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A NOTE ON SUMS OF TWO SQUARES AND SUM-OF-DIVISORS FUNCTIONS

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Abstract
Let $\sigma_k(n)$ be the sum of positive divisors d of n satisfying $d \equiv k \pmod 4$ (for $k \in \{1,3\}$) and let $\nu_2(m)$ denote the 2-adic exponent of a natural number m. We prove that n is a sum of two squares if and only if $\nu_2(\sigma_1(n)) \neq \nu_2(\sigma_3(n))$.

1. Results

Let $d(n)$ and $d_k(n)$ be, respectively, the number of all positive divisors of n and the number of those positive divisors d which satisfy $d \equiv k \pmod 4$ (for $k \in \{1,3\}$). It is well-known and easy to prove that

(A) *$d(n)$ is odd if and only if n is a square.*

A classical result [1] states that

(B) *the number of pairs (x,y) of integers satisfying $x^2+y^2=n$ equals* $4(d_1(n)-d_3(n))$.

The sole goal of this note is to provide a certain characterization of sums of two squares in terms of sum-of-divisors functions instead of number-of-divisors functions. Let $\sigma_k(n)$ be the sum of divisors d of n satisfying $d \equiv k \pmod 4$ (for $k \in \{1,3\}$). We start with a lemma.

Lemma 1. *For odd n with prime factorization $n=\prod_i q_i^{a_i}$, the following formula holds:*

$$\frac{\sigma_1(n)-\sigma_3(n)}{\sigma_1(n)+\sigma_3(n)} = \prod_{q_i\equiv 3 \pmod 4,\, a_i\, odd} \frac{1-q_i}{1+q_i} \prod_{q_i\equiv 3 \pmod 4,\, a_i\ even} \frac{1-q_i+\ldots+q^{a_i}}{1+q_i+\ldots+q_i^{a_i}}. \tag{1}$$

Proof. Let χ be the unique non-trivial Dirichlet character mod 4:

$$\chi(4k+1)=1,\quad \chi(4k+3)=-1,\quad \chi(2k)=0.$$

Then we have

$$\sigma_1(n) - \sigma_3(n) = \prod_i (1 + \chi(q_i)q_i + \ldots + \chi(q_i^{a_i})q_i^{a_i}),$$

and obviously

$$\sigma_1(n) + \sigma_3(n) = \sigma(n) = \prod_i (1 + q_i + \ldots + q_i^{a_i}).$$

Hence

$$\frac{\sigma_1(n) - \sigma_3(n)}{\sigma_1(n) + \sigma_3(n)} = \prod_{q_i \equiv 3 \pmod 4} \frac{1 + \chi(q_i)q_i + \ldots + \chi(q_i^{a_i})q_i^{a_i}}{1 + q_i + \ldots + q_i^{a_i}},$$

and the formula (1) does follow in virtue of the observation that for odd a_i

$$\frac{1 - q_i + q_i^2 - \ldots - q^{a_i}}{1 + q_i + \ldots + q_i^{a_i}} = \frac{1 - q_i}{1 + q_i}.$$

□

Our main result reads as follows.

Theorem 1. *A natural number n is a sum of two squares if and only if*

$$\nu_2(\sigma_1(n)) \neq \nu_2(\sigma_3(n)), \tag{2}$$

where $\nu_2(r)$ is the 2–adic exponent of a rational number r ($\nu_2(0) = \infty$ by definition).

Proof. One can assume from the beginning that n is odd. By a well-known classical theorem [1], if n is a sum of two squares then for all $q_i \equiv 3 \pmod 4$ the relevant exponent a_i is even. By Lemma 1 we get

$$\frac{\sigma_1(n) - \sigma_3(n)}{\sigma_1(n) + \sigma_3(n)} = \prod_{q_i \equiv 3 \pmod 4, a_i \text{ even}} \frac{1 - q_i + q_i^2 - \ldots + q^{a_i}}{1 + q_i + \ldots + q_i^{a_i}}.$$

Both numerators and denominators on the right-hand side are odd, hence

$$\nu_2(\sigma_1(n) - \sigma_3(n)) = \nu_2(\sigma_1(n) + \sigma_3(n)) \tag{3}$$

and (2) easily follows.

In the opposite direction we assume that (2) does hold. Then (3) follows, hence by Lemma 1 the first product on the right-hand side of the formula (1) has to be empty: if not, each factor $(1-q_i)/(1+q_i)$ would contribute to 2 in the denominator, because of

$$\nu_2\left(\frac{1 - q_i}{1 + q_i}\right) \leq -1.$$

Hence n is a sum of two squares. □

Remark 1. Our theorem is a generalization of statement (A) in the following sense. For an odd n one has $\sigma(n) \equiv d(n) \pmod 2$ and $\sigma(n) = \sigma_1(n) + \sigma_3(n)$. Hence (A) (for an odd n) can be reformulated as follows

An odd n is a square if and only if $\sigma_1(n) \not\equiv \sigma_3(n) \pmod 2$.

The condition $\sigma_1(n) \not\equiv \sigma_3(n) \pmod 2$ is a very special case of the condition $\nu_2(\sigma_1(n)) \neq \nu_2(\sigma_3(n))$ and being a square is a very special case of being a sum of two squares.

Remark 2. A naive mimicking of (B) in terms of sum-of-divisors functions is elusive, because by the formula (1) we obtain

$$(-1)^{\frac{n-1}{2}}(\sigma_1(n) - \sigma_3(n)) > 0 \quad \text{for odd } n.$$

On the other hand the representabilty of n as a sum of two squares lies much deeper than residue of n mod 4.

Corollary 1. *The following asymptotics holds*

$$\#\{n \leq x | \nu_2(\sigma_1(n)) \neq \nu_2(\sigma_3(n))\} \approx c\frac{x}{\sqrt{\log x}},$$

and for almost all natural numbers n (in the sense of natural density) one has

$$\nu_2(\sigma_1(n)) = \nu_2(\sigma_3(n)).$$

Proof. It follows immediately from Theorem 1 in virtue of famous Landau's theorem [2] giving asymptotics for the number of numbers below x which are sums of two squares. □

Corollary 2. *For any natural number n the following three conditions are equivalent*

1. $d_1(n) = d_3(n)$,
2. $\nu_2(\sigma_1(n)) = \nu_2(\sigma_3(n))$,
3. *n is not a sum of two squares.*

We have contributed only the condition 2.

References

[1] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Graduate Texts in Mathematics **84**, Springer, 1990.

[2] E. Landau, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, *Arch.Math. Phys.* **13** (1908), 305-312.