GRUNDY NUMBERS OF IMPARTIAL CHOCOLATE BAR GAMES

Ryohei Miyadera
Kwansei Gakuin High School, Nishinomiya City, Japan
runnerskg@gmail.com

Shunsuke Nakamura
Graduate School of Engineering Science, Osaka University, Toyonaka City, Japan
nakamura.stat@gmail.com

Yushi Nakaya
School of Engineering, Tohoku University, Sendai City, Japan
math271k@gmail.com

Received: 7/9/17, Revised: 5/7/19, Accepted: 2/7/20, Published: 2/14/20

Abstract

Chocolate bar games are a variant of CHOMP. A chocolate bar is a rectangular array of squares but with some squares removed. There is a poisoned square at the bottom of the bar (column 0), there are $z+1$ columns of squares, and the height of the i-th column is given by a non-decreasing function $f(i)+1$, with a maximum height of $y+1$. This is denoted by $CB(f,y,z)$. A move is to break the bar along a horizontal or vertical line and eat that part, not counting the poisoned square. In the case $f(i) = \lfloor \frac{i}{k} \rfloor$ for some even number k, the authors have previously proved that the Sprague–Grundy value of $CB(f,y,z)$ is $y \oplus z$. The case of an odd number k is still open. In this paper, the functions f such that the Sprague–Grundy value of $CB(f,y,z)$ is $y \oplus z$ are characterized, and this result is generalized to the case when the Sprague–Grundy value of $CB(f,y,z)$ is $(y \oplus (z+s)) - s$ for a natural number s.

1. Introduction

A chocolate bar is a rectangular array of squares, but with some squares removed. There is a poisoned square at the bottom of the bar.

The original chocolate bar game [3] had a rectangular bar with one bitter corner, as shown in Figure 1. Each player in turn breaks the bar in a straight line along the grooves and eats the piece he/she breaks off. The player who manages to leave his opponent with the single bitter block (black block) is the winner.

We now consider the chocolate bar game in Figure 2.

Example 1.1. Examples of chocolate bar games.

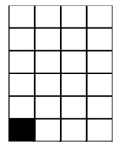

Figure 1.

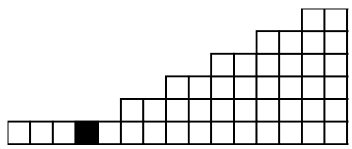

Figure 2.

To study this game, it suffices to study the chocolate bars in Figures 3 and 4 separately, as the game in Figure 2 is the sum of the game in Figure 3 and the game in Figure 4. The game in Figure 3 is mathematically equivalent to a pile of three stones; thus, we concentrate on the game in Figure 4.

Example 1.2.

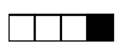

Figure 3.

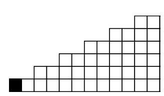

Figure 4.

In the game in Figure 4, a vertical break can reduce the number of horizontal breaks. We can think of this game as being played with two heaps, but now a move may change more than one heap.

In this study, we consider the Grundy numbers of a chocolate bar with a poisoned square at the bottom left of the bar. For a general bar, the strategies appear complicated. We focus on bars that grow regularly in height.

We now define the chocolate bars. First, we define a function that determines the shape of the bar. Let $Z_{\geq 0}$ be the set of non-negative integers.

Definition 1.1. Let f be a function that satisfies the following two conditions:
(i) $f(t) \in Z_{\geq 0}$ for $t \in Z_{\geq 0}$;
(ii) f is monotonically increasing, i.e., we have $f(u) \leq f(v)$ for $u, v \in Z_{\geq 0}$ with $u \leq v$.

Definition 1.2. Let f be a function that satisfies the conditions in Definition 1.1. For $y, z \in Z_{\geq 0}$, the chocolate bar will consist of $z+1$ columns where the 0-th column is the bitter square, and the height of the i-th column is $t(i) = \min(f(i), y) + 1$ for i = 0,1,...,z. We will denote this by $CB(f, y, z)$.

Thus, the height of the i-th column is determined by f, i, and y.

Example 1.3. We give some examples of chocolate bars of the type $CB(f, y, z)$. We note that the function f defines the shape of the bar, and the two coordinates y and z stand for the number of grooves above and to the right of the bitter square, respectively.

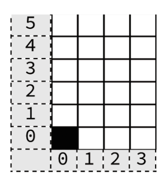

Figure 5: $CB(f, 5, 3)$ with $f(t) = 5$.

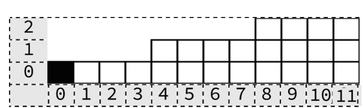

Figure 6: $CB(f, 2, 11)$ with $f(t) = \lfloor \frac{t}{4} \rfloor$.

Before we study these chocolate bars, we briefly review some necessary concepts in combinatorial game theory; see [1] or [4] for more details.

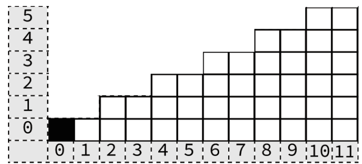

Figure 7: $CB(f,5,11)$ with $f(t) = \lfloor \frac{t}{2} \rfloor$.

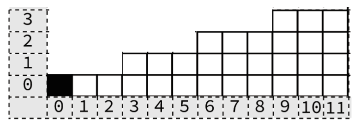

Figure 8: $CB(f,3,11)$ with $f(t) = \lfloor \frac{t}{3} \rfloor$.

Definition 1.3. Let x, y be non-negative integers. We represent them in base 2, so that $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0,1\}$. We define the *nim-sum* $x \oplus y$ by

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i,$$

where $w_i = x_i + y_i \ (mod\ 2)$.

As chocolate bar games are impartial games without draws, there will be only two kinds of positions.

Definition 1.4. (a) A position is called a $\mathcal{P}$-*position*, if it is a winning position for the previous player (the player who just moved), as long as he/she plays correctly at every stage.
(b) A position is called an $\mathcal{N}$-*position*, if it is a winning position for the next player, as long as he/she plays correctly at every stage.

Definition 1.5. The *disjunctive sum* of two games, denoted by $\mathbf{G} + \mathbf{H}$, is a super-game where a player may move either in $\mathbf{G}$ or in $\mathbf{H}$, but not in both.

Definition 1.6. For any position $\mathbf{p}$ of a game $\mathbf{G}$, there is a set of positions that can be reached by making precisely one move in $\mathbf{G}$; this will be denoted by $move(\mathbf{p})$.

Remark 1.1. For examples of a *move*, see Example 2.1.

Definition 1.7. (i) The *minimum excluded value* (mex) of a set S of non-negative integers is the least non-negative integer that is not in S.
(ii) Let $\mathbf{p}$ be a position of an impartial game. The associated *Grundy number* is denoted by $G(\mathbf{p})$ and is recursively defined by $G(\mathbf{p}) = mex\{G(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.

We now present some lemmas regarding the minimum excluded value *mex* and the Grundy number.

Lemma 1. *Let $x \in Z_{\geq 0}$ and $y_k \in Z_{\geq 0}$ for $k = 1, 2, ..., n$. Then the following conditions are equivalent:*
(i) $x = mex(\{y_k : k = 1, 2, ..., n\})$.
(ii) $x \neq y_k$ *for any* k *and* $\{0, 1, 2, 3, ..., x-1\} \subset \{y_k : k = 1, 2, ..., n\}$.

Proof. This follows directly from Definition 1.7. □

Lemma 2. *Let S be a set and $\{1, 2, 3, ..., m-1\} \subset S$. Then $mex(S) \geq m$.*

Proof. This also follows directly from Definition 1.7. □

Lemma 3. *If $G(\mathbf{p}) > x$ for some $x \in Z_{\geq 0}$, then there exists $\mathbf{h} \in move(\mathbf{p})$ such that $G(\mathbf{h}) = x$.*

Proof. This follows directly from (ii) of Lemma 1 and Definition 1.7. □

The next result demonstrates the usefulness of Sprague–Grundy theory in impartial games.

Theorem 1. *Let $\mathbf{G}$ and $\mathbf{H}$ be impartial rulesets and let $G_{\mathbf{G}}$ and $G_{\mathbf{H}}$, respectively, be the Grundy numbers of game $\mathbf{g}$ played with the rules of $\mathbf{G}$, and of $\mathbf{h}$ played under the rules of $\mathbf{H}$. Then we have the following.*
(i) For any position $\mathbf{g}$ of $\mathbf{G}$, we have $G_{\mathbf{G}}(\mathbf{g}) = 0$ if and only if $\mathbf{g}$ is a $\mathcal{P}$-position.
(ii) The Grundy number of a position $\{\mathbf{g}, \mathbf{h}\}$ in the game $\mathbf{G}+\mathbf{H}$ is $G_{\mathbf{G}}(\mathbf{g}) \oplus G_{\mathbf{H}}(\mathbf{h})$.

For a proof of this theorem, see [1].

