BINOMIAL SERIES IDENTITIES INVOLVING GENERALIZED HARMONIC NUMBERS

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Received: 5/21/20, Revised: 10/10/20, Accepted: 11/4/20, Published: 12/4/20

Abstract

By means of Abel's lemma on summation by parts, we shall derive several infinite series identities involving generalized harmonic numbers and central binomial coefficients.

1. Introduction and Motivation

For central binomial coefficients, the following Maclaurin series [7] are well-known:

$$\sum_{n\geq 1}\frac{(2y)^{2n}}{n^2\binom{2n}{n}}=2(\arcsin y)^2, \tag{1}$$

$$\sum_{n\geq 1}\frac{(2y)^{2n}}{n\binom{2n}{n}}=\frac{2y\arcsin y}{\sqrt{1-y^2}}, \tag{2}$$

$$\sum_{n\geq 0}\frac{(2y)^{2n}}{\binom{2n}{n}}=\frac{1}{1-y^2}+\frac{y\arcsin y}{(1-y^2)^{3/2}}, \tag{3}$$

$$\sum_{n\geq 0}\frac{(2y)^{2n}}{(2n+1)\binom{2n}{n}}=\frac{\arcsin y}{y\sqrt{1-y^2}}, \tag{4}$$

$$\sum_{n\geq 1}\frac{(2y)^{2n}}{n(2n+1)\binom{2n}{n}}=2\Big(1-\frac{\sqrt{1-y^2}}{y}\arcsin y\Big). \tag{5}$$

In fact, starting from (1), one can derive, without difficulty, the others by differentiation and integration. When y is specified to particular values, these formulae

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have been used in [5, 7, 9] to produce numerous infinite series identities, that depend substantially on the evaluation of arcsin-function. By means of the elementary relations below

$$\arcsin y \;=\; \mathrm{i}\ln\Big(\sqrt{1-y^2}-\mathrm{i}y\Big) \quad\text{and}\quad \arcsin(\mathrm{i}y) \;=\; \mathrm{i}\ln\Big(\sqrt{1+y^2}+y\Big),$$

we construct the following short table that will be employed in this paper, together with Abel's lemma on summation by parts to evaluate a large class of infinite series.

θ	$\pi/2$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/5$	$2\pi/5$
$\sin\theta$	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$	$\frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$
θ	$\pi/8$	$3\pi/8$	$\pi/10$	$3\pi/10$	$\pi/12$	$5\pi/12$
$\sin\theta$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$

Table 1: Special values of the sine function

As a classical analytic instrument, Abel's lemma has been fundamental in convergence tests of infinite series. Recently, it has been utilized by Chu [2] to derive several infinite series identities involving the classical harmonic numbers and their variants. For subsequent applications, we reproduce this lemma as follows. With an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$\nabla\tau_k = \tau_k - \tau_{k-1} \quad\text{and}\quad \Delta\tau_k = \tau_k - \tau_{k+1}, \tag{6}$$

where Δ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign. Then **Abel's lemma** on summation by parts (see [2] for a proof) may be reformulated as follows. For a fixed integer ε and two complex sequences $\{U_k\}$ and $\{V_k\}$, we have

$$\sum_{k\geq\varepsilon} V_k\nabla U_k = [UV]_+ - U_{\varepsilon-1}V_\varepsilon + \sum_{k\geq\varepsilon} U_k\Delta V_k, \tag{7}$$

provided that one of both series is convergent and the following limit exists:

$$[UV]_+ \;=\; \lim_{n\to\infty} U_n V_{n+1}. \tag{8}$$

In a paper on binomial sums involving harmonic numbers, Genčev [6] found several remarkable identities. We reproduce his typical Example 2.1:

$$\sum_{k\geq1}\frac{(3k+5)\,H_k(\alpha)}{(2k+1)(2k+3)\binom{2k}{k}} = \begin{cases} 4-\frac{2\pi}{\sqrt{3}}, & \alpha=1;\\ 4-\frac{(1+\sqrt{2})\pi}{2}, & \alpha=1+1/\sqrt{2};\\ 4-\frac{(2+\sqrt{3})\pi}{3}, & \alpha=2+\sqrt{3}. \end{cases} \tag{9}$$

Here and forth, $H_k(\alpha)$ is the generalized harmonic number, with a real parameter α subject to $|\alpha| \geq 1$, defined by

$$H_n(\alpha) = \sum_{k=1}^{n} \frac{1}{k\alpha^k} \qquad \text{with} \qquad H_0(\alpha) = 0.$$

Motivated by Genčev's work, this paper will explore further applications of Abel's lemma on summation by parts to evaluate the following type infinite series:

$$\sum_{k\geq 1} H_k(\alpha) \frac{(4\beta)^k}{\binom{2k}{k}} R(k) \quad \text{with} \ \ |\alpha| \geq 1 \text{ and } |\beta| < 1, \tag{10}$$

which involves a rational function $R(k)$, central binomial coefficients and $H_k(\alpha)$.

Up to now, only very few similar series have been evaluated, which represent just the tip of the iceberg. For example, the formulae found by Genčev [6] correspond to the cases $\alpha = 1$ and $\beta \in \{1/2, 1/4, 3/4\}$ as well as $\beta = 1/4$ and $\alpha \in \{1, 2+\sqrt{3}, 1+1/\sqrt{2}\}$ that are covered by the following five choices $y = \sqrt{\beta/\alpha} \in \{\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{2-\sqrt{2}}}{2}\}$ out of twelve listed in our table.

