# ON A CONJECTURE OF VICTOR GUO 

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#### Abstract

In this paper we give a partial proof of the following conjecture of Victor Guo: If $n$ and $k$ are two positive integers with $2 \leq k \leq \frac{n}{2}$, then $$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$


## 1. Introduction

The binomial coefficient $\binom{n}{k}$ is the number of ways of picking $k$ unordered outcomes from $n$ possibilities. The value of the binomial coefficient for nonnegative $n$ and $k$ is given explicitly by

$$
\binom{n}{k}=\frac{n!}{(n-k)!k!}
$$

Since $\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$, we have $\operatorname{gcd}\left(\binom{n}{k}, n\right)>1$, where $\operatorname{gcd}(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$. If $\operatorname{gcd}\left(\binom{n}{k}, n-1\right)>1$, then, from $\operatorname{gcd}(n, n-1)=1$ it follows that $\binom{n}{k}$ has at least two different prime factors. In general $\operatorname{gcd}\left(\binom{n}{k}, n-1\right)$ is not always greater than one. For example, $\operatorname{gcd}\left(\binom{7}{3}, 6\right)=1$. We obtain a similar conclusion for $\operatorname{gcd}\left(\binom{n}{k}, n-2\right)$. The example $\operatorname{gcd}\left(\binom{14}{4}, 12\right)=1$

[^0]shows that $\left.\operatorname{gcd}\binom{n}{k}, n-2\right)$ is not always greater than one. However, it seems that one of the claims $\operatorname{gcd}\left(\binom{n}{k}, n-1\right)>1$ and $\operatorname{gcd}\left(\binom{n}{k}, n-2\right)>1$ is always true. In this direction, in [2] Guo stated the following conjecture:

If $n$ and $k$ are two positive integers with $2 \leq k \leq \frac{n}{2}$, then

$$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$

Based on a result from [1] and by applying elementary number theory, in this paper we prove that if $n-1$ is a prime power or $n-2$ is a prime power, double or triple of a prime power, then for any $2 \leq k \leq \frac{n}{2}$ we have $\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1$ (Theorems 1 and 3, respectively). Moreover if $2 \leq k<\frac{1+\sqrt{1+4(n-1)(1+\sqrt{n-1})}}{2}$, then the conjecture is true (Theorem 4). In addition, we derive two divisibility criteria which provide that $\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1$ (Propositions 1 and 2$)$.

## 2. Results

In [1] Gould and Schlesinger proved

$$
\left.\frac{n-i}{\operatorname{gcd}(n-i, k(k-1) \cdots(k-i))} \right\rvert\,\binom{ n}{k}
$$

for every $0 \leq i \leq k-1$.
The results in this paper are based on the above divisibility property setting $i=1$ and $i=2$. For that purpose, in the following lemma we present the proofs of these cases.

Lemma 1. 1. Let $2 \leq k \leq n$. Then,

$$
\left.\frac{n-1}{\operatorname{gcd}(n-1, k(k-1))} \right\rvert\,\binom{ n}{k} .
$$

2. Let $3 \leq k \leq n$. Then,

$$
\left.\frac{n-2}{\operatorname{gcd}(n-2, k(k-1)(k-2))} \right\rvert\,\binom{ n}{k} .
$$

Proof. 1. Let $\operatorname{gcd}(n-1, k(k-1))=d$. There exist integers $x$ and $y$ such that $(n-1) x+k(k-1) y=d$. It is enough to prove that $d\binom{n}{k}$ is divisible by $n-1$. We have

$$
\begin{gathered}
d\binom{n}{k}=((n-1) x+k(k-1) y)\binom{n}{k}=(n-1) x\binom{n}{k}+k(k-1) y \frac{n}{k} \frac{n-1}{k-1}\binom{n-2}{k-2} \\
=(n-1)\left(x\binom{n}{k}+n y\binom{n-2}{k-2}\right) .
\end{gathered}
$$

2. Let $\operatorname{gcd}(n-2, k(k-1)(k-2))=d$. Similarly like above, there exist integers $x$ and $y$ such that $(n-2) x+k(k-1)(k-2) y=d$. We need to show that $d\binom{n}{k}$ is divisible by $n-2$. We have

$$
\begin{gathered}
d\binom{n}{k}=((n-2) x+k(k-1)(k-2) y)\binom{n}{k} \\
=(n-2) x\binom{n}{k}+k(k-1)(k-2) y \frac{n}{k} \frac{n-1}{k-1} \frac{n-2}{k-2}\binom{n-3}{k-3} \\
=(n-2)\left(x\binom{n}{k}+n(n-1) y\binom{n-3}{k-3}\right) .
\end{gathered}
$$

