l), \exp(\exp(9)), 10s_0(l-1)+1\} \quad (1)$$

for each prime l in $\mathcal{P}$ and, for any $\epsilon > 0$, $C_0 = C_0(d, s, n, P, \epsilon)$ be the maximum among quantities $2(d+1)s, x_1(l)^{8.35}, L(\epsilon, n)$ and

$$\epsilon + \exp\left(\frac{(17.62196\varphi(P) + 129.5214(l-1))\,|\mathcal{P}|\log x_1}{(l-1)\log\frac{n}{d}}\right) \quad (2)$$

with l running over all primes in $\mathcal{P}$.

If $N = p_1^{\alpha_1}\cdots p_s^{\alpha_s} q_1^{\beta_1}\cdots q_t^{\beta_t}$ satisfies $h(N) = \frac{n}{d}$, then, for any $\epsilon > 0$, N has a prime factor smaller than C_0.

For fixed s and n, our upper bound is the order of exponential of $P\varphi(P)\,|\mathcal{P}|$, rather than double-exponential of $\varphi(P)\log P$ as in Theorem 3 of [9].

We note that no absolute upper bound is known for the smallest prime factor of a *general* odd perfect number if it exists at all; another known result is Grün's result [11] that the smallest prime factor must be smaller than $\frac{2}{3}\omega(N) + 2$, where $\omega(N)$ denotes the number of distinct prime factors of N.

We shall also give a few more applications of sieve methods to divisor-related numbers. Cattaneo [2] called a positive integer N quasiperfect if $\sigma(N) = 2N + 1$ and showed that such an integer must be an odd square and any divisor of $\sigma(N)$ must be congruent to 1 or 3 modulo 8. Hagis and Cohen [15] showed that if N is quasiperfect, then $N > 10^{35}$ and N has at least 7 distinct prime factors.

Cohen [3] showed that if $p_1, p_2, \ldots, p_t$ are distinct primes and $(p_1 p_2 \cdots p_t)^{2a}$ is quasiperfect, a must be congruent to $1, 3, 5, 9$ or 11 (mod 12). Moreover, if an integer of the form $p_1^{6a_1+2} p_2^{6a_2+2}\cdots p_t^{6a_t+2}$ is quasiperfect, then $t \geq 230876$. We shall show the following analogue of Theorem 1.

Theorem 2. *Let $\mathcal{P}$ be a finite set of primes and $\alpha_1, \alpha_2, \ldots, \alpha_t$ be positive integers such that for each $i = 1, \ldots, t$, $\alpha_i + 1$ is divisible by at least one prime in the set $\mathcal{P}$. If $N = p_1^{2\alpha_1} p_2^{2\alpha_2}\cdots p_t^{2\alpha_t}$ is quasiperfect, then N must have a prime factor smaller than an effectively computable constant C_1 depending only on $\mathcal{P}$, which can be made explicit as follows:*

$$x_3 = x_3(l) = \max\{\exp(8l), \exp(\exp(9))\}, C_1 = \max_{l\in\mathcal{P}} x_3^{2310|\mathcal{P}|^2}. \quad (3)$$

Our method can also be applied to special amicable pairs. A pair of integers m, n are called amicable if the two equations $n = \sigma(m) - m$ and $m = \sigma(n) - n$ hold simultaneously or, equivalently, $\sigma(m) = \sigma(n) = m + n$. It is unknown whether there exists a relatively prime amicable pair or even whether there exists an amicable pair of opposite parity.

Assume that m is even, n is odd and m, n are relatively prime amicable numbers. Kanold showed that $(m, n) = (2M^2, N^2)$ for some odd integers M, N in [19] and that mn must have at least 21 distinct prime factors in [20]. Hagis [12] showed that mn cannot be a multiple of 3 and $mn \geq 10^{74}$. Moreover, if 5 does not divide mn, then $mn \geq 10^{238}$ and mn must have at least 53 distinct prime factors. Later Hagis showed that both $m, n > 10^{60}$ and $mn > 10^{121}$ in [13] and that mn must have at least 22 distinct prime factors in [14].

Under a slightly more general condition that m, n are relatively prime and $\sigma(m)\sigma(n) = (m+n)^2$, Kishore [21] showed that 4 does not divide mn and mn must have at least 22 distinct prime factors.

We have the following analogue of Theorem 1.

Theorem 3. *Let $\mathcal{P}$ be a finite set of primes and $\beta_1, \beta_2, \dots, \beta_t$ be positive integers such that for each $i = 1, \dots, t$, $2\beta_i + 1$ is divisible by at least one prime in $\mathcal{P}$. If m is even, n is odd and m, n are relatively prime integers satisfying $\sigma(m)\sigma(n) = (m+n)^2$ and $mn = 2^\alpha p_1^{2\beta_1} p_2^{2\beta_2} \cdots p_t^{2\beta_t}$, then mn must have a prime factor less than C_1, where C_1 is the same as in the previous theorem.*

Indeed, both Theorems 2 and 3 follow from the following general result.