Throughout the paper, we shall fix the sequence $\{U_k\}$ by setting $U_k = H_k(\alpha)$. From this, it follows easily that $\nabla U_k = 1/(k\alpha^k)$ as well as $U_0 = 0$ and $U_1 = 1/\alpha$. Moreover, for $\varepsilon = 1$ the term $U_{\varepsilon-1}V_\varepsilon$ vanishes, and, consequently, can be ignored in (7). Because $|\alpha| > 1$, the limit of U_n exists as $n \to \infty$.

The rest of the paper will be devoted to evaluating nine classes of infinite series specified by particular cases of the sequence $\{V_k\}$ and will be divided into nine sections characterized by polynomial factors appearing in the denominators. In each section, we shall prove a general theorem by means of Abel's lemma on summation by parts and then derive from it some exemplified infinite series identities by specifying the two parameters α and β with particular values.

2. Series with $\binom{2k}{k}(2k+1)$ in Denominators

In this section, we consider $\varepsilon = 1$ and

$$V_k = \frac{(4\beta)^k}{\binom{2k}{k}} \quad \text{with} \quad \Delta V_k = \frac{(4\beta)^k(1+2k-2\beta k-2\beta)}{\binom{2k}{k}(2k+1)}.$$

This particular choice in (7) leads to the following theorem.

Theorem 1. *Let α and β be two real numbers subject to $|\alpha| \geq 1$ and $|\beta| < 1$. Then for $y = \sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1} H_k(\alpha) \frac{(4\beta)^k(1+2k-2\beta k-2\beta)}{\binom{2k}{k}(2k+1)} = \frac{2y \arcsin y}{\sqrt{1-y^2}}.$$

Proof. According to the modified Abel's lemma on summation by parts (7), we can formally reformulate the following infinite series:

$$\sum_{k\geq 1}\frac{(4\beta/\alpha)^k}{k\binom{2k}{k}}=\sum_{k\geq 1}V_k\nabla U_k=[UV]_+-U_0V_1+\sum_{k\geq 1}U_k\Delta V_k. \tag{11}$$

Now, recalling the Stirling asymptotic formula (cf. [4, Pages 267 and 292])

$$n!\sim\left(\frac{n}{e}\right)^n\sqrt{2n\pi}\quad\text{as}\quad n\to\infty,$$

we can determine the limit

$$\big|[UV]_+\big|=\lim_{n\to\infty}\left|H_n(\alpha)\frac{(4\beta)^{n+1}}{\binom{2n+2}{n+1}}\right|\leq\lim_{n\to\infty}|\beta|^{n+1}\sqrt{(n+1)\pi}\ln n=0$$

since $|\beta|<1$. Then by means of D'Alembert's test, we can check easily that the series on the left of (11) is convergent and can hence be evaluated by (2) as follows:

$$\sum_{k\geq 1}\frac{(4\beta/\alpha)^k}{k\binom{2k}{k}}=\frac{2y\arcsin y}{\sqrt{1-y^2}},$$

where $y=\sqrt{\beta/\alpha}$. Therefore, the equality (11) is valid under the condition $|\alpha|\geq 1$ and $|\beta|<1$. Writing explicitly the sum on the right, we get the formula stated in the theorem. $\square$

For the presence of two free parameters in Theorem 1, we can derive more concrete infinite series identities by specifying particular values for α and β so that $\sqrt{\beta/\alpha}$ equals one of $\sin\theta$ given in the table. We limit ourselves to highlight only some representative results.

Corollary 1 ($\beta=1/4$ in Theorem 1)**.**

$$\sum_{k\geq 1}\frac{H_k(\alpha)(3k+1)}{\binom{2k}{k}(2k+1)}=\begin{cases}\frac{2\pi}{3\sqrt{3}}, & \alpha=1: \quad \text{Genčev [6]};\\ \frac{\pi}{2(1+\sqrt{2})}, & \alpha=1+1/\sqrt{2};\\ \frac{\pi}{3(2+\sqrt{3})}, & \alpha=2+\sqrt{3}.\end{cases}$$

Corollary 2 ($\beta=1/2$ in Theorem 1)**.**

$$\sum_{k\geq 1}\frac{2^kH_k(\alpha)\,k}{\binom{2k}{k}(2k+1)}=\begin{cases}\frac{\pi}{2}, & \alpha=1: \quad \text{Genčev [6]};\\ \frac{\pi}{3\sqrt{3}}, & \alpha=2;\\ \frac{\pi}{6(2+\sqrt{3})}, & \alpha=4+2\sqrt{3}.\end{cases}$$

Corollary 3 ($\beta = 3/4$ in Theorem 1)**.**

$$\sum_{k\geq 1} \frac{3^k H_k(\alpha)\,(k-1)}{\binom{2k}{k}(2k+1)} = \begin{cases} \frac{4\pi}{\sqrt{3}}, & \alpha = 1: \quad \text{Genčev [6];} \\ \frac{2\pi}{3\sqrt{3}}, & \alpha = 3; \\ \pi, & \alpha = 3/2; \\ \frac{\pi}{3(2+\sqrt{3})}, & \alpha = 6+3\sqrt{3}. \end{cases}$$

Corollary 4 ($\beta = 2/3$ in Theorem 1)**.**

$$\sum_{k\geq 1} \Big(\frac{8}{3}\Big)^k \frac{H_k(\alpha)\,(2k-1)}{\binom{2k}{k}(2k+1)} = \begin{cases} \frac{3\pi}{2}, & \alpha = 4/3; \\ \frac{\pi}{\sqrt{3}}, & \alpha = 8/3; \\ \frac{3\pi}{4}(\sqrt{2}-1), & \alpha = \frac{4}{3}(2+\sqrt{2}); \\ \frac{\pi}{2}(2-\sqrt{3}), & \alpha = \frac{8}{3}(2+\sqrt{3}); \\ \frac{3\pi}{5}\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}}}, & \alpha = \frac{4}{3}(3+\sqrt{5}); \\ \frac{9\pi}{5}\sqrt{\frac{\sqrt{5}+2}{\sqrt{5}}}, & \alpha = \frac{4}{3}(3-\sqrt{5}). \end{cases}$$

3. Series with $\binom{2k}{k}k(2k+1)$ in Denominators

From now on, the proofs of all the subsequent theorems will be omitted due to the similarity with the proof of Theorem 1. Instead, we shall highlight only the sequence $\{V_k\}$, its difference ΔV_k and the equation corresponding to (7).

