UNIQUENESS FOR SUMS OF NONVANISHING SQUARES

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Received: 1/21/20, Revised: 8/28/20, Accepted: 10/22/20, Published: 11/2/20

Abstract

We address the issue of uniqueness for sums of nonvanishing squares; that is, we determine all positive integers N that can be represented as a sum of $k \geq 5$ nonvanishing squares in essentially only one way. Our methods are elementary and are based on a lower bound on the number of 1s that must be present in such a representation.

1. Introduction

We consider the problem of representing a positive integer N as a sum of k nonvanishing squares:

$$N = \sum_{j=1}^{k} m_j^2 \tag{1}$$

where $m_j \in \mathbf{Z}$ with $m_j \neq 0$, $j = 1, \ldots, k$. In the current work we are concerned with uniqueness of representations of the form (1) up to the order of the summands and the sign of the summands. With this in mind, we assume that $m_j \geq 1$ in (1). Given $k \geq 1$, our aim is to determine which integers $N > 0$ admit a unique representation as in (1) up to permutation of the m_j. We will focus primarily on the cases $k \geq 5$.

There is a remarkable body of work concerning equations of the form (1); see, for instance, [7] and the references therein. The majority of the results do not distinguish between solutions to (1) that involve zeros and those which involve only positive integers. We have the following Theorem concerning the existence of representations of the form (1) consisting only of positive integers (which we will refer to as a nonvanishing sum of squares).

Theorem 1 (Pall [8], Dubouis [3]). *Let* $\mathbf{B} = \{1, 2, 4, 5, 7, 10, 13\}$.

1. *Every integer* N *is the sum of* k *nonvanishing squares provided* $k \geq 6$ *and* $N \geq k$, $N \neq k + b$ *for some* $b \in \mathbf{B}$.

2. *Suppose that* $k = 5$. *Every integer* N *is the sum of five nonvanishing squares provided* $N \geq 5$, $N \neq 5 + b$ *for some* $b \in \mathbf{B} \cup \{28\}$.

3. *Every integer* N *is the sum of four nonvanishing squares provided* $N \geq 4$, $N \neq 4 + b$ *with* $b \in \mathbf{B} \cup \{25, 37\}$ *and* $N \notin \mathbf{E}_4$, *where*
$$\mathbf{E}_4 = \{4^\alpha p \mid \alpha \geq 0,\ p = 2, 6, 14\}.$$

While the content of Theorem 1 appears in Pall's work [8], we also found the discussion of Grosswald [4] to be helpful. We note that the existence result in the case $k = 3$ remains open; see Conjecture 1 below.

There are significantly fewer results that are concerned with uniqueness for representations as in (1). Grosswald and Bateman [2] have established a result in the case $k = 3$ depending on the classification of imaginary quadratic fields of class number 4. This classification problem was later resolved by Arno in [1]. This problem has emerged in connection with work of the present authors on the existence of small amplitude periodic solutions of some higher dimensional nonlinear wave equations that are symmetric in the spatial variables, see [6]. Indeed, the problem of essential distinctness for a Diophantine equation similar to (1) was considered but where N depends on k and a different type of solution was sought.

Our main result is the following.

Theorem 2. *Let* $N, k \in \mathbf{N}$ *with* $k \geq 5$ *and let* $N \geq k$. *The Diophantine equation* (1) *has a unique solution in each of the following cases.*

1. $5 \leq k \leq 9$, *and* N *is among the corresponding values of* N *listed in Table 1 below (see Section 4).*

2. $k \geq 10$ *and* $N \in \{k + b \mid b \in \mathscr{G}\}$, *where*
$$\mathscr{G} = \{0, 3, 6, 8, 9, 11, 12, 14, 16, 17, 19, 20, 22, 25, 28\}.$$

If (N, k) *satisfies the hypotheses of the theorem and is not one of the cases listed in 1-2, then either the Diophantine equation* (1) *has no solution or the Diophantine equation* (1) *has two or more solutions.*

Our argument for Theorem 2 pivots on the observation that any solution $(m_j)_{j=1}^k$ of (1) has at least $\lambda_1 = (4k - N)/3 > 0$ entries for which $m_j = 1$, provided $N < 4k$.