Theorem 4. *Let $\mathcal{P}$ be a finite set of primes. If $N = p_1^{2\beta_1} p_2^{2\beta_2} \cdots p_t^{2\beta_t}$, with each $2\beta_i + 1$ divisible by some prime in $\mathcal{P}$, is an odd integer such that $\sigma(N)/N \geq 2$ and $\sigma(N)$ has no prime factor congruent to 5 or 7 modulo 8, then N must have a prime factor smaller than C_1.*

For quasiperfect numbers of the form $(p_1 p_2 \cdots p_t)^{2\beta}$, we obtain stronger results. In [32] we showed that if $N = (p_1 p_2 \cdots p_t)^{2\beta}$ with $p_1 < p_2 < \cdots < p_t$ is quasiperfect, then $2\beta + 1$ must be divisible by 3 and $p_1 < \exp(721.85) < 3.129477 \cdot 10^{313}$. This upper bound is still considerably large and we cannot even prove that $p_1 > 7$.

2. Upper Bound Sieve

Our main tool is a standard result in large sieve theory. However, for convenience to compute explicit bounds, we must use an explicit (but a little sophisticated) upper bound sieve formula. There are several explicit upper bound sieve formulae to obtain an explicit upper bound for the implied constant in an upper bound sieve. In [30], the author used the upper bound formula following from Selberg's sieve. But here we shall use the large sieve formula used by Fletcher, Nielsen and Ochem [9], which enabled them to obtain a considerably stronger estimate than in the author's paper [30].

Firstly, we would like to introduce some notations. Let X be a positive number and A be a set of integers contained in an interval of length at most X. For each prime p, let Ω_p be a set of residue classes modulo p and $\rho(p)$ denote the number of residue classes in Ω_p. Define $P(z) = \prod_{p<z} p$ to be the product of primes less than z, $g(m)$ to be the multiplicative function over the squarefree integers m with $g(p) = \rho(p)/(p - \rho(p))$ for each prime p,

$$V(Q) = \prod_{p|Q} \left(1 - \frac{\rho(p)}{p}\right)$$

for any real Q, where p runs over primes, and

$$G_z(T) = \sum_{d \le T, d|P(z)} g(d), G(T) = G_T(T).$$

Finally, we define $S(A, z) = S(A, z, \Omega)$ to be the number of integers in A that do not belong to Ω_p for any prime p dividing $P(z)$.

Now we introduce two lemmas concerning the large sieve inequality. These inequalities allow us to calculate an upper bound in Theorem 1 explicitly.

Lemma 1. *Assume that $\rho(p) < p$ for any prime p. Then it holds for any $w \ge 1$ that*

$$S(A, w) \le \frac{X + w^2}{G(w)}. \tag{4}$$

Proof. It immediately follows from Theorem 7.14 in [17] applied with Ω_p restricted to primes $p < z$ and $h(m)$ the multiplicative function over squarefree integers m defined by

$$h(p) = \begin{cases} g(p) & \text{for all primes } p < z. \\ 0 & \text{for other primes.} \end{cases}$$

□

Lemma 2. *Let us denote*

$$B(z) = \frac{1}{\log z} \sum_{p<z} \frac{\rho(p) \log p}{p} \tag{5}$$

and

$$\psi_1(K, t) = \max\left\{0, t \log \frac{t}{K} - t + K\right\}. \tag{6}$$

If $z \ge 2$ and $v = (\log x)/(\log z) \ge uB(z)$, then we have

$$G_z(x^{1/u}) \ge \frac{\psi_0(v, u)}{V(P(z))}, \tag{7}$$

where

$$\psi_0(v, u) = 1 - \exp(-\psi_1(B(z), v/u)). \tag{8}$$

Proof. This is Theorem 2.2.1 in [10] if we take $B = \sup_t B(t)$ instead of $B(z)$. But we can see that this theorem still holds with $B(z)$ in place of B whether the supremum B exists or not. Indeed, it follows from the argument on pages 53–54 in [10] that

$$1 - V(P(z))G_z(x^{1/u}) \leq \exp\left(-c\frac{\log x}{u\log z} + B(z)(e^c - 1)\right) \tag{9}$$

for any constant $c \geq 0$. Setting $c = \log(v/u) - \log B(z)$, we obtain the lemma. □

3. Proof of Theorem 1

In this section, we shall give a proof of Theorem 1 without making constants explicit. Explicit constants shall be given in the next section.

We may assume that $P \geq 21$ by virtue of the result in [9] concerning the case $\mathcal{P} = \{3, 5\}$ mentioned in the introduction of this paper. Let $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s} q_1^{\beta_1} \cdots q_t^{\beta_t}$ be a solution of $h(N) = \frac{n}{d}$. Let us denote by T the set of primes $\equiv 1 \pmod P$ and by T_y the set of primes congruent to $1 \pmod P$ or congruent to $1 \pmod l$ and not exceeding y. If N has a prime divisor in $\mathcal{P}$, then clearly N has a prime factor smaller than C_0. We may assume without loss of generality that N has no prime divisor in $\mathcal{P}$ and therefore $\Omega_{\mathcal{P}}(N) = 0$.

Let Q_l denote by the set of primes q_i with $\beta_i + 1$ divisible by l and $\pi_l(x)$ denote the number of primes not exceeding x that belong to Q_l. By assumption, any q_i belongs to Q_l for some l in $\mathcal{P}$.

Now we shall prove a result concerning the distribution of prime factors of N, which is the most important lemma in the proof of Theorem 1.