Specify the sequence $\{V_k\}$ and then compute its difference by

$$V_k = \frac{(4\beta)^k}{k\binom{2k}{k}} \quad \text{and} \quad \Delta V_k = \frac{(4\beta)^k(1+2k-2\beta k)}{k\binom{2k}{k}(2k+1)}.$$

By means of (7), we can reformulate the following infinite series:

$$\sum_{k\geq 1} \frac{(4\beta/\alpha)^k}{k^2\binom{2k}{k}} = \sum_{k\geq 1} V_k \nabla U_k = [UV]_+ - U_0 V_1 + \sum_{k\geq 1} U_k \Delta V_k.$$

According to (1), we establish the following summation theorem.

Theorem 2. *Let α and β be two real numbers subject to $|\alpha| \geq 1$ and $|\beta| < 1$. Then for $y = \sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1} H_k(\alpha) \frac{(4\beta)^k(1+2k-2\beta k)}{k\binom{2k}{k}(2k+1)} = 2(\arcsin y)^2.$$

Corollary 5 ($\beta = 1/4$ in Theorem 2)**.**

$$\sum_{k\geq 1}\frac{H_k(\alpha)\,(3k+2)}{k\binom{2k}{k}(2k+1)} = \begin{cases} \frac{\pi^2}{9}, & \alpha = 1; \\ \frac{\pi^2}{16}, & \alpha = 1+1/\sqrt{2}; \\ \frac{\pi^2}{36}, & \alpha = 2+\sqrt{3}. \end{cases}$$

Corollary 6 ($\beta = 1/2$ in Theorem 2)**.**

$$\sum_{k\geq 1}\frac{2^k H_k(\alpha)(k+1)}{k\binom{2k}{k}(2k+1)} = \begin{cases} \frac{\pi^2}{8}, & \alpha = 1; \\ \frac{\pi^2}{18}, & \alpha = 2; \\ \frac{\pi^2}{32}, & \alpha = 2+\sqrt{2}. \end{cases}$$

Corollary 7 ($\beta = 3/4$ in Theorem 2)**.**

$$\sum_{k\geq 1}\frac{3^k H_k(\alpha)(k+2)}{k\binom{2k}{k}(2k+1)} = \begin{cases} \frac{4\pi^2}{9}, & \alpha = 1; \\ \frac{\pi^2}{9}, & \alpha = 3; \\ \frac{\pi^2}{4}, & \alpha = 3/2. \end{cases}$$

Corollary 8 ($\beta = 5/6$ in Theorem 2)**.**

$$\sum_{k\geq 1}\Big(\frac{10}{3}\Big)^k\frac{H_k(\alpha)\,(k+3)}{k\binom{2k}{k}(2k+1)} = \begin{cases} \frac{3\pi^2}{8}, & \alpha = 5/3; \\ \frac{\pi^2}{6}, & \alpha = 10/3; \\ \frac{2\pi^2}{3}, & \alpha = 10/9. \end{cases}$$

4. Series with $\binom{2k}{k}(2k+1)(2k+3)$ in Denominators

Define the sequence $\{V_k\}$ and then compute its difference by

$$V_k = \frac{(4\beta)^k}{\binom{2k}{k}(2k+1)} \quad \text{and} \quad \Delta V_k = \frac{(4\beta)^k(3+2k-2\beta k-2\beta)}{\binom{2k}{k}(2k+1)(2k+3)}.$$

Then the equation corresponding to (7) becomes

$$\sum_{k\geq 1}\frac{(4\beta/\alpha)^k}{k\binom{2k}{k}(2k+1)} = \sum_{k\geq 1} V_k\nabla U_k = [UV]_+ - U_0V_1 + \sum_{k\geq 1} U_k\Delta V_k.$$

Because the series on left can be evaluated by (5), we prove consequently the general summation theorem below.

Theorem 3. *Let α and β be two real numbers subject to $|\alpha| \geq 1$ and $|\beta| < 1$. Then for $y = \sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1} H_k(\alpha)\frac{(4\beta)^k(3+2k-2\beta k-2\beta)}{\binom{2k}{k}(2k+1)(2k+3)} = 2 - \frac{2\arcsin y}{y}\sqrt{1-y^2}. \tag{12}$$

Letting $\beta = 1/4$ in this theorem, we recover Genčev's formulae [6, Example 2.1], anticipated in equation (9). Other formulae are given below.