After reordering the terms in the solution, if necessary, the sum (1) can now be rewritten

$$N - \lambda_1 = \sum_{j=\lambda_1+1}^{k} m_j^2. \tag{2}$$

It transpires that (2) is of one of the following three special forms for some positive integer ℓ:

$$4\ell = \sum_{j=1}^{\ell} m_j^2, \quad \text{or} \quad 4\ell + 1 = \sum_{j=1}^{\ell} m_j^2, \quad \text{or} \quad 4\ell + 2 = \sum_{j=1}^{\ell} m_j^2.$$

We are thus left to treat each of these special forms individually; this is accomplished using simple modular arithmetic. We are then left to consider the cases for which $N \geq 4k$. We use Theorem 1 to show that the Diophantine equation (1) has at least two solutions when $N > 2k + 48 + 8\sqrt{k+20}$. This means that we need only check the cases for which $4k \leq N \leq 2k + 48 + 8\sqrt{k+20}$. It follows that we must have $k \leq 59$. It is now possible to check (using a computer) that in each of these cases the only pairs (N, k) for which there is a unique solution to (1) are those listed in the statement of Theorem 2.

1.1. Open Problems

The following conjecture concerning the problem of existence also remains open. Some evidence in favor of this conjecture is collected in [5].

Conjecture 1. Let $\mathbf{T} = \{1, 2, 5, 10, 13, 25, 37, 58, 85, 130\}$ and let $\mathbf{M} = \{4^\alpha(8m + 7) \mid m \in \mathbf{Z}, \alpha \geq 0\}$. Every integer $N \notin \mathbf{M}$ and not of the form $4^\alpha p$ for some $p \in \mathbf{T}$ is a sum of three nonvanishing squares.

A key aspect of the current work is the use of the existence of representations for sums of k nonvaninishing squares with the uniqueness problem for sums of $k+1$ nonvanishing squares. Related to Conjecture 1 (for which $k = 3$), then, is the problem of characterizing which N admit an essentially distinct representation as a sum of four nonvanishing squares; this problem is not addressed in the current work and remains open.

1.2. Organization

The paper is organized as follows. In Section 2 we present the notation to be used throughout the paper along with the fundamental observations that are required in the proof of Theorem 2. Section 3 quantifies how large N must be compared with k to ensure that equation (1) has at least two solutions. The proof of Theorem 2 is carried out in Section 4.

2. Preliminaries

In this section we introduce the notation and terminology that will be used throughout the paper. We also develop some basic lemmas that will be used in the proof of Theorem 2.

Definition 1. We say that two solutions $\mathbf{n} = (n_j)_{j=1}^k$ and $\mathbf{m} = (m_j)_{j=1}^k$ of (1) are *equivalent* if there is a $\sigma \in S_k$, the group of permutations on k letters, such that

$$\mathbf{m} = (n_{\sigma(j)})_{j=1}^k.$$

If two solutions $\mathbf{m}$ and $\mathbf{n}$ are not equivalent, then we say that they are *distinct*.

If $\mathbf{m} = (m_j)_{j=1}^k$ is a solution of (1), we denote the equivalence class of $\mathbf{m}$ by $[\![m]\!]$. We are interested in counting the number of solutions of (1) up to equivalence for various values of N and k. With this in mind we define

$$p_k(N) = \#\left\{ [\![(m_j)_{j=1}^k]\!] \mid N = \sum_{j=1}^k m_j^2 \right\}.$$

Observe that since we only expect uniqueness up to the order of the summands, we may assume that the entries in a solution vector are arranged in increasing order. That is, if $\mathbf{m} = (m_j)_{j=1}^k$ is a solution of (1), we may assume that $m_1 \le m_2 \le \cdots \le m_k$. We will frequently make this assumption in what follows.