Lemma 3. *Choose any l from $\mathcal{P}$. Let $\kappa = \frac{l-1}{\varphi(P)}$ and y be a sufficiently large real number. There exist three constants B_0, B_1 and $X_1 = X_1(s_0; l, P)$ depending only on s_0, l and P, which shall be made explicit later, such that if u, v and y are real numbers with $u \geq 2, v > B_0 u$ and $y \geq X_1$ and N has no prime factor $\leq y$, then we have*

$$\pi_l(x) \leq \Omega(n) + \begin{cases} \dfrac{B_1(1 + x^{2/u-1})v^2 x}{\xi(v, u)\log^2 x} & \text{for } \max\{y, X_1^v\} \leq x < y^v, \\ \dfrac{B_1(1 + x^{2/u-1})v^{1+\kappa} x}{\xi(v, u)\log^{1-\kappa} y \log^{1+\kappa} x} & \text{for } x \geq y^v, \end{cases} \tag{10}$$

where $\xi(v, u) = \psi_0(B_0, v/u)$.

Proof. Let $\pi_l^*(x)$ denote the number of primes $q_i \leq x$ that belong to Q_l such that $\sigma(q_i^{l-1})$ has no common prime factor smaller than X_1 with n. By assumption, N has no prime factor smaller than X_1 and therefore there exist at most $\Omega(n)$ prime factors q_i such that $\sigma(q_i^{l-1})$ has any common prime factor smaller than X_1 with n. This immediately gives that

$$\pi_l(x) < \pi_l^*(x) + \Omega(n). \tag{11}$$

Now, let $U = U_l$ be the set of primes congruent to 1 (mod P) or congruent to 1 (mod l) and not exceeding y except primes dividing N or primes above or equal to X_1 dividing n. Namely, we set $U_l = T_y \backslash (\mathrm{pf}(N) \cup (\mathrm{pf}(n) \cap [X_1, \infty)))$, where $\mathrm{pf}(m)$ denotes the set of prime factors of an integer m. So that, if a prime divisor r of nN belongs to U, then r divides n and $r \geq X_1$. Hence, we see that if $q_i \in Q_l$ and $\sigma(q_i^{l-1})$ has no common prime factor smaller than X_1 with n, then $\sigma(q_i^{l-1})$ is divisible by no prime in U.

Let r be a prime in U. Then, since $r \equiv 1 \pmod{l}$, there are exactly $l-1$ congruence classes $g_1(r), \ldots, g_{l-1}(r) \pmod{r}$ that belong to order l. Since r does not divide $\sigma(q_i^{l-1})$, q_i belongs to none of the l classes $0, g_1, \ldots, g_{l-1} \pmod{r}$.

Now we can apply the sieve method described in the previous section with A the set of integers not exceeding x, $X = x$, $\Omega_r^{(l)}$ the set of integers not exceeding x that belong to any of congruence classes $0, g_1, \ldots, g_{l-1} \pmod{r}$ for $r \in U$ and $0 \pmod{r}$ for $r \notin U$, $\rho(r) = l$ for $r \in U$ and $\rho(r) = 1$ for $r \notin U$. Thus we see that if q is a prime greater than $x^{1/u}$ in Q_l counted by π_l^*, then q belongs to none of the congruence classes $\Omega_r^{(l)}$ with $r \leq x^{1/u}$. Hence, letting A the set of integers not exceeding x, we have,

$$\pi_l^*(x) \leq S(A, x^{1/u}, \Omega^{(l)}) + x^{1/u} \tag{12}$$

and, using (11),

$$\pi_l(x) \leq S(A, x^{1/u}, \Omega^{(l)}) + x^{1/u} + \Omega(n). \tag{13}$$

We can easily see that $\rho(r) < r$ for any prime r and, provided that $v/u \geq B(z)$, Lemmas 1 and 2 with $w = x^{1/u}$ give

$$\pi_l(x) \leq \frac{x + x^{2/u}}{G(x^{1/u})} + x^{1/u} \leq \frac{x(1 + x^{2/u-1}) V(P(z))}{\psi_0(v, u)} + x^{1/u} + \Omega(n), \tag{14}$$

where we put $z = x^{1/v}$, observing that $G_w(w) \geq G_z(w)$ for $w \geq z$.

Now we need to confirm that $v/u \geq B(z)$ and obtain an upper bound for the quantity $V(P(z))/\psi_0(v, u)$. There are two cases: $x \geq y^v$, i.e. $z \geq y$ and $x < y^v$, i.e. $z < y$. In both cases, we shall obtain an upper bound for $B(z)$ and then $V(P(z))$.

We begin by considering the case $z \geq y$. We see that

$$\begin{aligned} &\sum_{p \leq z} \frac{\rho(p) \log p}{p} \\ &\leq \sum_{p \leq z} \frac{\log p}{p} + \sum_{\substack{p \leq y, \\ p \equiv 1 \pmod{l}}} \frac{(l-1) \log p}{p} + \sum_{\substack{y < p \leq z, \\ r \equiv 1 \pmod{P}}} \frac{(l-1) \log p}{p}. \end{aligned} \tag{15}$$