Corollary 9 ($\beta = 1/2$ in Theorem 3)**.**

$$\sum_{k\geq 1} \frac{2^k H_k(\alpha)\,(k+2)}{\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} 2-\frac{\pi}{2}, & \alpha = 1; \\ 2-\frac{\pi}{\sqrt{3}}, & \alpha = 2; \\ 2-\frac{(2+\sqrt{3})\pi}{6}, & \alpha = 4+2\sqrt{3}. \end{cases}$$

Corollary 10 ($\beta = 3/4$ in Theorem 3)**.**

$$\sum_{k\geq 1} \frac{3^k H_k(\alpha)\,(k+3)}{\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} 4-\frac{4\pi}{3\sqrt{3}}, & \alpha = 1; \\ 4-\frac{2\pi}{\sqrt{3}}, & \alpha = 3; \\ 4-\pi, & \alpha = 3/2; \\ 4-\frac{(2+\sqrt{3})\pi}{3}, & \alpha = 6+3\sqrt{3}. \end{cases}$$

Corollary 11 ($\beta = 1/3$ in Theorem 3)**.**

$$\sum_{k\geq 1} \left(\frac{4}{3}\right)^k \frac{H_k(\alpha)\,(4k+7)}{\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} 6-\pi\sqrt{3}, & \alpha = \frac{4}{3}; \\ 6-\frac{3\pi}{4}(1+\sqrt{2}), & \alpha = \frac{2}{3}(2+\sqrt{2}); \\ 6-\frac{\pi}{2}(2+\sqrt{3}), & \alpha = \frac{4}{3}(2+\sqrt{3}); \\ 6-\frac{3\pi}{5}\sqrt{5+2\sqrt{5}}, & \alpha = \frac{2}{3}(3+\sqrt{5}). \end{cases}$$

Corollary 12 ($\beta = 5/8$ in Theorem 3)**.**

$$\sum_{k\geq 1} \left(\frac{5}{2}\right)^k \frac{H_k(\alpha)\,(3k+7)}{\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} 8-2\pi, & \alpha = \frac{5}{4}; \\ 8-\frac{4\pi}{\sqrt{3}}, & \alpha = \frac{5}{2}; \\ 8-\pi(1+\sqrt{2}), & \alpha = \frac{5}{4}(2+\sqrt{2}); \\ 8-\frac{2\pi}{3}(2+\sqrt{3}), & \alpha = \frac{5}{2}(2+\sqrt{3}); \\ 8-\frac{4\pi}{5}\sqrt{5+2\sqrt{5}}, & \alpha = \frac{5}{4}(3+\sqrt{5}). \end{cases}$$

5. Series with $\binom{2k}{k}k(2k+1)(2k+3)$ in Denominators

Define the sequence $\{V_k\}$ and then compute its difference by

$$V_k = \frac{(4\beta)^k}{k\binom{2k}{k}(2k+1)} \quad \text{and} \quad \Delta V_k = \frac{(4\beta)^k(3+2k-2\beta k)}{k\binom{2k}{k}(2k+1)(2k+3)}.$$

Applying the modified Abel's lemma on summation by parts (7), we can reformulate the following infinite series:

$$\sum_{k\geq 1} \frac{(4\beta/\alpha)^k}{k^2\binom{2k}{k}(2k+1)} = \sum_{k\geq 1} V_k \nabla U_k = [UV]_+ - U_0 V_1 + \sum_{k\geq 1} U_k \Delta V_k.$$

Keeping in mind of the equality

$$\frac{1}{k^2(2k+1)} = \frac{1}{k^2} - \frac{2}{k(2k+1)},$$

we can evaluate the series on the left by (1) and (5). This results in the following general summation theorem.

Theorem 4. *Let α and β be two real numbers subject to $|\alpha| \geq 1$ and $|\beta| < 1$. Then for $y = \sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1} H_k(\alpha) \frac{(4\beta)^k(3+2k-2\beta k)}{k\binom{2k}{k}(2k+1)(2k+3)} = \frac{4\sqrt{1-y^2}}{y} \arcsin y + 2(\arcsin y)^2 - 4.$$

Corollary 13 ($\beta = 1/4$ in Theorem 4)**.**

$$\sum_{k\geq 1} \frac{H_k(\alpha)\,(k+2)}{k\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} \frac{\pi^2}{27} + \frac{4\pi}{3\sqrt{3}} - \frac{8}{3}, & \alpha = 1; \\ \frac{\pi^2}{48} + \frac{(1+\sqrt{2})\pi}{3} - \frac{8}{3}, & \alpha = 1 + 1/\sqrt{2}; \\ \frac{\pi^2}{108} + \frac{(4+2\sqrt{3})\pi}{9} - \frac{8}{3}, & \alpha = 2 + \sqrt{3}. \end{cases}$$

Corollary 14 ($\beta = 1/2$ in Theorem 4)**.**

$$\sum_{k\geq 1} \frac{2^k H_k(\alpha)(k+3)}{k\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} \frac{\pi^2}{8} + \pi - 4, & \alpha = 1; \\ \frac{\pi^2}{18} + \frac{2\pi}{\sqrt{3}} - 4, & \alpha = 2; \\ \frac{\pi^2}{32} + \frac{(1+\sqrt{2})\pi}{2} - 4, & \alpha = 2 + \sqrt{2}; \\ \frac{\pi^2}{72} + \frac{(2+\sqrt{3})\pi}{3} - 4, & \alpha = 4 + 2\sqrt{3}. \end{cases}$$

Corollary 15 ($\beta = 3/4$ in Theorem 4)**.**

$$\sum_{k\geq 1}\frac{3^k H_k(\alpha)(k+6)}{k\binom{2k}{k}(2k+1)(2k+3)}=\begin{cases}\frac{4\pi^2}{9}+\frac{8\pi}{3\sqrt{3}}-8, & \alpha=1;\\ \frac{\pi^2}{9}+\frac{4\pi}{\sqrt{3}}-8, & \alpha=3;\\ \frac{\pi^2}{4}+2\pi-8, & \alpha=3/2;\\ \frac{\pi^2}{36}+\frac{2\pi}{3}(2+\sqrt{3})-8, & \alpha=6+3\sqrt{3}.\end{cases}$$