Now we are ready to give the first result in this paper.
Proposition 1. Let $2 \leq k \leq \frac{n}{2}$. If $k(k-1)$ is not divisible by $n-1$ or $k(k-1)(k-2)$ is not divisible by $n-2$, then

$$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$

Proof. Let $\operatorname{gcd}(n-1, k(k-1))=d<n-1$. From Lemma 1 we have $\frac{n-1}{d} \left\lvert\,\binom{ n}{k}\right.$.
If $n$ is an even number, then $\frac{n-2}{2}$ is a positive integer. Therefore $\left.\frac{n-1}{d} \right\rvert\,(n-1)$. $\frac{n-2}{2}=\binom{n-1}{2}$. Since $\frac{n-1}{d} \left\lvert\,\binom{ n}{k}\right.$ and $\frac{n-1}{d} \left\lvert\,\binom{ n-1}{2}\right.$ we have $\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right) \geq \frac{n-1}{d}>1$.

If $n$ is an odd number, then $\operatorname{gcd}(n-1, k(k-1))=d=2 d^{\prime}$. Thus, $\left.\frac{n-1}{d} \right\rvert\, \frac{n-1}{2}$, that is, $\frac{n-1}{d} \left\lvert\,\binom{ n-1}{2}\right.$. Again we obtain $\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right) \geq \frac{n-1}{d}>1$.

Following the same reasoning as above we easily prove that if $\operatorname{gcd}(n-2, k(k-$ $1)(k-2))=d<n-2$, then $\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right) \geq \frac{n-2}{d}>1$.

From now on we assume that $k(k-1)$ is divisible by $n-1$ and $k(k-1)(k-2)$ is divisible by $n-2$. This assumption leads to the following result.

Theorem 1. If $n-1$ or $n-2$ is a prime power, then for any $k$ such that $2 \leq k \leq \frac{n}{2}$,

$$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$

Proof. Let $p$ be a prime number and let $n-1=p^{e}$ or $n-2=p^{e}$. Since $n-1 \mid k(k-1)$ and $n-2 \mid k(k-1)(k-2)$ we obtain $p^{e} \mid k(k-1)$ or $p^{e} \mid k(k-1)(k-2)$.

If $n-1=p^{e}$, then from $\operatorname{gcd}(k, k-1)=1$ we get $p^{e} \mid k$ or $p^{e} \mid k-1$. This is in contradiction to $p^{e}=n-1 \geq 2 k-1$.

If $n-2=p^{e}$, then we consider two cases. If $p \geq 3$, then from $p^{e} \mid k(k-1)(k-2)$ and $\operatorname{gcd}(k, k-1, k-2)=1, \operatorname{gcd}(k, k-2) \leq 2$ we have $p^{e} \mid k$ or $p^{e} \mid k-1$ or $p^{e} \mid k-2$. As above, these three divisibilities are not possible. If $p=2$, then $n-2=2^{e}$ and $2^{e} \mid k(k-1)(k-2)$. Since $k \leq \frac{n}{2}$ we get $k \leq 2^{e-1}+1$.

If $k$ is an odd number, then $2^{e} \mid k-1$; this divisibility is not possible since $k-1 \leq$ $2^{e-1}$.

If $k$ is an even number, then $2^{e-1} \mid k$ or $2^{e-1} \mid k-2$. If $k=2^{e-1}$, then from Lemma 1 we have

$$
\left.\frac{2^{e}+1}{\operatorname{gcd}\left(2^{e}+1,2^{e-1}\left(2^{e-1}-1\right)\right)} \right\rvert\,\binom{ 2^{e}+2}{2^{e-1}} .
$$

Since $\operatorname{gcd}\left(2^{e}+1,2^{e-1}\left(2^{e-1}-1\right)\right) \in\{1,3\}$ we get $2^{e}+1 \left\lvert\,\binom{ 2^{e}+2}{2^{e-1}}\right.$ or $\frac{2^{e}+1}{3} \left\lvert\,\binom{ 2^{e}+2}{2^{e-1}}\right.$.
On the other hand

$$
\binom{n-1}{2}=\binom{2^{e}+1}{2}=\frac{\left(2^{e}+1\right) 2^{e}}{2}=\left(2^{e}+1\right) 2^{e-1}
$$

Thus we obtain

$$
\operatorname{gcd}\left(\binom{2^{e}+2}{2^{e-1}},\binom{2^{e}+1}{2}\right) \geq \frac{2^{e}+1}{3}>1
$$

If $k \leq 2^{e-1}$, then $2^{e-1}$ does not divide $k$ and $k-2$.