From the theory of the distribution of primes in arithmetic progressions we see that

$$\sum_{p\le y, p\equiv a \pmod{l}} \frac{\log p}{p} < \frac{\log y + A_1}{l-1} \tag{16}$$

and

$$\sum_{y<p\le z, p\equiv a \pmod{P}} \frac{\log p}{p} < \frac{1}{\varphi(P)}\left(\log\frac{z}{y} + \frac{A_2}{\log y}\right) \tag{17}$$

for some constants A_1 and A_2 if y and z are sufficiently large. Hence, using the estimate $\sum_{p\le z}(\log p)/p < \log z$ in [27, (3.24), p. 70], we obtain

$$\begin{aligned}\sum_{p\le z}\frac{\rho(p)\log p}{p} &\le \log z + \log y + A_1 + \kappa(\log z - \log y) + \frac{A_2\kappa}{\log y}\\ &\le (1+\kappa)\log z + (1-\kappa)\log y + A_1 + \frac{A_2\kappa}{\log y},\end{aligned} \tag{18}$$

which is at most $B_0 \log z$ recalling that $z \ge y \ge X_1$ now. In other words, we have

$$B(z) < B_0. \tag{19}$$

Hence, the assumption $v > B_0 u$ implies that $v/u > B(z)$.

Nextly, we shall obtain an upper bound for $V(P(z))$. There can be at most $\Omega_{\mathcal{P}}(nN) = \Omega_{\mathcal{P}}(n)$ prime factors q_i in T since if $q_i \in T$, then $\sigma(q_i^{\beta_i})$ must be divisible by $\beta_i + 1$ and therefore by some l in $\mathcal{P}$. Hence, there exist at most $s + \omega_{\mathcal{P}}(n)$ prime factors of N in T, which must be larger than $y \ge X_1$ since N is assumed to have no prime factor not exceeding y. Moreover, if $r < y$ is a prime $\equiv 1 \pmod{l}$ which does not belong to U, then r must divide n and therefore $r \ge X_1$.

Thus we conclude that U consists of all primes in T_y except at most $s_0 = s + \omega(n) + \Omega_{\mathcal{P}}(n)$ primes, which are larger than X_1. Hence, we obtain

$$\begin{aligned}\prod_{r<z, r\in U}\left(1-\frac{1}{r}\right) &\le \prod_{\substack{r<z,\\ r\equiv 1 \pmod{l},\\ r\notin U}} \frac{r}{r-1} \prod_{\substack{X_1\le r<z,\\ r\in U}}\left(1-\frac{1}{r}\right)\\ &\le \left(1+\frac{1}{X_1-1}\right)^{s_0} \prod_{\substack{X_1\le r<z,\\ r\in U}}\left(1-\frac{1}{r}\right)\\ &< \exp\frac{s_0}{X_1-1} \prod_{\substack{X_1\le r<y,\\ r\equiv 1 \pmod{l}}}\left(1-\frac{1}{r}\right) \prod_{\substack{y\le r<z,\\ r\equiv 1 \pmod{P}}}\left(1-\frac{1}{r}\right).\end{aligned} \tag{20}$$

We see that if $k \ge 1$ and Y, Z with $Z \ge Y$ are sufficiently large compared to k, then

$$\prod_{Y\le p<Z, p\equiv 1 \pmod{k}}\left(1-\frac{1}{p}\right) < \left(\frac{\log Y}{\log Z}\right)^{1/\varphi(k)} \exp\left(\frac{A_3}{\varphi(k)\log^2 Y}\right) \tag{21}$$

for some constant A_3. Since $z \geq y \geq X_1 = X_1(s_0; l, P)$, we can apply (21) with $k = l$ and $k = P$ and obtain

$$\prod_{r<z, r\in U} \left(1 - \frac{1}{r}\right) < \left(\frac{\log X_1}{\log y}\right)^{1/(l-1)} \left(\frac{\log y}{\log z}\right)^{1/\varphi(P)} \times \exp\left(\frac{s_0}{X_1 - 1} + \frac{A_3}{(l-1)\log^2 X_1} + \frac{A_3}{\varphi(P)\log^2 y}\right). \tag{22}$$

For $k = 1$, an explicit formula of Mertens has been obtained in the form $\prod_{p<z}(1 - 1/p) < e^{-\gamma} \log^{-1} z(1 + 1/(2\log^2 z))$ by [27, (3.26), p. 70]. Hence,

$$\begin{aligned} V(P(z)) &= \prod_{r<z} \left(1 - \frac{\rho(r)}{r}\right) \leq \prod_{r<z} \left(1 - \frac{1}{r}\right)^{\rho(r)} \\ &= \prod_{r<z} \left(1 - \frac{1}{r}\right) \prod_{r<z, r\in U} \left(1 - \frac{1}{r}\right)^{l-1} \\ &< \frac{e^{-\gamma} \log X_1}{\log^{1-\kappa} y \log^{1+\kappa} z} \left(1 + \frac{1}{2\log^2 z}\right) \\ &\quad \times \exp\left(\frac{s_0(l-1)}{X_1 - 1} + \frac{A_3}{\log^2 X_1} + \frac{A_3 \kappa}{\log^2 y}\right). \end{aligned} \tag{23}$$

Provided that X_1 is sufficiently large compared to s_0 and l, we have

$$V(P(z)) < \frac{A_4 \log X_1}{\log^{1-\kappa} y \log^{1+\kappa} z} \tag{24}$$

for some constant A_4. Since $B(z) < B_0 \leq v/u$ by (19), we have $\psi_0(v, u) = 1 - \exp(-\psi_1(B(z), v/u)) > 1 - \exp(-\psi_1(B_0, v/u)) = \xi(v, u)$ and therefore

$$\frac{V(P(z))}{\psi_0(v, u)} \leq \frac{A_4 \log X_1 (1 + x^{2/u-1}) v^{1+\kappa}}{\psi_0(v, u) \log^{1-\kappa} y \log^{1+\kappa} x} \leq \frac{A_4 \log X_1 (1 + x^{2/u-1}) v^{1+\kappa}}{\xi(v, u) \log^{1-\kappa} y \log^{1+\kappa} x}. \tag{25}$$