The chocolate bar game was introduced by A.C. Robin in [3], but it is essentially the same as the game "NIM". An example treated by A.C. Robin is the chocolate bar in Figure 5, which is equivalent to NIM with a heap of five stones and a heap of three stones. Here, 5 and 3 are the numbers of grooves above and to the right of the bitter square, respectively. As the Grundy number of NIM with a heap of five stones and a heap of three stones is $5 \oplus 3$, the Grundy number of the game in Figure 5 is $5 \oplus 3$. The chocolate bar games that are considered in this study were introduced by the authors in [2], where the following theorem was proved.

Theorem 2. *Let $f(t) = \lfloor \frac{t}{2k} \rfloor$ for a fixed natural number k, where $\lfloor \quad \rfloor$ is the floor function. Then, the Grundy number of $CB(f, y, z)$ is $y \oplus z$.*

This is Theorem 2.1 in [2].

Figures 6 and 7 show examples of chocolate bars that satisfy the condition of Theorem 2. When $f(t) = \lfloor \frac{t}{k} \rfloor$ for a fixed odd integer k, the Grundy number of the chocolate bar is not $y \oplus z$. Figure 8 shows an example of this type of chocolate bar.

Therefore, it is natural to search for a necessary and sufficient condition whereby a chocolate bar may have Grundy number $G(y, z) = y \oplus z$, where y and z are the coordinates of the bar. The aim of this article is to answer the following question and generalize the result.

Question. *What is a necessary and sufficient condition whereby a chocolate bar $CB(f, y, z)$ may have Grundy number $G(y, z) = y \oplus z$?*

There are other types of chocolate bar games, such as CHOMP. CHOMP is a game with a rectangular chocolate bar. The players take turns, and they choose one block and eat it, together with those that are below it and to its right. The top left block is bitter, and the players cannot eat it. Although this game has been extensively studied, the winning strategy is still unknown. For an overview of related research, see [5].

2. Grundy Numbers of Chocolate Bars

For a fixed function f, the position of $CB(f, y, z)$ will be denoted by the coordinates $\{y, z\}$, and f will be supressed. Moreover, we define $move_f$ as follows.

Definition 2.1. For $y, z \in Z_{\geq 0}$, we define
$move_f(\{y, z\}) = \{\{v, z\} : v < y\} \cup \{\{\min(y, f(w)), w\} : w < z\}$, where $v, w \in Z_{\geq 0}$.

This definition is a special case of *move* in Definition 1.6.

Example 2.1. Here, we explain $move_f$ when $f(t) = \lfloor \frac{t}{2} \rfloor$. If we start with the chocolate bar in Figure 9, whose coordinates are $\{y, z\} = \{5, 11\}$, and reduce $y = 5$ to $y = 3$ by cutting horizontally, then we have the bar in 10, whose coordinates are $\{y, z\} = \{3, 11\}$.

If we start with the chocolate bar in Figure 9, and reduce $z = 11$ to $z = 10$ by cutting vertically, then we have the bar in Figure 11, whose coordinates are $\{y, z\} = \{5, 10\}$. We note that $\{\min(5, \lfloor \frac{10}{2} \rfloor), 10\} = \{5, 10\}$, and the reduction of z does not affect y.

If we start with the chocolate bar in Figure 9, and reduce $z = 11$ to $z = 8$ by cutting vertically, then we have the bar in Figure 12, whose coordinates are $\{y, z\} = \{4, 8\}$. We note that $\{\min(5, \lfloor \frac{8}{2} \rfloor), 8\} = \{4, 8\}$, and the reduction of z decreases the value of y.

Therefore, we have $\{3, 11\}$, $\{5, 10\}$, $\{4, 8\} \in move_f(\{5, 11\})$.

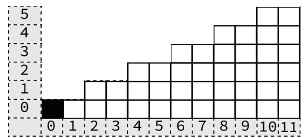

Figure 9: $CB(f,5,11)$.

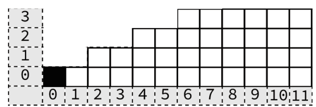

Figure 10: $CB(f,3,11)$.

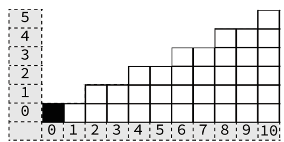

Figure 11: $CB(f,5,10)$.

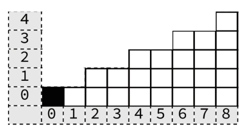

Figure 12: $CB(f,4,8)$.

3. Chocolate Bar Game $CB(f,y,z)$ Whose Grundy Number is $G(\{y,z\}) = y \oplus z$

3.1. Sufficient Condition Whereby a Chocolate Bar $CB(f,y,z)$ May Have Grundy Number $G(\{y,z\}) = y \oplus z$

In this subsection, we study a sufficient condition whereby a chocolate bar $CB(f,y,z)$ may have a Grundy number $G(\{y,z\}) = y \oplus z$.

In the proofs, it will be useful to consider the disjunctive sum of a chocolate bar $CB(f,y,z)$ to the right of the bitter square and a single chocolate strip to the left, as in Figure 13.

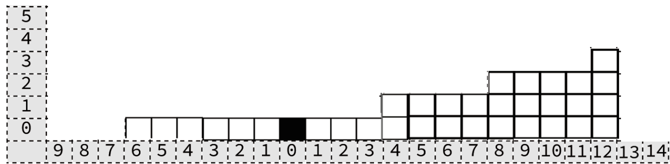

Figure 13: Single chocolate strip of length 6 and the chocolate bar $CB(f, 3, 12)$.

We will denote the position of such a chocolate bar by $\{x, y, z\}$, where x is the length of the single chocolate strip and y, z are the coordinates of $CB(f, y, z)$. Figure 13 shows an example where the coordinates are $\{6, 3, 12\}$. For the disjunctive sum of the chocolate bar game $CB(f, y, z)$ to the right of the bitter square and a single chocolate strip to the left, we will show that the $\mathcal{P}$-positions are obtained when $x \oplus y \oplus z = 0$, so that the Grundy number of $CB(f, y, z)$ is $x = y \oplus z$.

We define the set $move_f(\{x, y, z\})$ that contains all the positions that can be reached from the position $\{x, y, z\}$ in one step (directly).

Definition 3.1. For $x, y, z \in Z_{\geq 0}$, we define
$move_f(\{x, y, z\}) = \{\{u, y, z\} : u < x\} \cup \{\{x, v, z\} : v < y\} \cup \{\{x, \min(y, f(w)), w\} : w < z\}$, where $u, v, w \in Z_{\geq 0}$.

Remark 3.1. In this article, we define two types of $move_f$. One is in Definition 2.1, and pertains to chocolate bars with two coordinates $\{y, z\}$. The other is in Definition 3.1, and pertains to chocolate bars with three coordinates $\{x, y, z\}$.

Example 3.1. Examples of moves of the disjunctive sum of a chocolate bar $CB(f, y, z)$ to the right of the bitter square and a single chocolate strip to the left, where $f(t) = \lfloor \frac{t}{4} \rfloor$. The coordinates of this chocolate bar are denoted by $\{x, y, z\}$.
(i) If we start with the chocolate bar in Figure 14, whose coordinates are $\{6, 2, 9\}$, and reduce the first coordinate x to 3 by cutting the bar vertically on the left side of the bitter square, then we obtain the bar in Figure 15. Here, $\{3, 2, 9\} \in move_f(\{6, 2, 9\})$.
(ii) If we start with the chocolate bar in Figure 14 and reduce the second coordinate y to 1 by cutting the bar horizontally, then we obtain the bar in Figure 16. Here, $\{6, 1, 9\} \in move_f(\{6, 2, 9\})$.
(iii) If we start with the chocolate bar in Figure 14 and reduce the third coordinate z to 8 by cutting the bar vertically, then we obtain the bar in Figure 17. Here, $\{6, \min(2, \lfloor \frac{8}{4} \rfloor), 8\} = \{6, 2, 8\} \in move_f(\{6, 2, 9\})$, and the reduction of z does not affect the second coordinate y.
(iv) If we start with the chocolate bar in Figure 14 and reduce the third coordinate z to 6 by cutting the bar vertically, then the second coordinate is also reduced

to 1. Then, we obtain the bar in Figure 18. Here, $\{6, \min(2, \lfloor \frac{6}{4} \rfloor), 6\} = \{6, 1, 6\}$ $\in move_f(\{6, 2, 9\})$, and the reduction of the third coordinate z affects the second coordinate y.

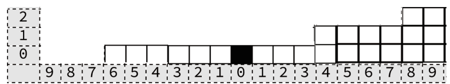

Figure 14: $\{6, 2, 9\}$.

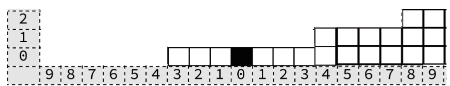

Figure 15: $\{3, 2, 9\}$.

Figure 16: $\{6, 1, 9\}$.

In this subsection, we prove that (a) in the following definition is a sufficient condition whereby a chocolate bar $CB(f, y, z)$ may have Grundy number $G(\{y, z\})$ $= y \oplus z$.

