A NOTE ON SUMS OF TWO SQUARES AND SUM-OF-DIVISORS FUNCTIONS

Mariusz Skalba
Department of Mathematics, University of Warsaw, Warsaw, Poland
skalba@mimuw.edu.pl

Received: 4/5/20, Accepted: 10/22/20, Published: 11/2/20

Abstract
Let \( \sigma_k(n) \) be the sum of positive divisors \( d \) of \( n \) satisfying \( d \equiv k \mod 4 \) (for \( k \in \{1, 3\} \)) and let \( \nu_2(m) \) denote the 2-adic exponent of a natural number \( m \). We prove that \( n \) is a sum of two squares if and only if \( \nu_2(\sigma_1(n)) \neq \nu_2(\sigma_3(n)) \).

1. Results
Let \( d(n) \) and \( d_k(n) \) be, respectively, the number of all positive divisors of \( n \) and the number of those positive divisors \( d \) which satisfy \( d \equiv k \mod 4 \) (for \( k \in \{1, 3\} \)). It is well-known and easy to prove that

(A) \( d(n) \) is odd if and only if \( n \) is a square.

A classical result [1] states that

(B) the number of pairs \( (x, y) \) of integers satisfying \( x^2 + y^2 = n \) equals \( 4(d_1(n) - d_3(n)) \).

The sole goal of this note is to provide a certain characterization of sums of two squares in terms of sum-of-divisors functions instead of number-of-divisors functions. Let \( \sigma_k(n) \) be the sum of divisors \( d \) of \( n \) satisfying \( d \equiv k \mod 4 \) (for \( k \in \{1, 3\} \)). We start with a lemma.

Lemma 1. For odd \( n \) with prime factorization \( n = \prod_i q_i^{a_i} \), the following formula holds:

\[
\frac{\sigma_1(n) - \sigma_3(n)}{\sigma_1(n) + \sigma_3(n)} = \prod_{q_i \equiv 3 \pmod{4}, a_i \text{ odd}} \frac{1 - q_i}{1 + q_i} \prod_{q_i \equiv 3 \pmod{4}, a_i \text{ even}} \frac{1 - q_i + \ldots + q_i^{a_i}}{1 + q_i + \ldots + q_i^{a_i}}.
\]

(1)

Proof. Let \( \chi \) be the unique non-trivial Dirichlet character mod 4:

\[\chi(4k + 1) = 1, \quad \chi(4k + 3) = -1, \quad \chi(2k) = 0.\]
Then we have
\[ \sigma_1(n) - \sigma_3(n) = \prod_i (1 + \chi(q_i) q_i + \ldots + \chi(q_i^{a_i}) q_i^{a_i}), \]
and obviously
\[ \sigma_1(n) + \sigma_3(n) = \sigma(n) = \prod_i (1 + q_i + \ldots + q_i^{a_i}). \]
Hence
\[ \frac{\sigma_1(n) - \sigma_3(n)}{\sigma_1(n) + \sigma_3(n)} = \prod_{q_i \equiv 3 \pmod{4}, a_i \text{ even}} \frac{1 + \chi(q_i) q_i + \ldots + \chi(q_i^{a_i}) q_i^{a_i}}{1 + q_i + \ldots + q_i^{a_i}}, \]
and the formula (1) does follow in virtue of the observation that for odd \( a_i \)
\[ \frac{1 - q_i + q_i^2 - \ldots - q_i^{a_i}}{1 + q_i + \ldots + q_i^{a_i}} = \frac{1 - q_i}{1 + q_i}. \]

Our main result reads as follows.

**Theorem 1.** A natural number \( n \) is a sum of two squares if and only if
\[ \nu_2(\sigma_1(n)) \neq \nu_2(\sigma_3(n)), \] (2)
where \( \nu_2(r) \) is the 2–adic exponent of a rational number \( r \) (\( \nu_2(0) = \infty \) by definition).

**Proof.** One can assume from the beginning that \( n \) is odd. By a well-known classical theorem [1], if \( n \) is a sum of two squares then for all \( q_i \equiv 3 \pmod{4} \) the relevant exponent \( a_i \) is even. By Lemma 1 we get
\[ \frac{\sigma_1(n) - \sigma_3(n)}{\sigma_1(n) + \sigma_3(n)} = \prod_{q_i \equiv 3 \pmod{4}, a_i \text{ even}} \frac{1 - q_i + q_i^2 - \ldots - q_i^{a_i}}{1 + q_i + \ldots + q_i^{a_i}}. \]
Both numerators and denominators on the right-hand side are odd, hence
\[ \nu_2(\sigma_1(n) - \sigma_3(n)) = \nu_2(\sigma_1(n) + \sigma_3(n)) \] (3)
and (2) easily follows.
In the opposite direction we assume that (2) does hold. Then (3) follows, hence by Lemma 1 the first product on the right-hand side of the formula (1) has to be empty: if not, each factor \( (1 - q_i)/(1 + q_i) \) would contribute to 2 in the denominator, because of
\[ \nu_2 \left( \frac{1 - q_i}{1 + q_i} \right) \leq -1. \]
Hence \( n \) is a sum of two squares. \( \Box \)
Remark 1. Our theorem is a generalization of statement (A) in the following sense. For an odd \( n \) one has \( \sigma(n) \equiv d(n) \pmod{2} \) and \( \sigma(n) = \sigma_1(n) + \sigma_3(n) \). Hence (A) (for an odd \( n \)) can be reformulated as follows

\[
\text{An odd } n \text{ is a square if and only if } \sigma_1(n) \neq \sigma_3(n) \pmod{2}.
\]

The condition \( \sigma_1(n) \neq \sigma_3(n) \pmod{2} \) is a very special case of the condition \( \nu_2(\sigma_1(n)) \neq \nu_2(\sigma_3(n)) \) and being a square is a very special case of being a sum of two squares.

Remark 2. A naive mimicking of (B) in terms of sum-of-divisors functions is elusive, because by the formula (1) we obtain

\[
(-1)^{\frac{n-1}{2}}(\sigma_1(n) - \sigma_3(n)) > 0 \quad \text{for odd } n.
\]

On the other hand the representability of \( n \) as a sum of two squares lies much deeper than residue of \( n \) mod 4.

Corollary 1. The following asymptotics holds

\[
\# \{ n \leq x | \nu_2(\sigma_1(n)) \neq \nu_2(\sigma_3(n)) \} \approx c x \sqrt{\log x},
\]

and for almost all natural numbers \( n \) (in the sense of natural density) one has

\[
\nu_2(\sigma_1(n)) = \nu_2(\sigma_3(n)).
\]

Proof. It follows immediately from Theorem 1 in virtue of famous Landau’s theorem [2] giving asymptotics for the number of numbers below \( x \) which are sums of two squares.

Corollary 2. For any natural number \( n \) the following three conditions are equivalent

1. \( d_1(n) = d_3(n) \),
2. \( \nu_2(\sigma_1(n)) = \nu_2(\sigma_3(n)) \),
3. \( n \) is not a sum of two squares.

We have contributed only the condition 2.

References