The key to our results is the following simple lemma.

Lemma 1. *In a representation $N = \sum_{j=1}^k m_j^2$ where $m_j \ge 1$ for each $1 \le j \le k$, there are at least $\lambda_1(N,k)$ entries in $\mathbf{m} = (m_j)_{j=1}^k$ with $m_j = 1$, where*

$$\lambda_1(N,k) = \max\left\{ \frac{4k-N}{3}, 0 \right\}.$$

Proof. Let λ be the number of entries for which $m_j = 1$ in a representation $N = \sum_{j=1}^k m_j^2$. We must have

$$N = \lambda + \sum_{j=\lambda+1}^k m_j^2 \ge \lambda + 4(k-\lambda).$$

Solving for λ in this inequality reveals that $\lambda \ge \frac{1}{3}(4k-N)$, as desired. $\square$

The next three lemmas are concerned with a complete description of the quantities $p_k(N)$ where $N = 4k, 4k+1, 4k+2$. These results, though simple to check, form a crucial component of our argument for Theorem 2.

Lemma 2.

(i) If $k \geq 5$, then $p_k(4k) \geq 2$.

(ii) If $k \geq 10$, then $p_k(4k+1) \geq 2$.

(iii) If $k \geq 7$, then $p_k(4k+2) \geq 2$.

Proof. In each case we illustrate the conclusion by providing two distinct solutions to the relevant Diophantine equation. We find that $\mathbf{m}_1 = (m_{1,j})_{j=1}^k$ with $m_{1,j} = 2$ for each $1 \leq j \leq k$ and $\mathbf{m}_2 = (m_{2,j})_{j=1}^k$ where

$$m_{2,j} = \begin{cases} 1 & 1 \leq j \leq 4 \\ 2 & 5 \leq j \leq k-1 \\ 4 & j = k \end{cases}$$

satisfy $|\mathbf{m}_1|^2 = |\mathbf{m}_2|^2 = 4k$, thus establishing (i). The proof of (ii) is similar. We note that if

$$m_{1,j} = \begin{cases} 1 & 1 \leq j \leq 3 \\ 2 & 4 \leq j \leq k-2 \\ 3 & j = k-1, k \end{cases} \qquad \text{and} \qquad m_{2,j} = \begin{cases} 1 & 1 \leq j \leq 7 \\ 2 & 8 \leq j \leq k-3 \\ 3 & j = k-2, k-1 \\ 4 & j = k \end{cases}$$

then $\mathbf{m}_1 = (m_{1,j})_{j=1}^k$ and $\mathbf{m}_2 = (m_{2,j})_{j=1}^k$ satisfy $|\mathbf{m}_1|^2 = |\mathbf{m}_2|^2 = 4k+1$. Finally, to establish (iii), we see that if

$$m_{1,j} = \begin{cases} 1 & j = 1 \\ 2 & 2 \leq j \leq k-1 \\ 3 & j = k \end{cases} \qquad \text{and} \qquad m_{2,j} = \begin{cases} 1 & 1 \leq j \leq 5 \\ 2 & 6 \leq j \leq k-2 \\ 3 & j = k-1 \\ 4 & j = k \end{cases}$$

then $\mathbf{m}_1 = (m_{1,j})_{j=1}^k$ and $\mathbf{m}_2 = (m_{2,j})_{j=1}^k$ satisfy $|\mathbf{m}_1|^2 = |\mathbf{m}_2|^2 = 4k+2$. □

In the same spirit as Lemma 2 we have the following positive result.

Lemma 3.

(i) If $1 \leq k \leq 4$, then $p_k(4k) = 1$.

(ii) If $5 \leq k \leq 9$, then $p_k(4k+1) = 1$.