In the remaining case $z < y$, we note that $z = x^{1/v} \geq X_1$ and a similar (but simpler) argument to the first case gives

$$\sum_{r\leq z} \frac{\rho(r)\log r}{r} \leq \sum_{r\leq z} \frac{\log r}{r} + \sum_{\substack{r\leq z, \\ r\equiv 1 \pmod{l}}} \frac{(l-1)\log r}{r} < B_0 \log z \tag{26}$$

and

$$\begin{aligned} V(P(z)) &\leq \prod_{r<z} \left(1 - \frac{1}{r}\right) \prod_{r<z, r\in U} \left(1 - \frac{1}{r}\right)^{l-1} \\ &< \frac{e^{-\gamma} \log X_1}{\log^2 z} \left(1 + \frac{1}{2\log^2 z}\right) \exp\left(\frac{s_0(l-1)}{X_1 - 1} + \frac{A_3}{\log^2 X_1}\right) \\ &< \frac{A_4 \log X_1}{\log^2 z}. \end{aligned} \tag{27}$$

By (26), we have $B(z) \le B_0 < v/u$ and therefore, as in the first case, (27) gives

$$\frac{V(P(z))}{\psi_0(v,u)} \le \frac{A_4 \log X_1 (1 + x^{2/u-1}) v^2}{\xi(v,u) \log^2 x}. \tag{28}$$

Now, with the aid of inequalities (25) and (28), the lemma easily follows from (14). □

Now we shall prove Theorem 1. Let q_0 be the smallest prime factor of N and assume that $q_0 \ge X_1(s_0; l, P)^v$ for any prime l dividing $\mathcal{P}$ and $q_0 \ge \max\{2(d+1)s, L(\epsilon, n)\}$.

Since $\prod_{i=1}^{s} h(p_i^{\alpha_i}) \le (q_0/(q_0-1))^s$, we obtain

$$\prod_{j=1}^{t} h(q_j^{2\beta_j}) \ge \frac{n}{d} \times \left(\frac{2(d+1)s-1}{2(d+1)s}\right)^s > \sqrt{\frac{n}{d}}. \tag{29}$$

Let $d_l = \prod_q q/(q-1)$, where q runs over all primes in Q_l. It follows from (29) that $\prod_{l \in \mathcal{P}} d_l \ge \sqrt{n/d}$. Hence, we have that $d_l \ge \delta_1 = \left(\frac{n}{d}\right)^{1/2|\mathcal{P}|}$ for some l in $\mathcal{P}$.

Recall that $\kappa = (l-1)/\varphi(P)$. Since N has no prime factor less than q_0, Lemma 3 gives that

$$\begin{aligned}
\log \delta_1 \le & \sum_{p \ge X_1^v, p \in \mathcal{P}} \frac{1}{p} \le \int_{q_0}^{\infty} \frac{\pi_l(t)}{t^2} dt \\
< & \frac{\epsilon}{\log q_0} + \int_{q_0}^{q_0^v} \frac{B_1 v^2 (1 + t^{2/u-1})}{\xi(v,u) t \log^2 t} dt + \int_{q_0^v}^{\infty} \frac{B_1 v^{1+\kappa} (1 + t^{2/u-1})}{\xi(v,u) t \log^{1+\kappa} t \log^{1-\kappa} q_0} dt \\
< & \frac{\epsilon}{\log q_0} + \frac{B_1 (1 + q_0^{2/u-1})}{\xi(v,u) \log q_0} \left(v^2 \left(1 - \frac{1}{v}\right) + \frac{v}{\kappa}\right) \\
< & \frac{\epsilon}{\log q_0} + \frac{\log X_1}{\log q_0} \left(\frac{B_2}{\kappa} + B_3\right)
\end{aligned} \tag{30}$$

for some constants B_2 and B_3. Hence, we have

$$\log q_0 < \epsilon + \frac{\log X_1}{\log \delta_1} \left(\frac{B_2}{\kappa} + B_3\right) = \epsilon + \frac{2 \log X_1 \left(\frac{B_2}{\kappa} + B_3\right)}{(l-1) \log \frac{n}{d}}. \tag{31}$$

In the next section, we shall show that we can take $X = x_1, B_2 = 17.62196$ and $B_3 = 129.5214$, which proves Theorem 1.

4. Distribution of Primes in Arithmetic Progressions

In order to complete the proof of Theorem 1, we must know some explicit estimates for the sum $\sum_p (\log p)/p$ and the product $\prod_p (1 - 1/p)$ with p running over primes in an arithmetic progression.

