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ON 5-REGULAR BIPARTITIONS INTO DISTINCT PARTS

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Abstract

Let $B_5(n)$ denote the number of 5-regular bipartitions of a positive integer n into distinct parts. In this paper, we establish several infinite families of congruences modulo powers of 2 for $B_5(n)$. For example,

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}n+\frac{11\cdot 2^{2\alpha+2}\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 4f_1f_{10} \pmod{2^3},$$

for all $n \geq 0$ and $\alpha, \beta \geq 0$.

1. Introduction

Throughout this paper, we let $|q| < 1$. We use the standard notation

$$f_k := (q^k;q^k)_\infty.$$

Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{3}$$

which are special cases of Ramanujan's general theta function [1]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{4}$$

In Ramanujan's notation, Jacobi's famous triple product identity becomes

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{5}$$

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An *ℓ-regular partition* is a partition in which none of its parts are divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n with $b_\ell(0) = 1$. The generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{f_\ell}{f_1}.$$

Recently, arithmetic properties of ℓ-regular partition functions have been studied by a number of mathematicians. Calkin et al. [2] established many congruences for 5-regular partitions modulo 2 and 13-regular partitions modulo 2 and 3 by using the theory of modular forms. For more details, one can see [3, 5].

Mahadeva Naika and Hemanthkumar [6] obtained many infinite families of congruences for 5-regular bipartitions. In [9], the authors established some infinite families of congruences for (ℓ, m)-regular partitions with distinct parts. For more details, one can see [7].

Let $B_5(n)$ denote the number of 5-regular bipartitions of n into distinct parts; its generating function is given by

$$\sum_{n=0}^{\infty} B_5(n) q^n = \frac{(-q;q)_\infty^2}{(-q^5;q^5)_\infty^2} = \frac{f_2^2 f_5^2}{f_1^2 f_{10}^2}. \tag{6}$$

2. Preliminary Results

In this section, we collect some identities which are useful in proving our results.

Lemma 1. [3, Theorem 2.2]. For any prime $p \geq 5$,

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq (\pm p-1)/6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}. \tag{7}$$

Furthermore, for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p-1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

Lemma 2. The following 2-dissections hold:

$$\frac{f_1}{f_5}=\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2} \tag{8}$$

and

$$\frac{f_5}{f_1}=\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}. \tag{9}$$

Equation (8) was proved by Hirschhorn and Sellers [5]; see also [10]. Replacing q by $-q$ in (8) and using the fact that $(-q;-q)_\infty=\frac{f_2^3}{f_1f_4}$, we obtain (9).

Lemma 3. [8]. We have

$$f_1f_5^3=f_2^3f_{10}-q\frac{f_2^2f_{10}^2f_{20}}{f_4}+2q^2f_4f_{20}^3-2q^3\frac{f_4^4f_{10}f_{40}^2}{f_2f_8^2}, \tag{10}$$

$$f_1^3f_5=\frac{f_2^2f_4f_{10}^2}{f_{20}}+2qf_4^3f_{20}-5qf_2f_{10}^3+2q^2\frac{f_4^6f_{10}f_{40}^2}{f_2f_8^2f_{20}^2}. \tag{11}$$

We shall prove the following theorems:

Theorem 1. *For all $n\geq 0$ and $\alpha,\beta\geq 0$, we have*

$$\sum_{n=0}^{\infty}B_5\left(2^{2\alpha+1}\cdot 5^{2\beta}n+\frac{2^{2\alpha+1}\cdot 5^{2\beta}+1}{3}\right)q^n\equiv 2f_1^3f_5 \pmod{2^2}, \tag{12}$$

$$\sum_{n=0}^{\infty}B_5\left(2^{2\alpha+2}\cdot 5^{2\beta}n+\frac{2^{2\alpha+3}\cdot 5^{2\beta}+1}{3}\right)q^n\equiv 2f_1f_5^3 \pmod{2^2}, \tag{13}$$

$$\sum_{n=0}^{\infty}B_5\left(2^{2\alpha+1}\cdot 5^{2\beta+1}n+\frac{2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)q^n\equiv 2f_1f_5^3 \pmod{2^2}, \tag{14}$$

$$\sum_{n=0}^{\infty}B_5\left(2^{2\alpha+2}\cdot 5^{2\beta+1}n+\frac{2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)q^n\equiv 2f_1^3f_5 \pmod{2^2}. \tag{15}$$