Corollary 16 ($\beta = 1/7$ in Theorem 4)**.**

$$\sum_{k\geq 1}\Big(\frac{4}{7}\Big)^k\frac{H_k(\alpha)(12k+21)}{k\binom{2k}{k}(2k+1)(2k+3)}=\begin{cases}\frac{7\pi^2}{72}+\frac{7\pi}{3}(2+\sqrt{3})-28, & \alpha=\frac{4}{7}(2+\sqrt{3});\\ \frac{7\pi^2}{50}+\frac{14\pi}{5}\sqrt{5+2\sqrt{5}}-28, & \alpha=\frac{2}{7}(3+\sqrt{5}).\end{cases}$$

6. Series with $\binom{2k+2}{k+1}(2k+3)$ in Denominators

Define the sequence $\{V_k\}$, a shifted one of that in §2, and then compute its difference by

$$V_k=\frac{(4\beta)^k}{\binom{2k+2}{k+1}}\quad\text{and}\quad\Delta V_k=\frac{(4\beta)^k(3+2k-2\beta k-4\beta)}{\binom{2k+2}{k+1}(2k+3)}.$$

According to the modified Abel's lemma on summation by parts (7), we can reformulate the following infinite series:

$$\sum_{k\geq 1}\frac{(4\beta/\alpha)^k}{k\binom{2k+2}{k+1}}=\sum_{k\geq 1}V_k\nabla U_k=[UV]_+-U_0V_1+\sum_{k\geq 1}U_k\Delta V_k.$$

In view of the identity

$$\frac{1}{k\binom{2k+2}{k+1}}=\frac{1}{2\binom{2k}{k}}\Big(\frac{1}{k}-\frac{1}{2k+1}\Big),$$

we can evaluate the left series displayed in the penultimate equation by (2) and (4). As observed by an anonymous referee, this can also be done by means of (4) and (5) taking into account that

$$\frac{1}{k\binom{2k+2}{k+1}}=\frac{k+1}{2k(2k+1)\binom{2k}{k}}=\frac{1}{2(2k+1)\binom{2k}{k}}+\frac{1}{2k(2k+1)\binom{2k}{k}}.$$

This leads to the following general summation theorem.

Theorem 5. *Let α and β be two real numbers subject to $|\alpha|\geq 1$ and $|\beta|<1$. Then for $y=\sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1}H_k(\alpha)\frac{(4\beta)^k(3+2k-2\beta k-4\beta)}{\binom{2k+2}{k+1}(2k+3)}=\frac{1}{2}-\frac{(1-2y^2)\arcsin y}{2y\sqrt{1-y^2}}.$$

Corollary 17 ($\beta = 1/4$ in Theorem 5)**.**

$$\sum_{k\geq 1} \frac{H_k(\alpha)\,(3k+4)}{\binom{2k+2}{k+1}(2k+3)} = \begin{cases} 1-\frac{\pi}{3\sqrt{3}}, & \alpha = 1; \\ 1-\frac{\pi}{4}, & \alpha = 1+1/\sqrt{2}; \\ 1-\frac{\pi}{2\sqrt{3}}, & \alpha = 2+\sqrt{3}. \end{cases}$$

Corollary 18 ($\beta = 1/2$ in Theorem 5)**.**

$$\sum_{k\geq 1} \frac{2^k H_k(\alpha)(k+1)}{\binom{2k+2}{k+1}(2k+3)} = \begin{cases} \frac{1}{2}, & \alpha = 1; \\ \frac{1}{2}-\frac{\pi}{6\sqrt{3}}, & \alpha = 2; \\ \frac{1}{2}-\frac{\pi}{8}, & \alpha = 2+\sqrt{2}; \\ \frac{1}{2}-\frac{\pi}{4\sqrt{3}}, & \alpha = 4+2\sqrt{3}. \end{cases}$$

Corollary 19 ($\beta = 3/4$ in Theorem 5)**.**

$$\sum_{k\geq 1} \frac{3^k H_k(\alpha)k}{\binom{2k+2}{k+1}(2k+3)} = \begin{cases} 1+\frac{2\pi}{3\sqrt{3}}, & \alpha = 1; \\ 1-\frac{\pi}{3\sqrt{3}}, & \alpha = 3; \\ 1, & \alpha = 3/2; \\ 1-\frac{\pi}{4}, & \alpha = 3+3/\sqrt{2}. \end{cases}$$

Corollary 20 ($\beta = 1/5$ in Theorem 5)**.**

$$\sum_{k\geq 1} \Big(\frac{4}{5}\Big)^k \frac{H_k(\alpha)(8k+11)}{\binom{2k+2}{k+1}(2k+3)} = \begin{cases} \frac{5}{2}-\frac{5\pi}{8}, & \alpha = \frac{2}{5}(2+\sqrt{2}); \\ \frac{5}{2}-\frac{5\pi}{4\sqrt{3}}, & \alpha = \frac{4}{5}(2+\sqrt{3}); \\ \frac{5}{2}-\frac{\pi}{2}\sqrt{\frac{2+\sqrt{5}}{\sqrt{5}}}, & \alpha = \frac{2}{5}(3+\sqrt{5}). \end{cases}$$

7. Series with $\binom{2k+4}{k+2}(2k+5)$ in Denominators

Define the sequence $\{V_k\}$ and then compute its difference by

$$V_k = \frac{(4\beta)^k}{\binom{2k+4}{k+2}} \quad \text{and} \quad \Delta V_k = \frac{(4\beta)^k(5+2k-2\beta k-6\beta)}{\binom{2k+4}{k+2}(2k+5)}.$$

According to the modified Abel's lemma on summation by parts (7), we can reformulate the following infinite series:

$$\sum_{k\geq 1} \frac{(4\beta/\alpha)^k}{k\binom{2k+4}{k+2}} = \sum_{k\geq 1} V_k \nabla U_k = [UV]_+ - U_0V_1 + \sum_{k\geq 1} U_k \Delta V_k.$$

The series on the left can be evaluated by (2) and (4) because of the equality

$$\frac{1}{k\binom{2k+4}{k+2}}=\frac{1}{6}\left(\frac{1}{k\binom{2k}{k}}-\frac{1}{(2k+1)\binom{2k}{k}}-\frac{1}{(2k+3)\binom{2k+2}{k+1}}\right).$$

This proves the following general summation theorem.