We begin by introducing Chebyshev prime-counting functions for arithmetic progressions:

$$\psi(x;k,a) = \sum_{n \le x, n \equiv a \pmod{k}} \Lambda(n) \tag{32}$$

$$\theta(x;k,a) = \sum_{p \le x, p \equiv a \pmod{k}} \log p. \tag{33}$$

It is well-known that, for any modulus $k \le \log x$ and congruence class $a \pmod{k}$ with $\gcd(a,k) = 1$, $\psi(x;k,a)$ is asymptotic to $x/\varphi(k)$ with an error term

$$O\left(\frac{x}{\varphi(k)\log x}\right).$$

Namely, we have

$$\left|\psi(x;k,a) - \frac{x}{\varphi(k)}\right| \le \frac{A_0 x}{\varphi(k)\log x} \tag{34}$$

for $x \ge x_0$ with x_0 sufficiently large, where A_0 denotes some constant. Indeed, we shall show the following explicit estimate.

Lemma 4. *If $x \ge \exp(\exp(9))$ is a real number and k is a positive integer coprime to a not exceeding $\log x$, then*

$$\left|\psi(x;k,a) - \frac{x}{\varphi(k)}\right| \le \frac{0.00009x}{\varphi(k)\log^2 x}. \tag{35}$$

In other words, putting $A_0 = 0.00009$ and $x_0 = \max\{\exp k, \exp(\exp(9))\}$, the inequality (34) holds for $x \ge x_0$.

Proof. For $k \ge 10^5$, Theorem 1.2 of [1] gives

$$\left|\psi(x,k,a) - \frac{x}{\varphi(k)}\right| \le \frac{1.012x^{1-40/(\sqrt{k}\log^2 k)}}{\varphi(k)} + 1.4579x\sqrt{X}\exp(-X), \tag{36}$$

where $X = \sqrt{\log x/9.645908801}$. Hence, we have

$$\left|\psi(x,k,a) - \frac{x}{\varphi(k)}\right| \le \frac{10^{-30}x}{\varphi(k)\log^2 x} \tag{37}$$

for $x \ge e^k$ and $k \ge 10^5$. Similar estimates have also been given in the author's preprint [31] and another estimate is implicit in [5].

For $k < 10^5$, we know from [26] that no Dirichlet L-function modulo k has a zero $s = \sigma + it$ with $s > 1/2$ and $|t| \le 1000$. Now, putting $C_1(\chi, 1000) = 9.14$, we can confirm the conditions in Theorem 5 of [7] for $x \ge x_0$. Hence, we apply this theorem to obtain

$$\left|\psi(x,k,a) - \frac{x}{\varphi(k)}\right| \le 3x\sqrt{\frac{kX}{9.14\varphi(k)}}\exp(-X) < \frac{0.00009x}{\log^2 x}. \tag{38}$$

Thus the lemma is proved. □

Based on this inequality, we shall prove the following estimates.

Lemma 5. *Let w and z be arbitrary real numbers with $z \geq w \geq x_0$. Then the inequalities*

$$\sum_{w<p\leq z, p\equiv a \pmod k} \frac{\log p}{p} < \frac{1}{\varphi(k)}\left(\log\frac{z}{w} + \frac{10^{-4}}{\log^2 w} + \frac{10^{-4}}{\log^2 z} + \frac{10^{-4}}{\log w}\right) \tag{39}$$

and

$$\prod_{w\leq p<z, p\equiv a \pmod k}\left(1-\frac{1}{p}\right) < \left(\frac{\log w}{\log z}\right)^{1/\varphi(k)} \exp\left(\frac{1}{4000\varphi(k)\log^2 x_0}\right). \tag{40}$$

hold.

Moreover, if $z \geq x_0^{100.7}$, then we have

$$\sum_{p\leq z, p\equiv a \pmod k} \frac{\log p}{p} < \frac{\log z}{\varphi(k)} + 1.007\log x_0. \tag{41}$$

Proof. We begin by noting that Lemma 4 yields

$$\left|\theta(x,k,a) - \frac{x}{\varphi(k)}\right| < \frac{10^{-4}}{\varphi(k)\log^2 x} \tag{42}$$

for $x \geq x_0$.

Now we shall prove (40). By partial summation, we have

$$\begin{aligned}\sum_{\substack{w<p<z,\\ p\equiv a \pmod k}} \frac{1}{p} &> \frac{\log\frac{\log z}{\log w}}{\varphi(k)} - \frac{10^{-4}}{\varphi(k)}\left(\frac{1}{\log^2 z} + \frac{1}{\log^2 w} + \int_w^z \frac{(1+\log t)dt}{t\log^4 t}\right)\\ &> \frac{\log\frac{\log z}{\log w}}{\varphi(k)} - \frac{10^{-4}}{\varphi(k)}\left(\frac{3}{2\log^2 w} + \frac{1}{\log^2 z} + \frac{1}{3\log^3 w} - \frac{1}{3\log^3 z}\right)\\ &> \frac{\log\frac{\log z}{\log w}}{\varphi(k)} - \frac{1}{4000\varphi(k)\log^2 w}\end{aligned} \tag{43}$$

for $z > w \geq x_0$ and therefore

$$\begin{aligned}\prod_{\substack{w\leq p<z,\\ p\equiv a \pmod k}}\left(1-\frac{1}{p}\right)^{-1} &= \exp\sum_{\substack{w\leq p<z,\\ p\equiv a \pmod k}}\left(\frac{1}{p}+\frac{1}{2p^2}+\cdots\right)\\ &> \exp\sum_{\substack{w\leq p<z,\\ p\equiv a \pmod k}}\frac{1}{p}\\ &> \left(\frac{\log z}{\log w}\right)^{1/\varphi(k)}\exp\left(-\frac{1}{4000\varphi(k)\log^2 w}\right)\end{aligned} \tag{44}$$

for $z \geq w \geq x_0$, which gives (40).