Corollary 1. *Let $a\in\{7,13\}$ and $b\in\{11,14\}$. Then for all $n\geq 0$ and $\alpha,\beta\geq 0$, we have*

$$B_5\left(2^{2\alpha+1}\cdot 5^{2\beta+1}n+\frac{a\cdot 2^{2\alpha+1}\cdot 5^{2\beta}+1}{3}\right)\equiv 0 \pmod{2^2}, \tag{16}$$

$$B_5\left(2^{2\alpha+2}\cdot 5^{2\beta+1}n+\frac{b\cdot 2^{2\alpha+2}\cdot 5^{2\beta}+1}{3}\right)\equiv 0 \pmod{2^2}. \tag{17}$$

Theorem 2. *Let $c\in\{7,13,19\}$ and $d\in\{29,53,77,101\}$. Then for all non-negative integers α, β and n, we have*

$$B_5\left(2^{2\alpha+4}\cdot 5^{2\beta}n+\frac{c\cdot 2^{2\alpha+1}\cdot 5^{2\beta}+1}{3}\right)\equiv 0 \pmod{2^2}, \tag{18}$$

$$B_5\left(2^{2\alpha+4}\cdot 5^{2\beta}n+\frac{2^{2\alpha+1}\cdot 5^{2\beta}+1}{3}\right) \equiv \begin{cases} 2 \pmod{2^2} & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z}, \\ 0 \pmod{2^2} & \textit{otherwise,} \end{cases} \tag{19}$$

$$B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}n+\frac{c\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)\equiv 0 \pmod{2^2}, \tag{20}$$

$$B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+1}n+\frac{d\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)\equiv 0 \pmod{2^2}, \tag{21}$$

$$B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+1}n+\frac{2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right) \equiv \begin{cases} 2 \pmod{2^2} & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z}, \\ 0 \pmod{2^2} & \textit{otherwise.} \end{cases} \tag{22}$$

Theorem 3. *For all $n\geq 0$, $\alpha\geq 0$ and $i=1,2,3$, we have*

$$B_5\left(2^{2\alpha+2}n+\frac{2^{2\alpha+3}+1}{3}\right)\equiv B_5(4n+3) \pmod{2^3}, \tag{23}$$

$$B_5\left(2^{2\alpha+6}n+\frac{2^{2\alpha+3}(6i+1)+1}{3}\right)\equiv B_5(16n+4i+1) \pmod{2^3}, \tag{24}$$

$$B_5\left(2^{2\alpha+6}n+\frac{2^{2\alpha+3}+1}{3}\right) \equiv \begin{cases} B_5(16n+1)+4 \pmod{2^3} & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z}, \\ B_5(16n+1) \pmod{2^3} & \textit{otherwise.} \end{cases} \tag{25}$$

Theorem 4. *Let $g\in\{83,107\}$ and $h\in\{31,79\}$. Then for all $n\geq 0$ and $\alpha,\beta\geq 0$, we have*

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}n+\frac{11\cdot 2^{2\alpha+2}\cdot 5^{2\beta}+1}{3}\right)q^n\equiv 4f_1f_{10} \pmod{2^3}, \tag{26}$$

$$B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+1}n+\frac{g\cdot 2^{2\alpha+2}\cdot 5^{2\beta}+1}{3}\right)\equiv 0 \pmod{2^3}, \tag{27}$$

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+1}n+\frac{7\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)q^n\equiv 4f_2f_5 \pmod{2^3}, \tag{28}$$

$$B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+2}n+\frac{h\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)\equiv 0 \pmod{2^3}. \tag{29}$$

Theorem 5. *For a prime $p > 5$ and for all $n, \alpha, \beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}\cdot p^{2\gamma}n + \frac{11\cdot 2^{2\alpha+2}\cdot 5^{2\beta}\cdot p^{2\gamma}+1}{3}\right)q^n \equiv 4f_1f_{10} \pmod{2^3}, \tag{30}$$

$$B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}\cdot p^{2\gamma+1}(pn+i) + \frac{11\cdot 2^{2\alpha+2}\cdot 5^{2\beta}\cdot p^{2\gamma+2}+1}{3}\right) \equiv 0 \pmod{2^3}, \tag{31}$$