Definition 3.2. Let h be a function of $Z_{\geq 0}$ into $Z_{\geq 0}$ that satisfies the conditions of Definition 1.1 and the following condition:
(a) We assume that

$$\lfloor \frac{z}{2^i} \rfloor = \lfloor \frac{z'}{2^i} \rfloor \tag{1}$$

for some $z, z' \in Z_{\geq 0}$ and some natural number i. Then we have

$$\lfloor \frac{h(z)}{2^{i-1}} \rfloor = \lfloor \frac{h(z')}{2^{i-1}} \rfloor. \tag{2}$$

As condition (a) of Definition 3.2 is rather abstract, we present an example of a function that satisfies it.

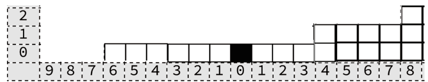

Figure 17: $\{6, 2, 8\}$.

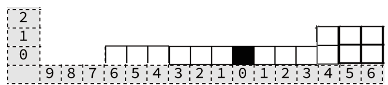

Figure 18: $\{6, 1, 6\}$.

Lemma 4. *Let* $h(z) = \lfloor \frac{z}{2k} \rfloor$ *for some natural number* k. *Then* $h(z)$ *satisfies the condition* (a) *of Definition 3.2.*

Proof. We provide a proof by contraposition. We suppose that Equation (2) in Definition 3.2 is false. Then there exist $z, z', u \in Z_{\geq 0}$ and a natural number i such that

$$\lfloor \frac{h(z)}{2^{i-1}} \rfloor = \lfloor \frac{\lfloor \frac{z}{2k} \rfloor}{2^{i-1}} \rfloor = u < u + 1 \leq \lfloor \frac{\lfloor \frac{z'}{2k} \rfloor}{2^{i-1}} \rfloor = \lfloor \frac{h(z')}{2^{i-1}} \rfloor. \tag{3}$$

We prove that Equation (1) in Definition 3.2 is false. From the inequality in (3), we have

$$\lfloor \frac{z}{2k} \rfloor \leq u2^{i-1} + 2^{i-1} - 1 < (u+1)2^{i-1} \leq \lfloor \frac{z'}{2k} \rfloor,$$

and hence

$$z \leq 2k(u2^{i-1} + 2^{i-1} - 1) + 2k - 1 < 2k(u+1)2^{i-1} \leq z'. \tag{4}$$

From the inequality in (4), we have

$$\frac{z}{2^i} \leq k(u+1) - \frac{1}{2^i} < k(u+1) \leq \frac{z'}{2^i}.$$

Therefore,

$$\lfloor \frac{z}{2^i} \rfloor < k(u+1) \leq \lfloor \frac{z'}{2^i} \rfloor.$$

This shows that Equation (1) in Definition 3.2 is false, and the proof is complete. □

In the remainder of this subsection, we assume that h is a function that satisfies the condition (a) in Definition 3.2. Our aim is to show that the disjunctive sum of the chocolate bar game with $CB(f, y, z)$ to the right of the bitter square and a single chocolate strip to the left have $\mathcal{P}$-positions when $x \oplus y \oplus z = 0$, so that the Grundy number of the chocolate bar $CB(f, y, z)$ is $x = y \oplus z$.

To this end, Lemmas 6 and 7 are required. Lemma 6 implies that from a position $\{x, y, z\}$ of the disjunctive sum such that $x \oplus y \oplus z \neq 0$, there is always an option leading to a position for which the nim-sum of the coordinates is 0. Lemma 7 implies that from a position $\{x, y, z\}$ of the disjunctive sum such that $x \oplus y \oplus z = 0$, any option leads to a position for which the nim-sum of the coordinates is not 0.
To prove Lemmas 6 and 7, the following is required.

Lemma 5. *We assume that*

$$h(\sum_{k=0}^{n} p_k 2^k) \geq \sum_{k=i-1}^{n} q_k 2^k. \tag{5}$$

Then

$$h(\sum_{k=i}^{n} p_k 2^k) \geq \sum_{k=i-1}^{n} q_k 2^k. \tag{6}$$

Proof. As

$$\lfloor \frac{\sum_{k=i}^{n} p_k 2^k}{2^i} \rfloor = \lfloor \frac{\sum_{k=0}^{n} p_k 2^k}{2^i} \rfloor,$$

by Definition 3.2 and the inequality in (5),

$$\lfloor \frac{h(\sum_{k=i}^{n} p_k 2^k)}{2^{i-1}} \rfloor = \lfloor \frac{h(\sum_{k=0}^{n} p_k)2^k}{2^{i-1}} \rfloor \geq \lfloor \frac{\sum_{k=i-1}^{n} q_k 2^k}{2^{i-1}} \rfloor = \frac{\sum_{k=i-1}^{n} q_k 2^k}{2^{i-1}}. \tag{7}$$

By the inequality in (7), we have the inequality in (6). □

If the nim-sum of the coordinates of a position is not 0, then by Definition 3.1 and the following Lemma 6, there is always an option that leads to a position whose nim-sum is 0.

Lemma 6. *We assume that $x \oplus y \oplus z \neq 0$ and*

$$y \leq h(z). \tag{8}$$

Then at least one of the following statements is true:
(1) *$u \oplus y \oplus z = 0$ for some $u \in Z_{\geq 0}$ such that $u < x$;*
(2) *$x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$;*
(3) *$x \oplus y \oplus w = 0$ for some $w \in Z_{\geq 0}$ such that $w < z$ and $y \leq h(w)$;*
(4) *$x \oplus v \oplus w' = 0$ for some $v, w' \in Z_{\geq 0}$ such that $v < y, w' < z$ and $v = h(w')$.*

Proof. Let $x = \sum_{k=0}^{n} x_k 2^k$, $y = \sum_{k=0}^{n} y_k 2^k$, and $z = \sum_{k=0}^{n} z_k 2^k$. If $n = 0$, then this lemma is obvious. We assume that $n \geq 1$ and that there exists a non-negative integer s such that $x_i + y_i + z_i = 0$ $(mod\ 2)$ for $i = n, n-1, ..., n-s$ and

$$x_{n-s-1} + y_{n-s-1} + z_{n-s-1} \neq 0 \ (mod\ 2). \tag{9}$$

Case (i) We assume that $x_{n-s-1} = 1$. Then, we define $u = \sum_{i=1}^{n} u_i 2^i$ by $u_i = x_i$ for $i = n, n-1, ..., n-s$, $u_{n-s-1} = 0 < x_{n-s-1}$ and $u_i = y_i + z_i$ $(mod\ 2)$ for $i = n-s-2, n-s-3, ..., 0$. Then we have $u \oplus y \oplus z = 0$ and $u < x$. Therefore, we have (1).

Case (ii) We assume that $y_{n-s-1} = 1$. Then, by an argument similar to that used in (i), we prove that $x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$. Therefore, we have (2).

Case (iii) We assume that

$$x_{n-s-1} = y_{n-s-1} = 0 < 1 = z_{n-s-1}.$$

For $i = n, n-1, ..., n-s$, let

$$w_i = z_i. \tag{10}$$

Let

$$w_{n-s-1} = 0 < 1 = z_{n-s-1}, \tag{11}$$

and $w_i = x_i + y_i$ $(mod\ 2)$ for $i = n-s-2, ..., 0$. Then $x \oplus y \oplus w = 0$. There are two subcases here.

Subcase $(iii.1)$ If $y \leq h(w)$, then we have (3).

Subcase $(iii.2)$ We assume that

$$y > h(w). \tag{12}$$

By the inequality in (8), we have

$$\sum_{k=n-s-1}^{n} y_k 2^k \leq \sum_{k=0}^{n} y_k 2^k = y \leq h(\sum_{k=0}^{n} z_k 2^k),$$

and hence by Lemma 5 and (10)

$$\sum_{k=n-s-1}^{n} y_k 2^k \leq h(\sum_{k=n-s}^{n} z_k 2^k) = h(\sum_{k=n-s}^{n} w_k 2^k) \leq h(w). \tag{13}$$

By the inequalities in (13) and (12), there exists a natural number j such that

$$\sum_{k=n-j}^{n} y_k 2^k \leq h(\sum_{k=0}^{n} w_k 2^k) = h(w) \tag{14}$$

and

$$\sum_{k=n-j-1}^{n} y_k 2^k > h(w) \geq h(\sum_{k=n-j}^{n} w_k 2^k). \tag{15}$$

By the inequalities in (14) and (15),

$$y_{n-j-1} = 1. \tag{16}$$

By the inequalities in (13) and (15),

$$n - j - 1 < n - s - 1.$$

By Lemma 5 and the inequality in (14)

$$\sum_{k=n-j}^{n} y_k 2^k \leq h(\sum_{k=n-j+1}^{n} w_k 2^k) \leq h(w). \tag{17}$$