(iii) If $2 \leq k \leq 6$, then $p_k(4k+2) = 1$.

Proof. These statements can be checked by hand. □

To complete the analysis of the cases considered in Lemmas 2 and 3, we have the following two negative results, the proofs of which are simple.

Lemma 4. *The following hold:*

(i) There are no values of k for which $p_k(4k) = 0$.

(ii) If $1 \le k \le 4$, then $p_k(4k+1) = 0$.

(iii) If $k = 1$, then $p_k(4k+2) = p_1(6) = 0$.

Lemma 5. *If $k \ge 2$, then $p_k(k+1) = p_k(k+2) = 0$.*

3. Large N

Intuitively, if N is sufficiently large compared to k, we expect that (1) will have many solutions. The aim of this section is to quantify how large N must be in order to guarantee $p_k(N) \ge 2$. We begin with a definition.

Definition 2. We say that a nonempty set $\mathbf{P} \subset \mathbf{N}$ is the *k-perforation* for (1) if it satisfies the following conditions:

(A) If $N \ge k$ and $N \ne k + b$ for each $b \in \mathbf{P}$, then there is a nonvanishing solution $\mathbf{m} = (m_1, \ldots, m_k)$ of (1),

(B) If $\widetilde{\mathbf{P}} \subset \mathbf{N}$ is nonempty and satisfies condition (A) above, then $\mathbf{P} \subseteq \widetilde{\mathbf{P}}$.

Condition (A) of the preceding definition guarantees the existence of a solution of (1) while condition (B) guarantees that the k-perforation of (1) is minimal. One easily verifies that the k-perforation exists and is unique whenever $k \ge 1$. For instance, the k-perforation is $\mathbf{P} = \{1, 2, 4, 5, 7, 10, 13\}$ whenever $k \ge 6$. We note that the k-perforation is finite provided $k \ge 5$.

Proposition 1. *Let $k \ge 6$. Let $\mathbf{P}$ be the $(k-1)$-perforation and set* $\max\ \mathbf{P} = p$. *If $N \ge 2(k+p) + 8\sqrt{k+p+7} + 22$, then $p_k(N) \ge 2$.*

Proof. Our strategy is to produce two essentially distinct representations of N having the form

$$\mathbf{m}_\alpha = (m_{\alpha,1}, \ldots, m_{\alpha,k-1}, \alpha) \quad \text{and} \quad \mathbf{m}_{\alpha+1} = (m_{\alpha+1,1}, \ldots, m_{\alpha+1,k-1}, \alpha+1),$$

for some positive integer α. For now we postpone a proof for the existence of such an α and the corresponding solutions $\mathbf{m}_\alpha$ and $\mathbf{m}_{\alpha+1}$. That $\mathbf{m}_\alpha$ and $\mathbf{m}_{\alpha+1}$ are representations of N as a sum of k nonvanishing squares means

$$N - \alpha^2 = \sum_{j=1}^{k-1} m_{\alpha,j}^2 \quad \text{and} \quad N - (\alpha+1)^2 = \sum_{j=1}^{k-1} m_{\alpha+1,j}^2.$$

Note that $(\alpha+1)$ is bounded above by $\sqrt{N-k-p+1}$. To see this, we observe that

$$N-(\alpha+1)^2=\sum_{j=1}^{k-1} m_j^2 \geq k-1,$$

which has a solution provided $(\alpha+1)^2 \leq N-k-p+1$. *Existence of* $\mathbf{m}_\alpha$ *and* $\mathbf{m}_{\alpha+1}$. In order to establish that $\mathbf{m}_\alpha$ and $\mathbf{m}_{\alpha+1}$ are essentially distinct, it will be helpful to require that $\alpha \geq \sqrt{N/2}$. Together with the upper bound on $(\alpha+1)$ above, we wish to find integers $\alpha, \alpha+1$ such that