Theorem 6. *Let α and β be two real numbers subject to $|\alpha|\geq 1$ and $|\beta|<1$. Then for $y=\sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1}H_k(\alpha)\frac{(4\beta)^k(5+2k-2\beta k-6\beta)}{\binom{2k+4}{k+2}(2k+5)}=\frac{3+14y^2}{72y^2}+\frac{\arcsin y}{y\sqrt{1-y^2}}\left(\frac{y^2}{3}-\frac{1}{24y^2}-\frac{1}{6}\right). \quad (13)$$

Corollary 21 ($\beta=1/4$ in Theorem 6)**.**

$$\sum_{k\geq 1}\frac{H_k(\alpha)\,(3k+7)}{\binom{2k+4}{k+2}(2k+5)}=\begin{cases}\frac{13}{18}-\frac{\pi}{3\sqrt{3}}, & \alpha=1;\\ \frac{13+3\sqrt{2}}{18}-\frac{\pi}{12}(2+\sqrt{2}), & \alpha=1+1/\sqrt{2};\\ \frac{19+6\sqrt{3}}{18}-\frac{\pi}{18}(4+3\sqrt{3}), & \alpha=2+\sqrt{3}.\end{cases}$$

Corollary 22 ($\beta=1/2$ in Theorem 6)**.**

$$\sum_{k\geq 1}\frac{2^kH_k(\alpha)(k+2)}{\binom{2k+4}{k+2}(2k+5)}=\begin{cases}\frac{5}{18}-\frac{\pi}{24}, & \alpha=1;\\ \frac{13}{36}-\frac{\pi}{6\sqrt{3}}, & \alpha=2;\\ \frac{13+3\sqrt{2}}{36}-\frac{\pi}{24}(2+\sqrt{2}), & \alpha=2+\sqrt{2};\\ \frac{19+6\sqrt{3}}{36}-\frac{\pi}{36}(4+3\sqrt{3}), & \alpha=4+2\sqrt{3}.\end{cases}$$

Corollary 23 ($\beta=3/4$ in Theorem 6)**.**

$$\sum_{k\geq 1}\frac{3^kH_k(\alpha)(k+1)}{\binom{2k+4}{k+2}(2k+5)}=\begin{cases}\frac{1}{2}+\frac{2\pi}{27\sqrt{3}}, & \alpha=1;\\ \frac{13}{18}-\frac{\pi}{3\sqrt{3}}, & \alpha=3;\\ \frac{5}{9}-\frac{\pi}{12}, & \alpha=3/2.\end{cases}$$

Corollary 24 ($\beta=1/6$ in Theorem 6)**.**

$$\sum_{k\geq 1}\left(\frac{2}{3}\right)^k\frac{H_k(\alpha)(5k+12)}{\binom{2k+4}{k+2}(2k+5)}=\begin{cases}\frac{13+3\sqrt{2}}{12}-\frac{\pi}{8}(2+\sqrt{2}), & \alpha=\frac{2+\sqrt{2}}{3};\\ \frac{19+6\sqrt{3}}{12}-\frac{\pi}{12}(4+3\sqrt{3}), & \alpha=\frac{4+2\sqrt{3}}{3};\\ \frac{16+3\sqrt{5}}{12}-\frac{\pi}{10}\sqrt{\frac{38+17\sqrt{5}}{\sqrt{5}}}, & \alpha=\frac{3+\sqrt{5}}{3}.\end{cases}$$

8. Further Series with $\binom{2k}{k}(2k+1)(2k+3)$ in Denominators

Define the sequence $\{V_k\}$, a variant of that treated in §4, and then compute its difference by

$$V_k = \frac{k(4\beta)^k}{(2k+1)\binom{2k}{k}} \quad \text{and} \quad \Delta V_k = \frac{(4\beta)^k\big\{k(2k+3)-2\beta(k+1)^2\big\}}{\binom{2k}{k}(2k+1)(2k+3)}.$$

According to the modified Abel's lemma on summation by parts (7), we can reformulate the following infinite series:

$$\sum_{k\geq 1}\frac{(4\beta/\alpha)^k}{\binom{2k}{k}(2k+1)} = \sum_{k\geq 1} V_k \nabla U_k = [UV]_+ - U_0V_1 + \sum_{k\geq 1} U_k \Delta V_k.$$

Evaluating the sum on the left via (4), we derive the general theorem below.

Theorem 7. *Let α and β be two real numbers subject to $|\alpha|\geq 1$ and $|\beta|<1$. Then for $y=\sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1} H_k(\alpha)\frac{(4\beta)^k\big\{k(2k+3)-2\beta(k+1)^2\big\}}{\binom{2k}{k}(2k+1)(2k+3)} = \frac{\arcsin y}{y\sqrt{1-y^2}} - 1. \tag{14}$$

Corollary 25 ($\beta = 1/4$ in Theorem 7)**.**

$$\sum_{k\geq 1}\frac{H_k(\alpha)\,(3k^2+4k-1)}{\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} \frac{4\pi}{3\sqrt{3}}-2, & \alpha=1;\\ \frac{\pi}{\sqrt{2}}-2, & \alpha=1+1/\sqrt{2};\\ \frac{2\pi}{3}-2, & \alpha=2+\sqrt{3}. \end{cases}$$