By Definition 3.2, we have

$$\lfloor \frac{h(\sum_{k=n-j+1}^{n} w_k 2^k)}{2^{n-j}} \rfloor = \lfloor \frac{h(\sum_{k=n-j}^{n} w_k 2^k)}{2^{n-j}} \rfloor,$$

and hence

$$\begin{aligned} \sum_{k=n-j}^{n} y_k 2^{k-n+j} &\leq \lfloor \frac{h(\sum_{k=n-j+1}^{n} w_k 2^k)}{2^{n-j}} \rfloor \\ &= \lfloor \frac{h(\sum_{k=n-j}^{n} w_k 2^k)}{2^{n-j}} \rfloor \\ &\leq \lfloor \frac{\sum_{k=n-j-1}^{n} y_k 2^k}{2^{n-j}} \rfloor \\ &= \sum_{k=n-j}^{n} y_k 2^{k-n+j}. \end{aligned}$$

Therefore, there exist $y'_{n-j-1} \in \{0,1\}$ and $R(n-j-1) \in Z_{\geq 0}$ such that $R(n-j-1) < 2^{n-j-1}$ and

$$h(\sum_{k=n-j}^{n} w_k 2^k) = \sum_{k=n-j}^{n} y_k 2^k + y'_{n-j-1} 2^{n-j-1} + R(n-j-1). \tag{18}$$

By the inequality in (15), Equation (16), and Equation (18), we have

$$y'_{n-j-1} = 0 < 1 = y_{n-j-1}. \tag{19}$$

Let

$$w'_{n-j-1} = x_{n-j-1} + y'_{n-j-1} = x_{n-j-1} \ (mod\ 2). \tag{20}$$

By Definition 3.2

$$\lfloor \frac{h(\sum\limits_{k=n-j}^{n} w_k 2^k)}{2^{n-j-1}} \rfloor = \lfloor \frac{h(\sum\limits_{k=n-j}^{n} w_k 2^k + w'_{n-j-1} 2^{n-j-1})}{2^{n-j-1}} \rfloor,$$

and hence by Equation (18)

$$\left| h(\sum_{k=n-j}^{n} w_k 2^k + w'_{n-j-1} 2^{n-j-1}) - (\sum_{k=n-j}^{n} y_k 2^k + y'_{n-j-1} 2^{n-j-1}) \right| < 2^{n-j-1}. \tag{21}$$

Therefore, there exist $y'_{n-j-2} \in \{0,1\}$ and $R(n-j-2) \in Z_{\geq 0}$ such that $R(n-j-2) < 2^{n-j-2}$ and

$$h(\sum_{k=n-j}^{n} w_k 2^k + w'_{n-j-1} 2^{n-j-1})$$

$$= \sum_{k=n-j}^{n} y_k 2^k + y'_{n-j-1} 2^{n-j-1} + y'_{n-j-2} 2^{n-j-2} + R(n-j-2).$$

Let

$$w'_{n-j-2} = x_{n-j-2} + y'_{n-j-2} \ (mod\ 2).$$

Thereby, we define y'_k for $k = n-j-3, n-j-4, \cdots, 0$ and w'_k for $k = n-j-3, n-j-4, \cdots, 1$ such that for $k = n-j-3, n-j-4, \cdots, 1$

$$w'_k = x_k + y'_k \ (mod\ 2) \tag{22}$$

and

$$h(\sum_{k=n-j}^{n} w_k 2^k + \sum_{k=1}^{n-j-1} w'_k 2^k) = \sum_{k=n-j}^{n} y_k 2^k + \sum_{k=0}^{n-j-1} y'_k 2^k. \tag{23}$$

Then, let

$$w'_0 = x_0 + y'_0 \ (mod\ 2), \tag{24}$$

$$w' = \sum_{k=n-j}^{n} w_k 2^k + \sum_{k=0}^{n-j-1} w'_k 2^k \tag{25}$$

and

$$v = \sum_{k=n-j}^{n} y_k 2^k + \sum_{k=0}^{n-j-1} y'_k 2^k. \tag{26}$$

By Definition 3.2 and Equations (23), (25), and (26), we have

$$h(w') = v.$$

By the inequalities in (11) and (19),

$$v < y \text{ and } w' < w \leq z.$$

By Equations (22) and (24), we have $x \oplus v \oplus w' = 0$. □

If the nim-sum of the coordinates of a position is 0, then by Definition 3.1 and the following lemma, any option from this position leads to a position whose nim-sum is not 0.

Lemma 7. *If $x \oplus y \oplus z = 0$ and $y \leq h(z)$, then the following hold:*
(i) $u \oplus y \oplus z \neq 0$ for any $u \in Z_{\geq 0}$ such that $u < x$;
(ii) $x \oplus v \oplus z \neq 0$ for any $v \in Z_{\geq 0}$ such that $v < y$;
(iii) $x \oplus y \oplus w \neq 0$ for any $w \in Z_{\geq 0}$ such that $w < z$;
(iv) $x \oplus v \oplus w \neq 0$ for any $v, w \in Z_{\geq 0}$ such that $v < y, w < z$ and $v = h(w)$.

Proof. If $u < x$, then there exists i such that $u_i = 0 < 1 = x_i$. As $x_i + y_i + z_i = 0$, we have $u_i + y_i + z_i \neq 0$. This proves (i). Similarly, (ii) and (iii) follow directly from Definition 1.3 (the definition of nim-sum).

We now prove (iv). We assume that $v = h(w)$ for some $w \in Z_{\geq 0}$ such that $v < y, w < z$. Furthermore,

$$w_i = z_i \text{ for } i = n, n-1, n-2, ..., j \text{ and } w_{j-1} < z_{j-1}. \tag{27}$$

Then,

$$\lfloor \frac{w}{2^j} \rfloor = \lfloor \frac{z}{2^j} \rfloor,$$

and hence by Definition 3.2,

$$\lfloor \frac{h(w)}{2^{j-1}} \rfloor = \lfloor \frac{h(z)}{2^{j-1}} \rfloor.$$

Therefore,

$$\lfloor \frac{v}{2^{j-1}} \rfloor = \lfloor \frac{h(w)}{2^{j-1}} \rfloor = \lfloor \frac{h(z)}{2^{j-1}} \rfloor \geq \lfloor \frac{y}{2^{j-1}} \rfloor. \tag{28}$$

As $y > v$, by the inequality in (28), we have

$$\lfloor \frac{v}{2^{j-1}} \rfloor = \lfloor \frac{y}{2^{j-1}} \rfloor. \tag{29}$$

Then, $v_{j-1} = y_{j-1}$, and hence by (27) and the fact that $x \oplus y \oplus z = 0$, we have $x_{j-1}+v_{j-1}+w_{j-1} = x_{j-1}+y_{j-1}+w_{j-1} \neq 0 \ (mod\ 2)$, and hence $x \oplus v \oplus w \neq 0$. □

Definition 3.3. Let $A_h = \{\{x, y, z\} : x, y, z \in Z_{\geq 0}, y \leq h(z)$ and $x \oplus y \oplus z = 0\}$ and $B_h = \{\{x, y, z\} : x, y, z \in Z_{\geq 0}, y \leq h(z)$ and $x \oplus y \oplus z \neq 0\}$.

Lemma 8. *Let A_h and B_h be the sets in Definition 3.3. Then the following hold:*
(i) If we start with a position in A_h, then any option (move) leads to a position in B_h.
(ii) If we start with a position in B_h, then there is at least one option (move) that leads to a position in A_h.

Proof. As $move_h(\{x, y, z\})$ in Definition 3.1 contains all the positions that can be reached from the position $\{x, y, z\}$ in one step, we have (i) and (ii) by Lemma 7 and Lemma 6, respectively. □

Theorem 3. *Let A_h and B_h be the sets in Definition 3.3. A_h is the set of $\mathcal{P}$-positions and B_h is the set of $\mathcal{N}$-positions of the disjunctive sum of the chocolate bar game with $CB(h, y, z)$ to the right of the bitter square and a single chocolate strip to the left.*

Proof. This follows directly from Lemma 8 and Theorem 2.13 in [1]. □

Theorem 4. *Let h be a function that satisfies the condition (a) in Definition 3.2. Then the Grundy number of $CB(h, y, z)$ is $y \oplus z$.*

Proof. By Theorem 3, a position $\{x, y, z\}$ of the sum of the chocolate bars is a $\mathcal{P}$-position when $x \oplus y \oplus z = 0$. Therefore, Theorem 1 implies that the Grundy number of the chocolate bar to the right is $x = y \oplus z$. □

By Theorem 4, the condition of Definition 3.2 is a sufficient condition whereby the chocolate bar $CB(h, y, z)$ may have Grundy number $G(\{y, z\}) = y \oplus z$. In the next subsection, we will prove that this condition is also necessary.

3.2. Necessary Condition Whereby a Chocolate Bar May Have Grundy Number $y \oplus z$

In Subsection 3.1, we proved that the Grundy number of $CB(h, y, z)$ is $G(\{y, z\}) = y \oplus z$ when the function h satisfies the condition (a) in Definition 3.2. In this subsection, we will prove that this condition on f is necessary for the chocolate bar $CB(f, y, z)$ to have Grundy number $y \oplus z$.

Definition 3.4. Let f be a monotonically increasing function of $Z_{\geq 0}$ into $Z_{\geq 0}$ that satisfies the following condition:
(a) We assume that $G(\{y,z\})$ is the Grundy number of the chocolate bar $CB(f,y,z)$. Then,

$$G(\{y,z\}) = y \oplus z.$$

Throughout this subsection, we assume that the function f satisfies the condition (a) in Definition 3.4, and G is the Grundy number for $CB(f,y,z)$. We will prove that f satisfies the condition (a) in Definition 3.2.