$$\sqrt{\frac{N}{2}} \leq \alpha < \alpha+1 \leq \sqrt{N-k-p+1}. \tag{3}$$

To accommodate this restriction we must have

$$\sqrt{N-k-p+1}-\sqrt{\frac{N}{2}} > 2. \tag{4}$$

We find that this inequality holds for $N > 2(k+p)+8\sqrt{k+p+7}+22$. Next we note that in order to insure the existence of solutions we also require that $N-\alpha^2$ and $N-(\alpha+1)^2$ can be represented as a sum of $k-1$ nonvanishing squares. This is possible since $N-(\alpha+1)^2 > k+p-1$. Together with (3), (4), we see that $\alpha+1 \leq \sqrt{N/2}+2$. It follows that we require

$$N-\left(\sqrt{\frac{N}{2}}+2\right)^2 > k+p-1. \tag{5}$$

We find that the inequality (5) is satisfied provided

$$N > 2(k+p)+8\sqrt{k+p+7}+22.$$

Essential Distinctness. To see that $[\![\widetilde{\mathbf{m}}_\alpha]\!] \neq [\![\widetilde{\mathbf{m}}_{\alpha+1}]\!]$, we proceed by contradiction. Suppose that $[\![\widetilde{\mathbf{m}}_\alpha]\!] = [\![\widetilde{\mathbf{m}}_{\alpha+1}]\!]$. This means that $\widetilde{\mathbf{m}}_\alpha$ and $\widetilde{\mathbf{m}}_{\alpha+1}$ are rearrangements of one another. In particular, it must be that α occurs as an entry in $\widetilde{\mathbf{m}}_{\alpha+1}$. Without loss of generality we may assume that $m_{\alpha+1,k-1}=\alpha$, so that $\widetilde{\mathbf{m}}_{\alpha+1} = (m_{\alpha+1,1}, \ldots, m_{\alpha+1,k-2}, \alpha, \alpha+1)$. Recalling that $\alpha+1 > \alpha \geq \sqrt{N/2}$, we note that

$$N=(\alpha+1)^2+\alpha^2+\sum_{j=1}^{k-2} m_{p+1,j}^2 > \frac{N}{2}+\frac{N}{2}+k-2 > N,$$

a contradiction. We conclude that the equivalence classes $[\![\widetilde{\mathbf{m}}_\alpha]\!]$ and $[\![\widetilde{\mathbf{m}}_{\alpha+1}]\!]$ are distinct, which completes the proof of the theorem. $\square$

The following corollary now follows immediately from Proposition 1 and Theorem 1.

Corollary 1. *If $k = 6$ and $N > 142$, then $p_6(N) \geq 2$. If $k \geq 7$ and $N > 2k + 48 + 8\sqrt{k+20}$, then $p_k(N) \geq 2$.*

In the next proposition we treat the case $k = 5$. We handle this case separately as the existence result for $k = 4$ is more complicated; see Theorem 1 above.

Proposition 2. *In the case when $k = 5$ we find that $p_5(N) \geq 2$ whenever $N > 229$.*

Proof. We proceed in much the same way as we did in the proof of Proposition 1 above, although we note that the 4-perforation is infinite. To make use of the existence result in Theorem 1 we see that we must avoid multiples of 4. To do so we consider the parity of N: if N is odd then we will choose N large enough to guarantee that there are consecutive even integers $\alpha, \alpha + 2$ such that $N - \alpha^2, N - (\alpha + 2)^2$ both admit representations as a sum of 4 nonvanishing squares. Similarly, if N is even, we choose N large enough so that there are consecutive odd integers $\alpha, \alpha + 2$ with the analogous property. Actually, the parity considerations are only required to inform the inequalities that must be satisfied by N as in the proof of Proposition 1. Indeed we find that we require that $\alpha + 2 \leq \sqrt{N/2} + 3$, leaving us with the inequality

$$N - \left(\sqrt{\frac{N}{2}} + 3\right)^2 > 41. \tag{6}$$

The inequality (6) is the analog of (5) in the proof of Proposition 1. Solving for N, we find that we require $N > 229$. □

4. Proof of Theorem 2

The proof of our main result is developed using the results of Section 2. We note that, of course, if $N = k$, then there is a unique solution $\mathbf{m} = (m_j)_{j=1}^k$ with $m_j = 1$ for each $j = 1, \ldots, k$.