Corollary 26 ($\beta = 1/2$ in Theorem 7)**.**

$$\sum_{k\geq 1}\frac{2^k H_k(\alpha)\,(k^2+k-1)}{\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} \frac{\pi}{2}-1, & \alpha=1;\\ \frac{2\pi}{3\sqrt{3}}-1, & \alpha=2;\\ \frac{\pi}{2\sqrt{2}}-1, & \alpha=2+\sqrt{2};\\ \frac{\pi}{3}-1, & \alpha=4+2\sqrt{3}. \end{cases}$$

Corollary 27 ($\beta = 3/4$ in Theorem 7)**.**

$$\sum_{k\geq 1}\frac{3^k H_k(\alpha)\,(k^2-3)}{\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} \frac{8\pi}{3\sqrt{3}}-2, & \alpha=1;\\ \frac{4\pi}{3\sqrt{3}}-2, & \alpha=3;\\ \frac{\pi}{\sqrt{2}}-2, & \alpha=3+3/\sqrt{2};\\ \frac{2\pi}{3}-2, & \alpha=6+3\sqrt{3}. \end{cases}$$

Corollary 28 ($\beta = 3/7$ in Theorem 7)**.**

$$\sum_{k\geq 1}\left(\frac{12}{7}\right)^k \frac{H_k(\alpha)\,(8k^2+9k-6)}{\binom{2k}{k}(2k+1)(2k+3)} = \begin{cases} \frac{14\pi}{3\sqrt{3}}-7, & \alpha = 12/7;\\ \frac{7\pi}{2\sqrt{2}}-7, & \alpha = \frac{6}{7}(2+\sqrt{2});\\ \frac{7\pi}{3}-7, & \alpha = \frac{12}{7}(2+\sqrt{3}). \end{cases}$$

9. Series with $\binom{2k}{k}(2k+1)(2k+3)(2k+5)$ in Denominators

Finally, by combining linearly the equations

$$20\mathrm{Eq}(13) - 3\mathrm{Eq}(12) - \mathrm{Eq}(14)$$

and then simplifying the result, we get the following general theorem.

Theorem 8. *Let α and β be two real numbers subject to $|\alpha| \geq 1$ and $|\beta| < 1$. Then for $y = \sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1} H_k(\alpha)\frac{(4\beta)^k\big\{(2k+5)(3k^2+6k+1)-2\beta(k+1)(3k^2+12k+10)\big\}}{\binom{2k}{k}(2k+1)(2k+3)(2k+5)}$$
$$= \frac{5\left(3-4y^2\right)}{18y^2} + \frac{\left(4y^4+10y^2-5\right)\arcsin y}{6y^3\sqrt{1-y^2}}.$$

Corollary 29 ($\beta = 1/4$ in Theorem 8: Genčev [6, Example 2.2])**.**

$$\sum_{k\geq 1}\frac{H_k(\alpha)}{\binom{2k}{k}}\frac{k(k+2)(3k+7)}{(2k+1)(2k+3)(2k+5)}$$
$$= \begin{cases} \frac{40}{27}-\frac{2\pi}{3\sqrt{3}}, & \alpha = 1;\\ \frac{40+30\sqrt{2}}{27}-\frac{\pi}{18}(11+4\sqrt{2}), & \alpha = 1+1/\sqrt{2};\\ \frac{100+60\sqrt{3}}{27}-\frac{7\pi}{27}(4+3\sqrt{3}), & \alpha = 2+\sqrt{3}. \end{cases}$$

Corollary 30 ($\beta = 1/2$ in Theorem 8)**.**

$$\sum_{k\geq 1}\frac{2^k H_k(\alpha)}{\binom{2k}{k}}\frac{3k^3+12k^2+10k-5}{(2k+1)(2k+3)(2k+5)}$$
$$= \begin{cases} \frac{5}{9}+\frac{\pi}{6}, & \alpha = 1;\\ \frac{20}{9}-\frac{\pi}{\sqrt{3}}, & \alpha = 2;\\ \frac{20+15\sqrt{2}}{9}-\frac{\pi}{12}(11+4\sqrt{2}), & \alpha = 2+\sqrt{2};\\ \frac{10}{9}(5+3\sqrt{3})-\frac{7\pi}{18}(4+3\sqrt{3}), & \alpha = 4+2\sqrt{3}. \end{cases}$$

Corollary 31 ($\beta = 3/4$ in Theorem 8)**.**

$$\sum_{k\geq 1} \frac{3^k H_k(\alpha)}{\binom{2k}{k}} \frac{3k^3+9k^2-2k-20}{(2k+1)(2k+3)(2k+5)}$$
$$= \begin{cases} \frac{76\pi}{27\sqrt{3}}, & \alpha = 1; \\ \frac{40}{9} - \frac{2\pi}{\sqrt{3}}, & \alpha = 3; \\ \frac{40+30\sqrt{2}}{9} - \frac{\pi}{6}(11+4\sqrt{2}), & \alpha = 3+3/\sqrt{2}; \\ \frac{20}{9}(5+3\sqrt{3}) - \frac{7\pi}{9}(4+3\sqrt{3}), & \alpha = 6+3\sqrt{3}. \end{cases}$$

Corollary 32 ($\beta = 3/8$ in Theorem 8)**.**

$$\sum_{k\geq 1} \Big(\frac{3}{2}\Big)^k \frac{H_k(\alpha)}{\binom{2k}{k}} \frac{15k^3+63k^2+62k-10}{(2k+1)(2k+3)(2k+5)}$$
$$= \begin{cases} \frac{80}{9} - \frac{4\pi}{\sqrt{3}}, & \alpha = 3/2; \\ \frac{80+60\sqrt{2}}{9} - \frac{\pi}{3}(11+4\sqrt{2}), & \alpha = \frac{3}{4}(2+\sqrt{2}); \\ \frac{200+120\sqrt{3}}{9} - \frac{14\pi}{9}(4+3\sqrt{3}), & \alpha = \frac{3}{2}(2+\sqrt{3}). \end{cases}$$