Lemma 9. *Let $y,z,y' \in Z_{\geq 0}$ such that $f(z) = y < y' \leq f(z+1)$. Then $G(\{y,z+1\}) < G(\{y',z+1\})$.*

Proof. As $f(w) \leq f(z) = y < y'$ for $w < z+1$, by Definition 2.1

$$\begin{aligned} move_f(\{y',z+1\}) =& \{\{v,z+1\} : v < y'\} \cup \{\{\min(y',f(w)),w\} : w < z+1\} \\ =& \{\{v,z+1\} : v < y'\} \cup \{\{f(w),w\} : w < z+1\} \\ \supset& \{\{v,z+1\} : v < y\} \cup \{\{f(w),w\} : w < z+1\} \\ =& \{\{v,z+1\} : v < y\} \cup \{\{\min(y,f(w)),w\} : w < z+1\} \\ =& move_f(\{y,z+1\}), \text{where } v,w \in Z_{\geq 0}. \end{aligned}$$

Therefore, by the definition of the Grundy number

$$\begin{aligned} G(\{y',z+1\}) &= mex(\{G(\{a,b\}) : \{a,b\} \in move_f(\{y',z+1\}\}) \\ &\geq mex(\{G(\{a,b\}) : \{a,b\} \in move_f(\{y,z+1\}\}) = G(\{y,z+1\}). \end{aligned} \tag{30}$$

As $\{y,z+1\} \in move_f(\{y',z+1\})$, $G(\{y',z+1\}) \neq G(\{y,z+1\})$. Therefore, the inequality in (30) implies $G(\{y,z+1\}) < G(\{y',z+1\})$. □

Lemma 10. *For any $y,z \in Z_{\geq 0}$ such that $y \leq f(z)$, we have $\{G(\{\min(y,f(w)),w\}) : w < z\} = \{y \oplus w : w < z\}$.*

Proof. Let m be a natural number such that $y,z < 2^m$. Then, for $w \in Z_{\geq 0}$ such that $w < z$, we have

$$y \oplus w < 2^m \leq y \oplus (z + 2^m) = G(\{y, z+2^m\}). \tag{31}$$

By the inequality in (31) and Lemma 3,

$$y \oplus w \in \{G(\{a,b\}) \in move_f(\{y,z+2^m\})\},$$

and hence by the definition of $move_f$, we have

$$y \oplus w \in \{G(\{y',z+2^m\}) : y' \in Z_{\geq 0} \text{ with } y' < y\} \tag{32}$$

or

$$y \oplus w \in \{G(\{\min(y, f(w')), w'\} : w' \in Z_{\geq 0} \text{ with } z \leq w' < z + 2^m\} \tag{33}$$

or

$$y \oplus w \in \{G(\{\min(y, f(w')), w'\} : w' \in Z_{\geq 0} \text{ with } w' < z\}. \tag{34}$$

As

$$G(\{y', z + 2^m\}) = y' \oplus (z + 2^m) \geq 2^m > y \oplus w \text{ for } y' < y,$$

(32) is not possible.

If

$$w' \geq z,$$

then $f(w') \geq f(z) \geq y$. Hence, by the fact that $w' \geq z > w$,

$$G(\{\min(y, f(w')), w'\}) = G(\{y, w'\}) = y \oplus w' \neq y \oplus w. \tag{35}$$

By the inequality in (35), (33) is false. Therefore, we have (34), and hence $\{y \oplus w : w < z\} \subset \{G(\{\min(y, f(w)), w\}) : w < z\}$. The number of elements in $\{y \oplus w : w < z\}$ is the same as the number of elements in $\{G(\{\min(y, f(w)), w\}) : w < z\}$, and hence $\{G(\{\min(y, f(w)), w\}) : w < z\} = \{y \oplus w : w < z\}$. □

Lemma 11. *Let $i \in Z_{\geq 0}$ and $z < z'$.*

(1)

$$\lfloor \frac{z}{2^i} \rfloor = \lfloor \frac{z'}{2^i} \rfloor$$

if and only if there exists $d \in Z_{\geq 0}$ such that

$$d \times 2^i \leq z < z' < (d+1) \times 2^i.$$

(2)

$$\lfloor \frac{z}{2^i} \rfloor < \lfloor \frac{z'}{2^i} \rfloor$$

if and only if there exist $c, s \in Z_{\geq 0}$ such that $s \geq i$ and

$$c \times 2^{s+1} \leq z < c \times 2^{s+1} + 2^s \leq z'.$$

Proof. (1) follows directly from the definition of the floor function.

(2)

$$\lfloor \frac{z}{2^i} \rfloor < \lfloor \frac{z'}{2^i} \rfloor$$

if and only if there exists s such that $s \geq i$ and $z_k = z'_k$ for $k = n, n-1, ..., s+1$ and $z_s = 0 < 1 = z'_s$. This condition is the same as

$$c \times 2^{s+1} \leq z < c \times 2^{s+1} + 2^s \leq z'$$

for some natural number c. □

Theorem 5. *Assume that a function f satisfies the condition (a) in Definition 3.4. Then f satisfies the condition (a) in Definition 3.2.*

Proof. It is sufficient to prove that

$$\lfloor \frac{f(a)}{2^{j-1}} \rfloor = \lfloor \frac{f(a+1)}{2^{j-1}} \rfloor$$

for $a \in Z_{\geq 0}$ such that

$$\lfloor \frac{a}{2^j} \rfloor = \lfloor \frac{a+1}{2^j} \rfloor. \tag{36}$$

We prove this by contradiction; thus, we assume

$$\lfloor \frac{f(a)}{2^{j-1}} \rfloor < \lfloor \frac{f(a+1)}{2^{j-1}} \rfloor \tag{37}$$

for $a \in Z_{\geq 0}$ that satisfies Equation (36). By the inequality in (37) and (2) of Lemma 11, there exist $i, c \in Z_{\geq 0}$ and $t \in R$ such that $i \geq j-1$, $0 \leq t < 2^i$, and

$$f(a) = c \times 2^{i+1} + t < c \times 2^{i+1} + 2^i \leq f(a+1). \tag{38}$$

By Equation (36) and $i + 1 \geq j$, we have

$$\lfloor \frac{a}{2^{i+1}} \rfloor = \lfloor \frac{a+1}{2^{i+1}} \rfloor.$$

Therefore, by (1) of Lemma 11, we have the following:

$$d \times 2^{i+1} \leq a < a+1 = d \times 2^{i+1} + 2^i + e < (d+1)2^{i+1} \tag{39}$$

or

$$d \times 2^{i+1} \leq a < a+1 = d \times 2^{i+1} + e < (d+1)2^{i+1} \tag{40}$$

for $d, e \in Z_{\geq 0}$ such that $0 < e < 2^i$.

`Case` (i) If we have the inequality in (39), then

$$\begin{aligned} G(\{c \times 2^{i+1} + 2^i, a+1\}) &= G(\{c \times 2^{i+1} + 2^i, d \times 2^{i+1} + 2^i + e\}) \\ &= (c \times 2^{i+1} + 2^i) \oplus (d \times 2^{i+1} + 2^i + e) \\ &= (c \oplus d)2^{i+1} + e < (c \oplus d)2^{i+1} + 2^i + (t \oplus e) \\ &= G(\{c \times 2^{i+1} + t, d \times 2^{i+1} + 2^i + e\}) \\ &= G(\{c \times 2^{i+1} + t, a+1\}). \end{aligned} \tag{41}$$

By the inequality in (38) and Lemma 9, we have $G(\{c \times 2^{i+1} + t, a+1\})$ $< G(\{c \times 2^{i+1} + 2^i, a+1\})$, which contradicts Equation (41).

`Case` (ii) If we have the inequality in (40), then by $e > 0$

$$\begin{aligned}G(\{c \times 2^{i+1} + 2^i, a+1\}) =&(c \times 2^{i+1} + 2^i) \oplus (d \times 2^{i+1} + e)\\ =&(c \oplus d)2^{i+1} + 2^i + e\\ >&(c \oplus d)2^{i+1} + 2^i.\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}(c \oplus d)2^{i+1} + 2^i \in&\{G(\{p,q\}) : \{p,q\} \in move_f(\{c \times 2^{i+1} + 2^i, a+1\})\}\\ =&\{G(\{v, a+1\}) : 0 \le v \le c \times 2^{i+1} + 2^i - 1\}\\ &\cup \{G(\{\min(c \times 2^{i+1} + 2^i, f(w)), w\}) : 0 \le w \le a\}.\end{aligned}$$

Therefore, we have the following two subcases:

`Subcase` $(ii.1)$ We assume that

$$(c \oplus d)2^{i+1} + 2^i \in \{G(\{v, a+1\}) : 0 \le v \le c \times 2^{i+1} + 2^i - 1\}. \tag{42}$$

Then, by Definition 3.4 and $a + 1 = d \times 2^{i+1} + e$,

$$\begin{aligned}&\{G(\{v, a+1\}) : 0 \le v \le c \times 2^{i+1} + 2^i - 1\}\\ =&\{v \oplus (d \times 2^{i+1} + e) : 0 \le v \le c \times 2^{i+1} + 2^i - 1\}.\end{aligned} \tag{43}$$

For v such that $0 \le v \le c \times 2^{i+1} + 2^i - 1$,

$$v \oplus (d \times 2^{i+1} + e) < (c \oplus d)2^{i+1} + 2^i,$$

and hence (42) and (43) lead to a contracicition.

