A COMBINATORIAL IDENTITY FOR THE SUM OF DIVISORS FUNCTION INVOLVING $\mathbf{p_r(n)}$

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Abstract

Let $\sum_{d|n} d$ denote the sum of divisors of a positive integer n, and let $\prod_{j=1}^{\infty}(1-q^j)^r = \sum_{n=0}^{\infty} p_r(n)q^n$. The aim of this note is to prove the following interesting combinatorial identity:

$$\sum_{d|n} d = n \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r} p_r(n).$$

1. Main Result

In the following, let $\sum_{d|n} d$ denote the sum of divisors of a positive integer n.

Definition 1 ([3])**.** The *Euler function* is the infinite product

$$E(q) := \prod_{j=1}^{\infty}(1-q^j) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}}$$

where $|q| < 1$.

Definition 2. For any positive integer r define $p_r(n)$ by

$$E(q)^r = \prod_{j=1}^{\infty}(1-q^j)^r = \sum_{n=0}^{\infty} p_r(n)q^n.$$

The function $p_r(n)$ is related to the partition function $p(n)$ (which counts the number of partitions of n), and its congruence properties have been studied like those of the partition function [1].

Our aim is to derive the following combinatorial identity.

Theorem 1. *For all positive integers n we have*

$$\sum_{d|n} d = n \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r} p_r(n). \tag{1}$$

We require following two lemmas for our proof.

Lemma 1. *For all positive integers n we have*

$$\sum_{d|n} \frac{1}{d} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^k \, (k-1)! \, B_{n,k} \left(E'(0), E''(0), \ldots, E^{(n-k+1)}(0) \right) \tag{2}$$

where $E(q)$ *is the Euler function, and* $B_{n,k} \equiv \mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ *are the partial Bell polynomials defined by [2, p. 206]*

$$\mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}.$$

Proof. It is easy to see that

$$\log(E(q)) = \sum_{j=1}^{\infty} \log(1 - q^j) = -\sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{q^{lj}}{l} = -\sum_{n=1}^{\infty} q^n \left(\sum_{d|n} \frac{1}{d} \right).$$

Let $f(q) = \log q$. Using Faà di Bruno's formula [2, p. 134] we have

$$\frac{d^n}{dq^n} f(E(q)) = \sum_{k=1}^{n} f^{(k)}(E(q)) \cdot B_{n,k} \left(E'(q), E''(q), \ldots, E^{(n-k+1)}(q) \right). \tag{3}$$

Since $f^{(k)}(q) = \frac{(-1)^{k-1}\,(k-1)!}{q^k}$ and $E(0) = 1$, letting $q \to 0$ in the above equation gives us Equation (2). □

Lemma 2. *We have, for positive integers* n, k*,*

$$B_{n,k} \left(E'(0), E''(0), \ldots, E^{(n-k+1)}(0) \right) = \frac{n!}{k!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} p_r(n). \tag{4}$$

Proof. We start with the generating function for the partial Bell polynomials as follows:

$$\begin{aligned}
\sum_{n=k}^{\infty} B_{n,k} \left(E'(0), E''(0), \ldots, E^{(n-k+1)}(0) \right) \frac{q^n}{n!} &= \frac{1}{k!} \left(\sum_{j=1}^{\infty} E^{(j)}(0) \frac{q^j}{j!} \right)^k \\
&= \frac{1}{k!} (E(q) - 1)^k \\
&= \frac{1}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} E(q)^r \\
&= \frac{1}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \sum_{n=0}^{\infty} p_r(n) q^n
\end{aligned}$$

to conclude Equation (4). □

Proof of Theorem 1. We combine Equation (2) and Equation (4) to obtain

$$\sum_{d|n} \frac{1}{d} = \sum_{k=1}^{n} \frac{1}{k} \sum_{r=1}^{k} (-1)^r \binom{k}{r} p_r(n)$$
$$= \sum_{r=1}^{n} (-1)^r p_r(n) \sum_{k=r}^{n} \frac{1}{k} \binom{k}{r}$$
$$= \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r} p_r(n).$$

Now we can deduce Equation (1) from the fact that $\sum_{d|n} \frac{n}{d} = \sum_{j|n} j$. □

Remark 1. It is interesting to note the following combinatorial identity for $p(n)$ [4],

$$p(n) = \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} p_r(n).$$

Remark 2. Let $p_{-r}(n)$ be defined by $\sum_{n=0}^{\infty} p_{-r}(n)\, q^n = \prod_{j=1}^{\infty} (1-q^j)^{-r}$. Then it can be proved that

$$\sum_{d|n} d = n \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \binom{n}{r} p_{-r}(n) \qquad [5].$$

References

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