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+1}\cdot p^{2\gamma}n + \frac{7\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}\cdot p^{2\gamma}+1}{3}\right)q^n \equiv 4f_2f_5 \pmod{2^3}, \tag{32}$$

$$B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+1}\cdot p^{2\gamma+1}(pn+i) + \frac{11\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}\cdot p^{2\gamma+2}+1}{3}\right) \equiv 0 \pmod{2^3}, \tag{33}$$

where $i = 1, 2, 3, \cdots, p-1$.

3. Proof of Theorem 1

From (6), we have

$$\sum_{n=0}^{\infty} B_5(n)q^n = \frac{f_2^2}{f_{10}^2} \times \frac{f_5^2}{f_1^2}. \tag{34}$$

Using (9) in (34) and then extracting the terms involving q^{2n+1} from both sides, we arrive at

$$\sum_{n=0}^{\infty} B_5(2n+1)q^n = 2\frac{f_2^3 f_{10}}{f_1^3 f_5}. \tag{35}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m} \equiv f_{2k}^m \pmod{2}, \tag{36}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{2^2}. \tag{37}$$

Using (36) in (35), we find that

$$\sum_{n=0}^{\infty} B_5(2n+1)q^n \equiv 2f_1^3 f_5 \pmod{2^2}, \tag{38}$$

which is the $\alpha = \beta = 0$ case of (12). Suppose that Congruence (12) holds for some integer $\beta > 0$ and $\alpha = 0$.
Ramanujan recorded the following identity in his notebooks without proof:

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2R(q^5)), \tag{39}$$

where $R(q) = \dfrac{f(-q,-q^4)}{f(-q^2,-q^3)}$.
For a proof of (39), one can see [4, 11].
Employing (39) in (12) with $\alpha = 0$, we find that

$$\sum_{n=0}^{\infty} B_5\left(2\cdot 5^{2\beta}n + \frac{2\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 2f_5f_{25}^3(R(q^5)^{-1} - q - q^2R(q^5))^3 \pmod{2^2}. \tag{40}$$

Extracting the coefficients of q^{5n+3} from (40), we find that

$$\sum_{n=0}^{\infty} B_5\left(2\cdot 5^{2\beta+1}n + \frac{4\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 2f_1f_5^3 \pmod{2^2}. \tag{41}$$

Again, using (39) in (41), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} B_5\left(2\cdot 5^{2\beta+1}n + \frac{4\cdot 5^{2\beta+1}+1}{3}\right)q^n\\ &\equiv 2f_5^3f_{25}(R(q^5)^{-1} - q - q^2R(q^5)) \pmod{2^2}.\end{aligned} \tag{42}$$

Extracting the coefficients of q^{5n+1} from (42), we get

$$\sum_{n=0}^{\infty} B_5\left(2\cdot 5^{2\beta+2}n + \frac{4\cdot 5^{2\beta+2}+1}{3}\right)q^n \equiv 2f_1f_5^3 \pmod{2^2}, \tag{43}$$

which implies that (12) is true for $\beta+1$. Hence, by induction, Congruence (12) is true for any non-negative integer β and $\alpha = 0$. Suppose that Congruence (12) holds for some integers $\alpha, \beta > 0$. Employing (11) in (12), we find that

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+1}\cdot 5^{2\beta}n + \frac{2^{2\alpha+1}\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 2f_8 + 2qf_2f_{10}^3 \pmod{2^2}, \tag{44}$$

which implies

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+2}\cdot 5^{2\beta}n + \frac{2^{2\alpha+3}\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 2f_1f_5^3 \pmod{2^2}, \tag{45}$$

which proves (13).