`Subcase` $(ii.2)$ We assume that

$$(c \oplus d)2^{i+1} + 2^i \in \{G(\{\min(c \times 2^{i+1} + 2^i, f(w)), w\}) : 0 \le w \le a\}. \tag{44}$$

For $w \le a$, we have

$$f(w) \le f(a) = c \times 2^{i+1} + t < c \times 2^{i+1} + 2^i. \tag{45}$$

Hence,

$$\begin{aligned}&\{G(\{\min(c \times 2^{i+1} + 2^i, f(w)), w\}) : 0 \le w \le a\}\\ =&\{G(\{f(w), w\}) : 0 \le w \le a\}\\ =&\{f(w) \oplus w : 0 \le w \le a\}.\end{aligned} \tag{46}$$

For w such that $0 \le w \le a = d \times 2^{i+1} + e - 1$, by the inequality in (45)

$$f(w) \oplus w < (c \oplus d)2^{i+1} + 2^i,$$

and hence (44) and Equation (46) lead to a contracicition. □

By Theorem 5, the condition (a) in Definition 3.2 is a necessary condition whereby $CB(f, y, z)$ may have Grundy number $G(\{y, z\}) = y \oplus z$.

Corollary 1. *Let $h(z)$ be a function that satisfies the condition (a) in Definition 3.2. We have a disjunctive sum of a chocolate bar $CB(h, y, z)$ to the right of the bitter square and a single chocolate strip of length x to the left. Then the following hold:*
(i) The Grundy number of this disjunctive sum is $G(\{x, y, z\}) = x \oplus y \oplus z$.
(ii) $\{x, y, z\}$ is a $\mathcal{P}$-position if and only if $x \oplus y \oplus z = 0$.

Proof. By Theorem 4, the Grundy number of $CB(h, y, z)$ is $y \oplus z$, and hence the Grundy number of this disjunctive sum is $x \oplus y \oplus z$. Therefore, we have (i).
By (i) of Theorem 1 and (i) of this Corollary, we have (ii). ☐

For an example of a chocolate bar as in Corollary 1, see Figure 19.

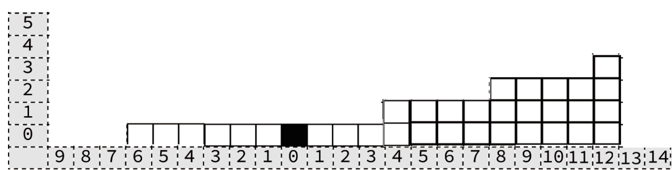

Figure 19: Single chocolate strip of length 6 and the chocolate bar $CB(f, 3, 12)$.

4. Chocolate Bar $CB(f_s, y, z)$ Whose Grundy Number is $G_{f_s}(\{y, z\}) = (y \oplus (z + s)) - s$ for a Fixed Natural Number s

In the previous sections, we studied chocolate bars $CB(f, y, z)$ whose Grundy number is $G(\{y, z\}) = y \oplus z$. The condition $G(\{y, z\}) = y \oplus z$ is quite strong, and thus it will be modified to $G(\{y, z\}) = (y \oplus (z + s)) - s$ for a fixed natural number s.

In this section, we will study a necessary and sufficient condition whereby a chocolate bar $CB(g, y, z)$ may have Grundy number $G_g(\{y, z\}) = (y \oplus (z + s)) - s$ for a fixed natural number s.

Remark 4.1. The difference between the two types of Grundy numbers $G(\{y, z\}) = y \oplus z$ and $G(\{y, z\}) = (y \oplus (z + s)) - s$ is demonstrated in Corollaries 1 and 3.

Example 4.1. By Lemma 4 and Theorem 4, the Grundy number of the chocolate bar in Figure 20 is

$$G_f(\{y, z\}) = y \oplus z, \tag{47}$$

where

$$f(t) = \lfloor \frac{t}{4} \rfloor. \tag{48}$$

However, Theorem 6 (which will be proved later) implies that the Grundy number of the chocolate bar in Figure 21 is

$$G_{f_{12}}(\{y, z\}) = (y \oplus (z + 12)) - 12, \tag{49}$$

where

$$f_{12}(t) = f(t + 12) = \lfloor \frac{t + 12}{4} \rfloor. \tag{50}$$

One should note the difference between Equations (47) and (49), and the difference between Equations (48) and (50).

Evidently, the chocolate bar in Figure 21 can be obtained by moving the bitter part horizontally and cutting the chocolate bar in Figure 20 vertically, as shown in Figure 22.

If we generalize this method, we obtain a necessary and sufficient condition whereby a chocolate bar $CB(g, y, z)$ may have Grundy number $G_g(\{y, z\}) = (y \oplus (z + s)) - s$ for a fixed natural number s.

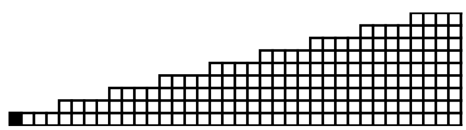

Figure 20: $CB(f, 8, 32)$ with $f(t) = \lfloor \frac{t}{4} \rfloor$.

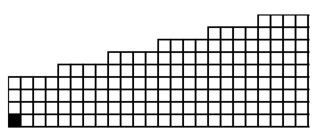

Figure 21: $CB(f_{12}, 8, 23)$ with $f_{12}(t) = f(t + 12) = \lfloor \frac{t+12}{4} \rfloor$.

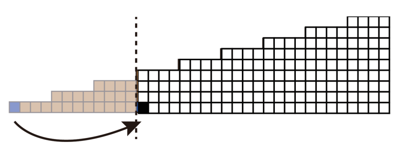

Figure 22.

As in Example 4.1, we can obtain chocolate bars with Grundy number $G_{f_s}(\{y, z\}) = (y \oplus (z+s)) - s$ from a chocolate bar whose Grundy number is $G_f(\{y, z\}) = y \oplus z$. First, we study a sufficient condition.

4.1. Sufficient Condition Whereby a Chocolate Bar May Have Grundy Number $(y \oplus (z + s)) - s$

Let h be a function that satisfies the condition (a) in Definition 3.2, and let $G_h(\{y, z\})$ be the Grundy number of $CB(h, y, z)$.

Definition 4.1. We define a function h_s as follows. Let s be a natural number such that for a natural number u and a non-negative integer v

$$\frac{s}{u} = 2^v > h(s). \tag{51}$$

Let $h_s(z) = h(z + s)$ for any $z \in Z_{\geq 0}$.

For an example of a number s that satisfies the inequality in (51) of Definition 4.1, see Corollary 2.

We will show that the condition in Definition 4.1 is a necessary and sufficient condition whereby the chocolate bar $CB(h_s, y, z)$ may have Grundy number $G_{h_s}(\{y, z\}) = (y \oplus (z + s)) - s$.

Lemma 12. *Let $p \in Z_{\geq 0}$ and s be a natural number. Then*

$$i \oplus s = i + s \text{ for } i = 0, 1, ..., p \tag{52}$$

if and only if there exist a natural number u and a non-negative integer v such that

$$\frac{s}{u} = 2^v > p. \tag{53}$$

Proof. We assume (52). When $p > 0$, there exists a natural number v such that

$$2^v > p \geq 2^{v-1} > \sum_{k=0}^{v-2} 2^k. \tag{54}$$

Let $s = \sum_{k=0}^{n} s_k 2^k$ with $s_k \in \{0, 1\}$. Then, by (52) and the inequality in (54), we have $2^{v-1} \oplus s = 2^{v-1} + s$ and $\sum_{k=0}^{v-2} 2^k \oplus s = \sum_{k=0}^{v-2} 2^k + s$, and hence we have $s_k = 0$ for $i = 0, 1, ..., v - 1$. Therefore, there exists a natural number u such that $s = u \times 2^v$, and we have the inequality in (53). When $p = 0$, we let $v = 0$ and $u = s$. Then we also have the inequality in (53).