Nextly, we shall prove (41). Partial summation similar to above gives

$$\begin{aligned}\sum_{\substack{p\leq z,\\ p\equiv a \pmod k}} \frac{\log p}{p} \leq & \frac{\log(k+1)}{k+1} + \frac{\theta(z;k,a)}{z} + \int_{2k+1}^{z} \frac{\theta(t;k,a)}{t^2}dt \\ < & \frac{1}{\varphi(k)}\left(\log(k+1)+1+\frac{10^{-4}}{\log^2 z}\right) + \int_{2k+1}^{z} \frac{\theta(t;k,a)}{t^2}dt.\end{aligned} \tag{45}$$

Recall that $\log x_0 = \max\{k, e^9\}$. Using the Brun-Titchmarsh theorem given in [25], we have

$$\begin{aligned}\int_{2k+1}^{x_0} \frac{\theta(t;k,a)}{t^2}dt < & \frac{2}{\varphi(k)} \int_{2k}^{x_0} \frac{\log t dt}{t \log \frac{t}{k}} \\ = & \frac{2}{\varphi(k)}\left(\log\frac{x_0}{2k} + \log k\left(\log\frac{\log x_0}{\log k} - \log\log 2\right)\right) \\ < & \frac{2.007}{\varphi(k)}\log x_0.\end{aligned} \tag{46}$$

We can easily see that (42) gives

$$\int_{x_0}^{z} \frac{\theta(t;k,a)}{t^2}dt < \frac{1}{\varphi(k)}\left(\log\frac{z}{x_0} + \frac{10^{-4}}{\log x_0}\right). \tag{47}$$

Inserting these upper bounds into (45) yields

$$\sum_{\substack{p< z,\\ p\equiv a \pmod k}} \frac{\log p}{p} < \frac{1}{\varphi(k)}\left(\log z + 1.007\log x_0\right) \tag{48}$$

for $z \geq x_0^{100.7}$, giving (41).

Finally, (39) immediately follows by using the partial summation

$$\sum_{\substack{w\leq p< z,\\ p\equiv a \pmod k}} \frac{\log p}{p} = \frac{\theta(z;k,a)}{z} - \frac{\theta(w;k,a)}{w} + \int_{w}^{z} \frac{\theta(z;k,a)}{t^2}dt \tag{49}$$

and (42). □

Now we shall complete the proof of Theorem 1. In Lemma 3, we shall take $X_1 = x_1(s_0; l, P)$ for each l in $\mathcal{P}$. In the case $z \geq y$, since $z \geq y \geq x_1(l,P) \geq \max\{x_0(P), x_0(l)^{100.7}\}$, (41) of Lemma 5 applied with $k = l$ allows us to take $A_1 = 1.007\log x_0 < 0.01\log z$ in (16) and (39) of Lemma 5 with $k = P$ allows us to take $A_2 = 1.1 \times 10^{-4}$ in (17). Hence, in the case $z \geq y$, we can take $B_0 = 2.01$ in (19). Since $z \geq X_1 = x_1 \geq x_0^{100.7}$, also in the case $z < y$, we can take $B_0 = 2.01$ in

(26). In our setting of x_1, we can take $A_3 = 1/4000$ in (21) with $k = l$ and $k = P$ from (40) of Lemma 5, noting that $x_1 \geq x_0 \geq \exp P$. Since $x_1 \geq 10s_0(l-1)+1$, we can take $A_4 = \exp(0.1 + 10^{-9} - \gamma)$ and $B_1 = e^{0.1+10^{-8}-\gamma} \log x_1$.

We choose $u = 2 + 10^{-7}, v = 8.35 > 4.03 > B_0 u$ and assume that $q_0 \geq x_1(l)^v$ for any l in $\mathcal{P}$ and $q_0 \geq \max\{2(d+1)s, L(\epsilon, n)\}$. Then the most right hand side of (30) is at most

$$\frac{\epsilon}{\log q_0} + \frac{\log x_1}{\log q_0}\left(\frac{8.81098}{\kappa} + 64.7607\right). \tag{50}$$

Hence, we obtain

$$\log q_0 < \epsilon + \frac{(17.62196\varphi(P) + 129.5214(l-1))\,|\mathcal{P}| \log x_1}{(l-1)\log \frac{n}{d}}. \tag{51}$$

This implies that $q_0 \leq C_0$ and the proof of Theorem 1 is complete.

5. Proof of Theorems 2-4

Firstly, we shall prove Theorem 4. Assume that $N = p_1^{2\alpha_1} p_2^{2\alpha_2} \cdots p_t^{2\alpha_t}$ satisfies that $\sigma(N) \geq 2N$ has no prime factor congruent to 5 or 7 modulo 8 and each $2\alpha_i + 1$ is divisible by some prime in $\mathcal{P}$.

For each $l \in \mathcal{P}$, let R_l denote the set of primes p_i with $2\alpha_i + 1$ divisible by l. By assumption, any p_i belongs to R_l for some l in $\mathcal{P}$. Let $a_1, a_2 \pmod{8l}$ be the congruence classes that are congruent to $1 \pmod{l}$ and $5, 7 \pmod{8}$ respectively. If $p \in R_l$, then $p^{l-1} + p^{l-2} + \cdots + 1$ has no prime factor congruent to a_1 or $a_2 \pmod{8l}$.