Proposition 2. Let $k, n$ be integers such that $2 \leq k \leq \frac{n}{2}$ and $\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)=1$. Then there exist positive integers $a$ and $c$ such that

$$
\begin{aligned}
& k(k-1)=a(n-1) \\
& a(k-2)=c(n-2)
\end{aligned}
$$

Moreover

$$
\frac{a^{2}}{c}-1<k<\frac{a^{2}}{c}-\frac{1}{2}
$$

and

$$
n-2 \mid a(a-2)
$$

Proof. Since $k=2$ does not satisfy the conditions we assume $k \geq 3$. Moreover, let us suppose that $\operatorname{gcd}(n-1, k(k-1))=n-1$ and $\operatorname{gcd}(n-2, k(k-1)(k-2))=n-2$. Hence, there exist positive integers $a$ and $b$ such that $k(k-1)=a(n-1)$ and $k(k-1)(k-2)=b(n-2)$. It is easy to see that $n-1$ divides $b(n-2)$. Since $\operatorname{gcd}(n-1, n-2)=1$, we obtain that $b$ is a multiple of $n-1$, that is, there exists a positive integer $c$ such that $b=c(n-1)$. Thus we obtain the following two equations

$$
\begin{align*}
k(k-1) & =a(n-1)  \tag{1}\\
k(k-1)(k-2) & =c(n-1)(n-2) \tag{2}
\end{align*}
$$

Replacing (1) in (2) and using $n-1=\frac{k(k-1)}{a}$ we obtain $a(k-2)=c\left(\frac{k(k-1)}{a}-1\right)$, that is, we obtain the following quadratic equation in $k$

$$
\begin{equation*}
c k^{2}-k\left(a^{2}+c\right)+2 a^{2}-a c=0 \tag{3}
\end{equation*}
$$

Solving (3) we get

$$
\begin{equation*}
k_{1,2}=\frac{a^{2}+c \pm \sqrt{a^{4}-6 a^{2} c+4 a c^{2}+c^{2}}}{2 c} . \tag{4}
\end{equation*}
$$

Since $k$ is a natural number, the discriminant of (3) is a perfect square. There exists a positive integer $t$ such that $a^{4}-6 a^{2} c+4 a c^{2}+c^{2}=t^{2}$. We will show that

$$
\begin{equation*}
0<a^{2}-3 c<t<a^{2}-2 c \tag{5}
\end{equation*}
$$

Dividing (2) by (1) we get $\frac{c}{a}=\frac{k-2}{n-2}<\frac{k}{n} \leq \frac{1}{2}$. Hence

$$
t^{2}<\left(a^{2}-2 c\right)^{2} \Leftrightarrow 0<a c(2 a-4 c)+3 c^{2} \Leftarrow 0<a c(2 a-4 c) \Leftrightarrow 2 c<a .
$$

From $a>2 c \geq 2$ and $t^{2}=\left(a^{2}-3 c\right)^{2}+4(a-2) c^{2}$ we derive the lower bound.
By (4) we obtain that $k$ is of the form $\frac{a^{2}+c \pm t}{2 c}$. If $k=\frac{a^{2}+c-t}{2 c}$, then we have

$$
\frac{3}{2}=\frac{a^{2}+c-a^{2}+2 c}{2 c}<k<\frac{a^{2}+c-a^{2}+3 c}{2 c}=2 .
$$

Therefore

$$
\begin{equation*}
k=\frac{a^{2}+c+t}{2 c} \tag{6}
\end{equation*}
$$

and

$$
\frac{a^{2}}{c}-1=\frac{a^{2}+c+a^{2}-3 c}{2 c}<k<\frac{a^{2}+c+a^{2}-2 c}{2 c}=\frac{a^{2}}{c}-\frac{1}{2} .
$$

Substituting $c=\frac{a^{2}(k-2)}{k^{2}-k-a}$ in (6) we obtain

$$
\begin{gathered}
t=2 c k-a^{2}-c=\frac{a^{2}\left(k^{2}-4 k+a+2\right)}{k^{2}-k-a}=\frac{a^{2}\left(k^{2}-4 k+a+2\right)}{k^{2}-k-a} \frac{k-2}{k-2} \\
=c \frac{k^{2}-4 k+a+2}{k-2}=c(k-2)+\frac{c(a-2)}{k-2}
\end{gathered}
$$