Proposition 3. *Suppose that $k \geq 5$, $N < 4k$ and $k \equiv N (\text{mod } 3)$. Then $p_k(N) = 1$ provided $N \in \{k, k+3, k+6, k+9, k+12\}$. Moreover, $p_k(N) \geq 2$ if $N \geq k + 15$.*

Proof. That $p_k(k) = 1$ is obvious. We consider $N > k$. Note that since $k \equiv N (\text{mod } 3)$, we have $4k - N \equiv 0 (\text{mod } 3)$. It follows that $\lambda_1(N, k) > 0$ is an integer. If $\mathbf{m} = (m_j)_{j=1}^k$ solves (1), then

$$N - \lambda_1(N, k) = \sum_{j=\lambda_1+1}^{k} m_j^2. \tag{7}$$

Now we observe that $N - \lambda_1(N, k) = 4(N - k)/3$ while there are $k - \lambda_1 = (N - k)/3$ terms remaining in the sum. It follows that (7) is a sum of the form addressed in part (i) of Lemmas 2, 3, and 4. In light of the aforementioned lemmas, we find that there is a unique solution to (7) if $1 \leq (N-k)/3 \leq 4$. It follows that we obtain a unique solution of (1) if $N \in \{k, k+3, k+6, k+9, k+12\}$, where we have included the case $N = k$ in light of our previous remarks. The bound $k \geq 5$ is required to ensure that $k+12 < 4k$. Finally, we point out that if $(N-k)/3 \geq 5$, then $p_k(N) \geq 2$ by Lemma 2, which means that $p_k(N) \geq 2$ whenever $N \geq k+15$. □

Proposition 4. *Suppose that $k \geq 7$, $N < 4k$ and $k \equiv N + 1 (\text{mod } 3)$. Then $p_k(N) = 1$ provided $N \in \{k+8, k+11, k+14, k+17, k+20\}$. Moreover, $p_k(N) = 0$ if $N = k+2$ or $N = k+5$, and $p_k(N) \geq 2$ if $N \geq k+23$.*

Proof. First note that by Lemma 5 we have that $p_k(k+2) = 0$. We proceed as in the proof of Proposition 3. Notice that $4k - N \equiv 1 (\text{mod } 3)$. It follows that

$$\lceil \lambda_1 \rceil = \frac{1}{3}(4k - N) + \frac{2}{3}.$$

If $\mathbf{m} = (m_j)_{j=1}^k$ is a solution of (1), then

$$N - \lceil \lambda_1 \rceil = \sum_{j=\lceil \lambda_1 \rceil + 1}^{k} m_j^2. \tag{8}$$

Observe that $N - \lceil \lambda_1 \rceil = [4(N-k) - 2]/3$ while $k - \lceil \lambda_1 \rceil = (N-k-2)/3$. Therefore, equation (8) is of the form considered in part (iii) of Lemmas 2, 3, and 4. We obtain the desired uniqueness result if $2 \leq (N-k-2)/3 \leq 6$, which means that $N \in \{k+8, k+11, k+14, k+17, k+20\}$. The bound $k \geq 7$ is required to ensure that $k + 20 < 4k$. We note that $p_k(N) = 0$ provided $(N-k-2)/3 = 1$, i.e. in the case when $N = k+5$. Finally, by Lemma 2 we find that $p_k(N) \geq 2$ provided $(N-k-2)/3 \geq 7$, which simplifies to $N \geq k+23$. □