We now assume that there exist a natural number u and a non-negative integer v that satisfy the inequality in (53). Then it is clear that we have (52). □

Lemma 13. *Let $p \in Z_{\geq 0}$ and s be a natural number such that $\frac{s}{u} = 2^v > p$ for a natural number u and $v \in Z_{\geq 0}$. Let $j \in Z_{\geq 0}$ such that $0 \leq j \leq p$. Then*

$$\{j \oplus i : i = 0, 1, 2, ..., s-1\} = \{0, 1, 2, ..., s-1\}. \tag{55}$$

Proof. Let $j \in Z_{\geq 0}$ such that $0 \leq j \leq p < 2^v$. For i such that $0 \leq i < s = u \times 2^v$, there exist $u', i' \in Z_{\geq 0}$ such that $i = u' \times 2^v + i'$, $u' < u$ and $i' < 2^v$. Then,

$$0 \leq j \oplus i = u' \times 2^v + j \oplus i' < u \times 2^v = s.$$

Therefore,

$$\{j \oplus i : i = 0, 1, 2, ..., s-1\} \subset \{0, 1, 2, ..., s-1\}. \tag{56}$$

As $j \oplus k \neq j \oplus h$ for $0 \leq k, h \leq s-1$ such that $k \neq h$, the number of elements of $\{j \oplus i : i = 0, 1, 2, ..., s-1\}$ is the same as that of $\{0, 1, 2, ..., s-1\}$. Therefore, (56) implies (55). □

Lemma 14. *Let s be a natural number, $x \in Z_{\geq 0}$ and $x_k \in Z_{\geq 0}$ for $k = 1, 2, ..., n$. We assume that $x, x_k \geq s$ for $k = 1, 2, ..., n$. Then $mex(\{x_k : k = 1, 2, ..., n\} \cup \{0, 1, 2, ..., s-1\}) = x$ if and only if $mex(\{x_k - s : k = 1, 2, ..., n\}) = x - s$.*

Proof. For $x \geq s$, we have

$$x = mex(\{x_k : k = 1, 2, ..., n\} \cup \{0, 1, 2, ..., s-1\})$$

if and only if

$$x \neq x_k \text{ for } k = 1, 2, ..., n \text{ and } \{0, 1, ..., x-1\} \subset \{x_k : k = 1, 2, ..., n\} \cup \{0, 1, 2, ..., s-1\}$$

if and only if

$$x_k \neq x \text{ for } k = 1, 2, ..., n \text{ and } \{s, s+1, ..., x-1\} \subset \{x_k : k = 1, 2, ..., n\}$$

if and only if

$$x - s \neq x_k - s \text{ for } k = 1, 2, ..., n \text{ and } \{0, 1, ..., x-s-1\} \subset \{x_k - s : k = 1, 2, ..., n\}$$

if and only if

$$x - s = mex(\{x_k - s : k = 1, 2, ..., n\}).$$

□

Lemma 15. *Let s be a natural number that satisfies the inequality in (51) of Definition 4.1. Then, for any $y, z \in Z_{\geq 0}$ such that $y \leq h(z+s)$, we have*

$$\begin{aligned} y \oplus (z+s) =& G_h(\{y, z+s\}) = mex(\{v \oplus (z+s) : v < y\} \cup \{0, 1, 2, ..., s-1\} \\ & \cup \{\min(y, h(w)) \oplus w : s \leq w < z+s\}). \end{aligned} \tag{57}$$

In particular, $y \oplus (z+s) \geq s$.

Proof. By Theorem 4 and the definition of the Grundy number,

$$\begin{aligned} y \oplus (z+s) =& G_h(\{y, z+s\}) \\ =& mex(\{G_h(\{v, z+s\}) : v < y\} \cup \{G_h(\{\min(y, h(w)), w\}) : w < z+s\}) \\ =& mex(\{G_h(\{v, z+s\}) : v < y\} \cup \{G_h(\{\min(y, h(w)), w\}) : w < s\} \\ & \cup \{G_h(\{\min(y, h(w)), w\}) : s \le w < z+s\}) \\ =& mex(\{v \oplus (z+s) : v < y\} \cup \{G_h(\{\min(y, h(w)), w\}) : w < s\} \\ & \cup \{\min(y, h(w)) \oplus w : s \le w < z+s\}). \end{aligned} \tag{58}$$

When $w < s$, we have $h(w) \le h(s)$, and hence

$$\{G_h(\{\min(y, h(w)), w\}) : w < s\} = \{G_h(\{\min(y, h(s), h(w)), w\}) : w < s\}. \tag{59}$$

Next, we use Lemma 10; however, instead of y, z, and f, we use $\min(y, h(s))$, s, and h. Then by $\min(y, h(s)) \le h(s)$, we have

$$\{G_h(\{\min(y, h(s), h(w)), w\}) : w < s\} = \{\min(y, h(s)) \oplus w : w < s\}. \tag{60}$$

Next, we use Lemma 13; however, instead of j and p, we use $\min(y, h(s))$ and $h(s)$. Then, by $\min(y, h(s)) \le h(s)$ and Definition 4.1, we have

$$\{\min(y, h(s)) \oplus w : w < s\} = \{0, 1, 2, ..., s-1\}. \tag{61}$$

By Equations (58), (59), (60), and (61), we have Equation (57). Therefore, by Lemma 2, we have $y \oplus (z+s) \ge s$. □

Lemma 16. *Let s be a natural number that satisfies the inequality in (51) of Definition 4.1. For any $y, z \in Z_{\ge 0}$ such that $y \le h(z+s)$, we have*

$$\begin{aligned} (y \oplus (z+s)) - s = mex(&\{(v \oplus (z+s)) - s : v < y\} \\ & \cup \{(\min(y, h(w)) \oplus w) - s : s \le w < z+s\}). \end{aligned} \tag{62}$$

Proof. By Lemma 15, we have

$$\begin{aligned} G_h(\{y, z+s\}) =& y \oplus (z+s) \\ =& mex(\{v \oplus (z+s) : v < y\} \cup \{0, 1, 2, ..., s-1\} \\ & \cup \{\min(y, h(w)) \oplus w : s \le w < z+s\}) \\ =& mex(\{v \oplus (z+s) : v < y\} \cup \{0, 1, 2, ..., s-1\} \\ & \cup \{\min(y, h(w'+s)) \oplus (w'+s) : 0 \le w' < z\}) \end{aligned} \tag{63}$$

and

$$y \oplus (z+s) \ge s. \tag{64}$$

for any $y, z \in Z_{\geq 0}$ such that $y \leq h(z+s)$.

Next, we use Lemma 15; however, instead of y, we use v such that $v < y$. Then, by $v < y \leq h(z+s)$, we have

$$v \oplus (z+s) \geq s. \tag{65}$$

We use Lemma 15 again. Instead of y and z used in this lemma, we use $\min(y, h(w'+s))$ and w'. Then, by $\min(y, h(w'+s)) \leq h(w'+s)$, we have for $0 \leq w' < z$

$$\min(y, h(w'+s)) \oplus (w'+s) \geq s. \tag{66}$$

Lemma 14, Equation (63), and the inequalities in (64), (65), and (66) imply Equation (62). We have completed the proof. □

Theorem 6. *Let s be a natural number that satisfies the inequality in (51) of Definition 4.1 and $h_s(z) = h(z+s)$ for any $z \in Z_{\geq 0}$. Let $G_{h_s}(\{y,z\})$ be the Grundy number of $CB(h_s, y, z)$. Then $G_{h_s}(\{y,z\}) = (y \oplus (z+s)) - s$ for any $y, z \in Z_{\geq 0}$ such that $y \leq h_s(z)$.*

Proof. Let $y, z \in Z_{\geq 0}$ such that $y \leq h_s(z)$. We proceed by mathematical induction, and we assume that $G_{h_s}(\{v,w\}) = (v \oplus (w+s)) - s$ for $v, w \in Z_{\geq 0}$ such that $v \leq y, w < z$ or $v < y, w \leq z$.

$$\begin{aligned}
G_{h_s}(\{y,z\}) =& mex(\{G_{h_s}(\{v,z\}) : v < y\} \cup \{G_{h_s}(\{\min(y, h_s(w)), w\}) : w < z\}) \\
=& mex(\{(v \oplus (z+s)) - s : v < y\} \\
& \cup \{(\min(y, h(w+s)) \oplus (w+s)) - s : w < z\}) \\
=& mex(\{(v \oplus (z+s)) - s : v < y\} \\
& \cup \{(\min(y, h(w+s)) \oplus (w+s)) - s : s \leq w+s < z+s\}) \\
=& mex(\{(v \oplus (z+s)) - s : v < y\} \\
& \cup \{(\min(y, h(w')) \oplus w') - s : s \leq w' < z+s\}) \\
=& (y \oplus (z+s)) - s,
\end{aligned}$$

where the last equation follows directly from Lemma 16. □

4.2. Necessary Condition Whereby a Chocolate Bar May Have Grundy Number $(y \oplus (z+s)) - s$.

In this subsection, we study a necessary condition whereby a chocolate bar may have Grundy number $(y \oplus (z+s)) - s$ for a natural number s.

Definition 4.2. Let s be a fixed natural number and g be a function that satisfies the following three conditions:
(i) $g(t) \in Z_{\geq 0}$ for $t \in Z_{\geq 0}$.
(ii) g is monotonically increasing.
(iii) The Grundy number of $CB(g, y, z)$ is $G_g(\{y,z\}) = (y \oplus (z+s)) - s$.