Hence $k-2 \mid c(a-2)$, that is, $a(k-2) \mid c a(a-2)$. From $a(k-2)=c(n-2)$ we get

$$
n-2 \mid a(a-2)
$$

The next result follows directly from Proposition 2.
Theorem 2. Let $2 \leq k \leq \frac{n}{2}$. If $\frac{k^{2}(k-1)^{2}-2 k(k-1)(n-1)}{(n-2)(n-1)^{2}}$ is not an integer, then

$$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$

Proposition 3. Let $p \geq 3$ be a prime number such that $p^{e} \mid n-2$. Then for any $k \in\left[2, \frac{1+\sqrt{1+4 p^{e}(n-1)}}{2}\right)$ we have

$$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$

Proof. According to Proposition 2 it suffices to prove that $n-2$ does not divide $a(a-2)$, where $a=\frac{k(k-1)}{n-1}$.

Let $n-2 \mid a(a-2)$. Hence $p^{e} \mid a(a-2)$. Since $p \geq 3$ and $\operatorname{gcd}(a, a-2) \leq 2$, we have $p^{e} \mid a$ or $p^{e} \mid a-2$. We will show that this is not possible by proving $p^{e}>a=\frac{k(k-1)}{n-1}$. We have

$$
\begin{gathered}
p^{e}>\frac{k(k-1)}{n-1} \Leftrightarrow k^{2}-k-p^{e}(n-1)<0 \Leftrightarrow \\
k \in\left(\frac{1-\sqrt{1+4 p^{e}(n-1)}}{2}, \frac{1+\sqrt{1+4 p^{e}(n-1)}}{2}\right) .
\end{gathered}
$$

The case $p=2$ is considered in the following proposition.
Proposition 4. Let $2^{e} \mid n-2$. Then for any $k \in\left[2, \frac{1+\sqrt{1+2^{e+1}(n-1)}}{2}\right)$ we have

$$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$

Proof. Similarly as in the previous preposition we assume $n-2 \mid a(a-2)$. Therefore $2^{e} \mid a(a-2)$. Clearly $a$ is an even number and $\operatorname{gcd}(a, a-2)=2$. Thus $2^{e-1} \mid a$ or $2^{e-1} \mid a-2$. On the other hand, from $k<\frac{1+\sqrt{1+2^{e+1}(n-1)}}{2}$ we obtain $2^{e-1}>\frac{k(k-1)}{n-1}=$ $a$. This conclusion is in contradiction to $2^{e-1} \mid a$ or $2^{e-1} \mid a-2$.

Theorem 3. Let $p \geq 3$ be a prime number. If $n=2 p^{e}+2$ or $n=3 p^{e}+2$ and $2 \leq k \leq \frac{n}{2}$, then

$$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$

Proof. The proof relies on Proposition 3 by showing $\frac{1+\sqrt{1+4 p^{e}(n-1)}}{2}>\frac{n}{2}$. Since $p^{e} \geq \frac{n-2}{3}$ and since $\frac{1+\sqrt{1+4 p^{e}(n-1)}}{2}>\frac{n}{2}$ is equivalent to $p^{e}>\frac{n-1}{4}-\frac{1}{4(n-1)}$, it is enough to prove

$$
\begin{equation*}
\frac{n-2}{3}>\frac{n-1}{4}-\frac{1}{4(n-1)} \tag{7}
\end{equation*}
$$

From the conditions we can take $n>5$. This assumption is equivalent to $\frac{n-2}{3}>\frac{n-1}{4}$, which leads to the inequality (7).

Theorem 4. If $2 \leq k<\frac{1+\sqrt{1+4(n-1)(1+\sqrt{n-1})}}{2}$, then

$$
\operatorname{gcd}\left(\binom{n}{k},\binom{n-1}{2}\right)>1
$$

Proof. If $k^{2}(k-1)^{2}-2 k(k-1)(n-1)<(n-2)(n-1)^{2}$, then $\frac{k^{2}(k-1)^{2}-2 k(k-1)(n-1)}{(n-2)(n-1)^{2}}$ is not an integer, which leads to the positive answer of the conjecture. Setting $t=k(k-1)$ and solving the quadratic inequality $t^{2}-2 t(n-1)-(n-2)(n-$ $1)^{2}<0$ we obtain $t \in((n-1)-(n-1) \sqrt{n-1},(n-1)+(n-1) \sqrt{n-1})$. Since $t=k(k-1)>0$ we have $k(k-1)<(n-1)(1+\sqrt{n-1})$. Solving the inequality $k^{2}-k-(n-1)(1+\sqrt{n-1})<0$ we obtain $k<\frac{1+\sqrt{1+4(n-1)(1+\sqrt{n-1})}}{2}$.

## References

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