Proposition 5. *Suppose that $k \geq 10$, $N < 4k$ and $k \equiv N + 2 (\text{mod } 3)$. Then $p_k(N) = 1$ provided $N \in \{k+16, k+19, k+22, k+25, k+28\}$. Moreover, $p_k(N) = 0$ if $N \in \{k+1, k+4, k+7, k+10, k+13\}$, and $p_k(N) \geq 2$ whenever $N \geq k+31$.*

Proof. Observe that by Lemma 5 we have that $p_k(k+1) = 0$. Here we note that $4k - N \equiv 2 (\text{mod } 3)$, meaning that

$$\lceil \lambda_1 \rceil = \frac{1}{3}(4k - N) + \frac{1}{3}.$$

As above, if $\mathbf{m} = (m_j)_{j=1}^k$ solves (1), then we can rewrite (1) as in (8). Noticing that $N - \lceil \lambda_1 \rceil = [4(N-k) - 1]/3$ and there are $k - \lceil \lambda_1 \rceil = (N-k-1)/3$ terms remaining in the sum. This means that we can apply part (ii) of Lemmas 2, 3, and 4 to obtain uniqueness provided $5 \leq (N-k-1)/3 \leq 9$. That is, we find that $p_k(N) = 1$ for $N \in \{k+16, k+19, k+22, k+25, k+28\}$. We find that $p_k(N) = 0$

if $1 \leq (N-k-1)/3 \leq 4$, meaning that $N \in \{k+4, k+7, k+10, k+13\}$. From Lemma 2 it follows that $p_k(N) \geq 2$ if $(N-k-1)/3 \geq 10$; that is, $p_k(N) \geq 2$ if $N \geq k+31$. □

Following Propositions 3, 4, and 5, we see that we are left to consider pairs (N,k) in which $k \geq 10$ and $N \geq 4k$, or $5 \leq k \leq 9$. For $5 \leq k \leq 9$, Corollary 1 and Proposition 2 provide us with upper bounds on the possible values of N to consider.

If $k \geq 10$ and $N \geq 4k$, Corollary 1 can again be used to see that we need only consider $N \leq 2k + 48 + 8\sqrt{k+20}$. That is, we need only consider N such that

$$4k \leq N < 2k + 48 + 8\sqrt{k+20},$$

for if $N \geq 2k + 48 + 8\sqrt{k+20}$, then $p_k(N) \geq 2$. Notice that if $k \geq 60$, then $4k > 2k + 48 + 8\sqrt{k+20}$, meaning that Propositions 3, 4, and 5 are sufficient to characterize N so that the solutions to (1) are unique. We are therefore left to consider pairs (N,k) with $10 \leq k \leq 59$ and $4k \leq N < 2k + 48 + 8\sqrt{k+20}$.

In checking the remaining cases we find the pairs described below in Table 1. The calculations summarized in this table were carried out using Matlab. In particular, there were no pairs (N,k) that occurred for $10 \leq k \leq 59$. The following table summarizes the relevant pairs (N,k) for $5 \leq k \leq 9$.

k	N
5	$5, 8, 11, 13, 14, 16, 17, 19, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 34, 36, 39, 42, 57, 60$
6	$6, 9, 12, 14, 15, 17, 18, 20, 22, 23, 25, 26, 27, 28, 31, 32, 34, 35, 37, 40, 43$
7	$7, 10, 13, 15, 16, 18, 19, 21, 23, 24, 26, 27, 29, 32, 35, 36, 41, 44$
8	$8, 11, 14, 16, 17, 19, 20, 22, 24, 25, 27, 28, 30, 33, 36, 45$
9	$9, 12, 15, 17, 18, 20, 21, 23, 25, 26, 28, 29, 31, 34, 37$

Table 1: Pairs (N,k) with $p_k(N) = 1$ and $5 \leq k \leq 59$

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