10. Series about Quadratic Harmonic Numbers

Before the end of this paper, we are going to show another different example. For the sequences $\{U_k\}$ and $\{V_k\}$ (where V_k is the same as that in §2) defined by

$$U_k = H''_k(\alpha) = \sum_{j=1}^{k} \frac{1}{j^2\alpha^j} \quad \text{and} \quad V_k = \frac{(4\beta)^k}{\binom{2k}{k}},$$

it is routine to check their differences

$$\nabla U_k = \frac{1}{k^2\alpha^k} \quad \text{and} \quad \Delta V_k = \frac{(4\beta)^k(1+2k-2\beta k-2\beta)}{\binom{2k}{k}(2k+1)}$$

as well as limiting relations

$$[UV]_+ = U_0V_1 = 0 \quad \text{provided that } |\alpha| \geq 1 \text{ and } |\beta| < 1.$$

According to the modified Abel's lemma on summation by parts (7), we can manipulate the following infinite series:

$$\sum_{k\geq 1} \frac{(4\beta/\alpha)^k}{k^2\binom{2k}{k}} = \sum_{k\geq 1} V_k \nabla U_k = [UV]_+ - U_0V_1 + \sum_{k\geq 1} U_k \Delta V_k. \tag{15}$$

Evaluating the series on the left by (1), we prove the general theorem below.

Theorem 9. *Let α and β be two real numbers subject to $|\alpha| \geq 1$ and $|\beta| < 1$. Then for $y = \sqrt{\beta/\alpha}$, the following summation formula holds:*

$$\sum_{k\geq 1} H_k''(\alpha) \frac{(4\beta)^k(1+2k-2\beta k-2\beta)}{\binom{2k}{k}(2k+1)} = 2\big(\arcsin y\big)^2.$$

Corollary 33 ($\beta = 1/4$ in Theorem 9)**.**

$$\sum_{k\geq 1} \frac{H_k''(\alpha)(3k+1)}{\binom{2k}{k}(2k+1)} = \begin{cases} \frac{\pi^2}{9}, & \alpha = 1; \\ \frac{\pi^2}{16}, & \alpha = 1+1/\sqrt{2}; \\ \frac{\pi^2}{36}, & \alpha = 2+\sqrt{3}. \end{cases}$$

Corollary 34 ($\beta = 1/2$ in Theorem 9)**.**

$$\sum_{k\geq 1} \frac{2^k H_k''(\alpha)\, k}{\binom{2k}{k}(2k+1)} = \begin{cases} \frac{\pi^2}{8}, & \alpha = 1; \\ \frac{\pi^2}{18}, & \alpha = 2; \\ \frac{\pi^2}{72}, & \alpha = 4+2\sqrt{3}. \end{cases}$$

Corollary 35 ($\beta = 3/4$ in Theorem 9)**.**

$$\sum_{k\geq 1} \frac{3^k H_k''(\alpha)\,(k-1)}{\binom{2k}{k}(2k+1)} = \begin{cases} \frac{4\pi^2}{9}, & \alpha = 1; \\ \frac{\pi^2}{9}, & \alpha = 3; \\ \frac{\pi^2}{4}, & \alpha = 3/2; \\ \frac{\pi^2}{36}, & \alpha = 6+3\sqrt{3}. \end{cases}$$

Corollary 36 ($\beta = 2/7$ in Theorem 9)**.**

$$\sum_{k\geq 1} \left(\frac{8}{7}\right)^k \frac{H_k''(\alpha)\,(10k+3)}{\binom{2k}{k}(2k+1)} = \begin{cases} \frac{7\pi^2}{18}, & \alpha = 8/7; \\ \frac{7\pi^2}{32}, & \alpha = \frac{4}{7}(2+\sqrt{2}); \\ \frac{7\pi^2}{72}, & \alpha = \frac{8}{7}(2+\sqrt{3}); \\ \frac{7\pi^2}{50}, & \alpha = \frac{4}{7}(3+\sqrt{5}). \end{cases}$$

Corollary 37 ($\beta = 2/9$ in Theorem 9)**.**

$$\sum_{k\geq 1} \left(\frac{8}{9}\right)^k \frac{H_k''(\alpha)\,(14k+5)}{\binom{2k}{k}(2k+1)} = \begin{cases} \frac{9\pi^2}{32}, & \alpha = \frac{4}{9}(2+\sqrt{2}); \\ \frac{\pi^2}{8}, & \alpha = \frac{8}{9}(2+\sqrt{3}); \\ \frac{9\pi^2}{50}, & \alpha = \frac{4}{9}(3+\sqrt{5}). \end{cases}$$

11. Concluding Remarks

The examples exhibited in this paper suggest that there may exist potentially infinite difference pairs $\{U_k\}$ and $\{V_k\}$ fitting into our scheme. Their reformulations carried out through Abel's lemma on summation by parts would produce infinitely many summation formulae involving both the generalized harmonic numbers and the central binomial coefficients. For example, one may construct other difference pairs $\{U_k\}$ and $\{V_k\}$ based on the series appeared in [1, 2, 3, 8] and [4, Page 89]. The interested reader is encouraged to explore further this approach and search for more significant infinite series identities.

Acknowledgement. The authors express their sincere gratitude to an anonymous referee for the careful reading, critical comments and valuable suggestions, that made considerable improvement to the manuscript during the revision. The first author was partially supported, during this research, by the Scientific Research Fund of Liaoning Provincial Education Department, China (No. JDL2019028).

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