We shall show that $\prod_{p \geq C_1, p \in R_l} \frac{p}{p-1} < 2^{1/|\mathcal{P}|}$ for all l in $\mathcal{P}$, which would imply that N must have some prime factor smaller than C_1 in order to satisfy $\sigma(N)/N \geq 2$. But, in order to prove Theorem 4, we shall apply our sieve argument setting $\Omega_p^{(l)} = \{n \mid n(n^{l-1} + n^{l-2} + \cdots + 1) \equiv 0 \pmod{p}\}$ for primes p congruent to a_1 or $a_2 \pmod{8l}$ and $\Omega_p^{(l)} = \{n \mid n \equiv 0 \pmod{p}\}$ for other primes.

Let $\pi_l'(x)$ denote the number of primes not exceeding x that belong to R_l. We have

$$\pi_l'(x) < S(A, y, \Omega^{(l)}) + y \tag{52}$$

for any y. Let z be an arbitrary real number not smaller than $x_4 = x_3^{100.7}$. Then, observing that $x_3 \geq x_0(8l)$, (41) gives

$$\sum_{\substack{p \leq z, \\ p \equiv a_1, a_2 \pmod{8l}}} \frac{\log p}{p} < \frac{1.01}{2\varphi(l)} \log z \tag{53}$$

and therefore

$$\sum_{r\le z}\frac{\rho(r)\log r}{r}\le\sum_{r\le z}\frac{\log r}{r}+\sum_{\substack{r\le z,\\ r\equiv a_1,a_2 \pmod{8l}}}\frac{(l-1)\log r}{r}<1.505\log z. \tag{54}$$

Observing that $z\ge x_4>\exp(\exp(9))$ and using (40), we have

$$\begin{aligned} V(P(z)) &\le \prod_{r<z}\left(1-\frac{1}{r}\right)\prod_{\substack{x_3\le r<z,\\ r\equiv a_1,a_2 \pmod{8l}}}\left(1-\frac{1}{r}\right)^{l-1} \\ &<\frac{e^{-\gamma}\log^{1/2}x_3}{\log^{3/2}z}\left(1+\frac{1}{\log z}\right)\exp\left(\frac{1}{4000\log^2 x_3}\right) \qquad (55)\\ &<\frac{0.56146\log^{1/2}x_3}{\log^{3/2}z}. \end{aligned}$$

From (54) and (55), the sieve inequality given in Lemma 2 with $B=1.505, u=2.000007$ and $v=7.538$ gives that if $x\ge x_4^{7.538}$, then

$$S(A,x^{1/u},\Omega^{(l)})\le\frac{16.65708x\log^{1/2}x_3}{\log^{3/2}x} \tag{56}$$

and therefore

$$\pi_l'(x)\le S(A,x^{1/u},\Omega^{(l)})+x^{1/u}<\frac{16.65709x\log^{1/2}x_3}{\log^{3/2}x}. \tag{57}$$

Since $C_1=x_3^{2310|\mathcal{P}|^2}>x_4^{7.538}$, we have, as in the proof of Theorem 1,

$$\prod_{p\ge C_1, p\in R_l}\frac{p}{p-1}<\exp\left(\frac{2}{C_1}+\frac{33.31418\log^{1/2}x_3}{\log^{1/2}C_1}\right)<2^{1/|\mathcal{P}|} \tag{58}$$

for each prime $l\in\mathcal{P}$, which proves Theorem 4.

Now, all that remains is to derive Theorems 2 and 3 from Theorem 4. If $N=p_1^{2\beta_1}p_2^{2\beta_2}\cdots p_t^{2\beta_t}$ satisfies $\sigma(N)=2N+1$ and each $2\beta_i+1$ is divisible by some prime in $\mathcal{P}$, then, as mentioned above, Cattaneo has shown that $\sigma(N)$ has no prime factor congruent to 5 or 7 modulo 8 and therefore N satisfies the hypothesis of Theorem 4. Hence, N must have a prime factor smaller than C_1. This proves Theorem 2.

If m is even, n is odd and m,n are relatively prime integers satisfying $\sigma(m)\sigma(n)=(m+n)^2$ and $mn=2^\alpha p_1^{2\beta_1}p_2^{2\beta_2}\cdots p_t^{2\beta_t}$, then we see that $m=2A^2$ and $n=B^2$ for some odd integers A,B from Kishore [21]. Since m,n are relatively prime, so are A,B. Hence, $\sigma(m)\sigma(n)=(m+n)^2=(2A^2+B^2)^2$ has no prime factor congruent to 5 or 7 modulo 8. Now, taking $N=mn/2=A^2B^2=p_1^{2\beta_1}p_2^{2\beta_2}\cdots p_t^{2\beta_t}$, we see that $\sigma(N)/N>\sigma(mn)/(2mn)=\sigma(m)\sigma(n)/(2mn)=(m+n)^2/(2mn)>2$ and $\sigma(N)=\sigma(m)\sigma(n)/3=(2A^2+B^2)^2/3$ has no prime factor congruent to 5 or 7 modulo 8. So that, Theorem 3 follows from Theorem 4.

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