Again, employing (10) in (45), we obtain

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+2}\cdot 5^{2\beta}n + \frac{2^{2\alpha+3}\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 2f_2^3f_{10} + 2qf_{40} \pmod{2^2}. \tag{46}$$

Extracting the coefficients of q^{2n} from (46), we get

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+3}\cdot 5^{2\beta}n + \frac{2^{2\alpha+3}\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 2f_1^3f_5 \pmod{2^2}, \tag{47}$$

which shows that (12) is true for $\alpha+1$. Hence, by induction, Congruence (12) is true for any non-negative integers α and β. This completes the proof.

Employing (39) in Equations (12) and (13), we obtain (14) and (15) respectively.

4. Proof of Corollary 1

Using Equations (12) and (13) along with Equation (39), we obtain (16) and (17) respectively.

5. Proof of Theorem 2

Extracting the coefficients of q^{2n} from (44), we get

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+2}\cdot 5^{2\beta}n+\frac{2^{2\alpha+1}\cdot 5^{2\beta}+1}{3}\right)q^n \equiv 2f_4 \pmod{2^2}, \tag{48}$$

which implies (18).

Extracting the coefficients of q^{4n} from (48), we obtain (19).

Extracting the coefficients of q^{2n+1} from (46), we get

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+3}\cdot 5^{2\beta}n+\frac{2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 2f_{20} \pmod{2^2}. \tag{49}$$

Collecting the terms involving the powers of q^{4n+i} for $i=1,2,3$ from (49), we obtain (20).

Extracting the coefficients q^{4n} from (49), we obtain

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}n+\frac{2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)q^n \equiv 2f_5 \pmod{2^2}, \tag{50}$$

which implies (21) and (22).

6. Proof of Theorem 3

Using (37), Equation (35) reduces to

$$\sum_{n=0}^{\infty} B_5(2n+1)q^n \equiv 2\frac{f_2}{f_{10}}\times f_1f_5^3 \pmod{2^3}. \tag{51}$$

Using (10) in (51) and then collecting the even and odd powers of q from the resultant equation, we obtain

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n \equiv 2f_2^2 + 4qf_2f_{10} \times f_1f_5^3 \pmod{2^3} \tag{52}$$

and

$$\sum_{n=0}^{\infty} B_5(4n+3)q^n \equiv -2\frac{f_{10}}{f_2} \times f_1^3f_5 + 4qf_{20}^2 \pmod{2^3}. \tag{53}$$

Employing (11) in (53), we obtain

$$\sum_{n=0}^{\infty} B_5(8n+3)q^n \equiv -2\frac{f_2}{f_{10}} \times f_1f_5^3 + 4qf_2f_{10}^3 \pmod{2^3} \tag{54}$$

and

$$\sum_{n=0}^{\infty} B_5(8n+7)q^n \equiv 4f_2f_{10} \times f_1^3f_5 + 6f_{10}^2 \pmod{2^3}. \tag{55}$$

Using (10) in (54) and then collecting the coefficients of q^{2n+1} from the resultant equation, we obtain

$$B_5(16n+11) \equiv B_5(4n+3) \pmod{2^3}, \tag{56}$$

which is the $\alpha = 1$ case of (23). Suppose that Congruence (23) holds for some integer $\alpha > 0$. Employing (11) in (23) and then collecting the even powers of q from the resultant equation, we get

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+3}n + \frac{2^{2\alpha+3}+1}{3}\right) q^n \equiv -2\frac{f_2}{f_{10}} \times f_1f_5^3 + 4qf_2f_{10}^3 \pmod{2^3}. \tag{57}$$

Using (10) in (57), we obtain

$$B_5\left(2^{2\alpha+4}n + \frac{2^{2\alpha+5}+1}{3}\right) \equiv B_5(4n+3) \pmod{2^3}, \tag{58}$$

which implies that (23) is true for $\alpha + 1$. Hence, by induction, Congruence (23) is true for any non-negative integer $\alpha \geq 0$. This completes the proof.

Using (10) in (57) and then collecting the even powers of q from the resultant equation, we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+4}n + \frac{2^{2\alpha+3}+1}{3}\right) q^n &\equiv -2f_2^2 + 4qf_2f_{10} \times f_1f_5^3 \\ &\equiv \sum_{n=0}^{\infty} B_5(4n+1)q^n + 4f_4 \pmod{2^3}.\end{aligned} \tag{59}$$

Comparing the coefficients of q^{4n+i} for $i = 1, 2, 3$ on both sides of the above equation, we arrive at (24).

Collecting the coefficients of q^{4n} from both sides of (59), we arrive at

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+6}n + \frac{2^{2\alpha+3}+1}{3}\right)q^n \equiv \sum_{n=0}^{\infty} B_5(16n+1)q^n + 4f_1 \pmod{2^3}, \tag{60}$$

which gives (25).

7. Proof of Theorem 4

Using (11) in (55) and then collecting the odd powers of q, we get

$$\sum_{n=0}^{\infty} B_5(16n+15)q^n \equiv 4f_2f_{20} \pmod{2^3}, \tag{61}$$

which implies

$$B_5(32n+31) \equiv 0 \pmod{2^3} \tag{62}$$

and

$$\sum_{n=0}^{\infty} B_5(32n+15)q^n \equiv 4f_1f_{10} \pmod{2^3}, \tag{63}$$

which is the $\alpha = \beta = 0$ case of (26). Let us consider the case $\beta = 0$ in (26) and prove by induction on α. Suppose that Congruence (26) holds for some integer $\alpha > 0$.

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}n + \frac{11\cdot 2^{2\alpha+2}+1}{3}\right)q^n \equiv B_5(32n+15). \tag{64}$$

The remaining proof is similar to the proof of (23). So, we omit the details.

Suppose that Congruence (26) holds for some integers $\alpha, \beta > 0$. Employing (39) in (26), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}n + \frac{11\cdot 2^{2\alpha+2}\cdot 5^{2\beta}+1}{3}\right)q^n\\ &\equiv 4f_{10}f_{25}(R(q^5)^{-1} - q - q^2R(q^5)) \pmod{2^3}.\end{aligned} \tag{65}$$

Collecting the coefficients of q^{5n+3} and q^{5n+4} from (65), we get (27).

Collecting the coefficients q^{5n+1} from (65), we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+1}n + \frac{7\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}+1}{3}\right)q^n\\ &\equiv 4f_2f_5 \equiv 4f_5f_{50}(R(q^{10})^{-1} - q^2 - q^4R(q^{10})) \pmod{2^3},\end{aligned} \tag{66}$$

which implies

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta+2}n+\frac{11\cdot 2^{2\alpha+5}\cdot 5^{2\beta+2}+1}{3}\right)q^n \equiv 4f_1f_{10}, \tag{67}$$

which implies that (26) is true for $\beta+1$ with $\alpha \geq 0$. By induction, Equation (26) holds for all $\alpha, \beta \geq 0$.

Equation (66) proves (28).

Collecting the coefficients of q^{5n+1} and q^{5n+3} from (66), we obtain (29).

8. Proof of Theorem 5

We prove the identity (30) by induction, Equation (26) is the $\gamma = 0$ case of Congruence (30). Suppose that Congruence (30) holds for some integers $\gamma > 0$. For a prime $p > 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2}+10\times\frac{3m^2+m}{2}\equiv\frac{11p^2-11}{24} \pmod{p}.$$

This is equivalent to

$$(6k+1)^2+10(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-10}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Using (7) in (26), we obtain

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}\cdot p^{2\gamma+1}n+\frac{11\cdot 2^{2\alpha+2}\cdot 5^{2\beta}\cdot p^{2\gamma+2}+1}{3}\right)q^n \equiv 4f_pf_{10p} \pmod{2^3}, \tag{68}$$

which implies

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2\beta}\cdot p^{2\gamma+2}n+\frac{11\cdot 2^{2\alpha+2}\cdot 5^{2\beta}\cdot p^{2\gamma+2}+1}{3}\right)q^n \equiv 4f_1f_{10} \pmod{2^3}, \tag{69}$$

which implies that (30) is true for $\gamma+1$ with $\alpha, \beta > 0$. Hence, by induction, Congruence (30) is true for all non-negative integers $\gamma > 0$. This proves (30).

Collecting the coefficients of q^{pn+i} from both sides of (68), we obtain (31).

Since the proofs of (32) and (33) are similar to the proof of (30). So, we omit the details.

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