We will show that there exists a function h such that $g(z) = h(z+s)$ for any $z \in Z_{\geq 0}$,

$$\frac{s}{u} = 2^v > h(s)$$

for $v \in Z_{\geq 0}$, and the Grundy number of $CB(h, y, z)$ is $G_h(\{y, z\}) = y \oplus z$.

Lemma 17. *Let s be a natural number and g a function such that the conditions of Definition 4.2 are satisfied. Then we have $i \oplus s = i + s$ for $i = 0, 1, 2, ..., g(0)$.*

Proof. First, we prove that

$$G_g(\{i, 0\}) = i \tag{67}$$

for $i = 0, 1, 2, ..., g(0)$ by mathematical induction. By the definition of the Grundy number, $G_g(\{0, 0\}) = 0$. We assume that $G_g(\{k, 0\}) = k$ for $k = 0, 1, 2, ...i-1$ and $i \leq g(0)$.

By the definition of the Grundy number, $G_g(\{i, 0\}) = mex(\{G_g(\{k, 0\}) : k = 0, 1, 2, ..., i-1\} = mex(\{0, 1, 2, ..., i-1\}) = i$. By the conditions of Definition 4.2, we have $G_g(\{i, 0\}) = (i \oplus s) - s$, and hence Equation (67) implies $(i \oplus s) - s = i$. Therefore, we have completed the proof. □

Theorem 7. *Let s be a natural number and g a function such that the conditions of Definition 4.2 are satisfied. We define a function g_{-s} by $g_{-s}(z) = g(z-s)$ for $z > s$ and $g_{-s}(z) = g(0)$ for $0 \leq z \leq s$. Let $G_{g_{-s}}(\{y, z\})$ be the Grundy number of $CB(g_{-s}, y, z)$. Then $G_{g_{-s}}(\{y, z\}) = y \oplus z$ for any $y, z \in Z_{\geq 0}$ such that $y \leq g_{-s}(z)$.*

Proof. Case (1) By the definition of g_{-s}, we have $g_{-s}(z) = g(0)$ for $z \leq s$, and hence the function g_{-s} is a constant function for $z \leq s$ and thus satisfies the condition of Definition 3.2. Therefore, $G_{g_{-s}}(\{y, z\}) = y \oplus z$ for any $y, z \in Z_{\geq 0}$ such that $y \leq g_{-s}(z)$ and $z \leq s$.

Case (2) We prove that $G_{g_{-s}}(\{y, z+s\}) = y \oplus (z+s)$ for $y \leq g_{-s}(z+s)$. We proceed by mathematical induction, and we assume that $G_{g_{-s}}(\{v, w\}) = v \oplus w$ for $v, w \in Z_{\geq 0}$ such that $v \leq y, w < z+s$ or $v < y, w \leq z+s$.

By Lemma 17, we have $i \oplus s = i + s$ for $i = 0, 1, 2, ..., g(0)$. By Lemma 12, there exist a natural number u and a non-negative integer v such that

$$\frac{s}{u} = 2^v > g(0).$$

Here, we use Lemma 13; however, instead of p and j, we use $g(0)$ and $\min(y, g(0))$. Then, by $\min(y, g(0)) \leq g(0)$, we have that

$$\text{the set}\{\min(y, g(0)) \oplus w : w < s\} \text{ is the same as the set } \{0, 1, 2, ..., s-1\}. \tag{68}$$

By Definition 4.2 and the definition of g_{-s}, for y, z such that $y \leq g(z)$

$$\begin{aligned}(y \oplus (z+s)) - s =& G_g(\{y,z\}) \\ =& mex(\{G_g(\{v,z\}) : v < y\} \cup \{G_g(\{\min(y, g(w)), w\}) : w < z\}) \\ =& mex(\{(v \oplus (z+s)) - s : v < y\} \\ & \cup \{(\min(y, g(w)) \oplus (w+s)) - s : w < z\}) \\ =& mex(\{(v \oplus (z+s)) - s : v < y\} \\ & \cup \{(\min(y, g_{-s}(w+s)) \oplus (w+s)) - s : w < z\}). \end{aligned} \tag{69}$$

As

$$(v \oplus (z+s)) - s = G_g(\{v,z\}) \geq 0 \text{ for } v < y$$

and

$$(\min(y, g_{-s}(w+s)) \oplus (w+s)) - s = G_g(\{\min(y, g(w)), w\}) \geq 0 \text{ for } w < z,$$

we have

$$(v \oplus (z+s)) \geq s \text{ for } v < y \tag{70}$$

and

$$(\min(y, g_{-s}(w+s)) \oplus (w+s)) \geq s \text{ for } w < z. \tag{71}$$

By Equation (69), the inequalities in (70) and (71), and Lemma 14, we have

$$\begin{aligned} y \oplus (z+s) =& mex(\{v \oplus (z+s) : v < y\} \\ & \cup \{\min(y, g_{-s}(w+s)) \oplus (w+s) : w < z\} \cup \{0, 1, ..., s-1\}). \end{aligned} \tag{72}$$

As $g_{-s}(w) = g(0)$ for $0 \leq w < s$, (68) implies

$$\{\min(y, g_{-s}(w)) \oplus w : w < s\} = \{\min(y, g(0)) \oplus w : w < s\} = \{0, 1, 2, ..., s-1\}. \tag{73}$$

By the inductive hypothesis and Equation (73),

$$\begin{aligned} & G_{g_{-s}}(\{y, z+s\}) \\ =& mex(\{G_{g_{-s}}(\{v, z+s\}) : v < y\} \\ & \cup \{G_{g_{-s}}(\{\min(y, g_{-s}(w)), w\}) : s \leq w < z+s\}) \\ & \cup \{G_{g_{-s}}(\{\min(y, g_{-s}(w)), w\}) : w < s\} \\ =& mex(\{v \oplus (z+s) : v < y\} \cup \{\min(y, g_{-s}(w)) \oplus w : s \leq w < z+s\} \\ & \cup \{\min(y, g_{-s}(w)) \oplus w : w < s\}) \\ =& mex(\{v \oplus (z+s) : v < y\} \cup \{\min(y, g_{-s}(w)) \oplus w : s \leq w < z+s\} \\ & \cup \{0, 1, 2, ..., s-1\}) \\ =& mex(\{v \oplus (z+s) : v < y\} \cup \{\min(y, g_{-s}(w+s)) \oplus (w+s) : w < z\} \\ & \cup \{0, 1, 2, ..., s-1\}). \end{aligned} \tag{74}$$

By Equations (72) and (74), we have $G_{g_{-s}}(\{y, z+s\}) = y \oplus (z+s)$. $\square$

Theorems 6 and 7 prove the following propositions (i) and (ii), respectively.
(i) Let h be a function such that the Grundy number of the chocolate bar $CB(h, y, z)$ is $G_h(\{y, z\}) = y \oplus z$. Then the Grundy number of the chocolate bar $CB(h_s, y, z)$ is $G_{h_s}(\{y, z\}) = (y \oplus (z+s)) - s$, where s satisfies the inequality in (51) and $h_s(z) = h(z+s)$.
(ii) Let g be a function such that the Grundy number of the chocolate bar $CB(g, y, z)$ is $G_g(\{y, z\}) = (y \oplus (z+s)) - s$. Then the Grundy number of the chocolate bar $CB(g_{-s}, y, z)$ is $G_{g_{-s}}(\{y, z\}) = y \oplus z$, where $g_{-s}(z) = h(z-s)$. We note that $g = (g_{-s})_s$.

Therefore, we have a necessary and sufficient condition whereby the chocolate bar $CB(h, y, z)$ may have Grundy number $G_{h_s}(\{y, z\}) = (y \oplus (z+s)) - s$.

An example of this condition is now presented for the function $h(z) = \lfloor \frac{z}{2k} \rfloor$. Notably, the condition is quite simple.

Corollary 2. *Let $h(z) = \lfloor \frac{z}{2k} \rfloor$ for a fixed natural number k. Then*

$$s = m2^v \text{ for } v, m \in Z_{\geq 0} \text{ such that } m = 0, 1, 2, ..., 2k-1 \tag{75}$$

if and only if the Grundy number of $CB(h_s, y, z)$ is $(y \oplus (z+s)) - s$, where $h_s(z) = \lfloor \frac{z+s}{2k} \rfloor$.

Proof. By Lemma 4, the function h satisfies the conditions of Definition 3.2. By Theorem 7 and 6,

$$G_{h_s}(\{y, z\}) = (y \oplus (z+s)) - s$$

if and only if there exists $u \in Z_{\geq 0}$ such that

$$\frac{s}{u} = 2^v > \lfloor \frac{s}{2k} \rfloor = h(s)$$

if and only if (75) holds. $\square$

Corollary 3. *Let $h_s(z)$ be a function as in Definition 4.1. We have a disjunctive sum of a chocolate bar $CB(h_s, y, z)$ to the right of the bitter square and a single chocolate strip of length x to the left. Then, $\{x, y, z\}$ is a $\mathcal{P}$-position if and only if*

$$(x+2) \oplus y \oplus (z+2) = 0. \tag{76}$$

Proof. From Theorem 6, the Grundy number of $CB(h_s, y, z)$ is $(y \oplus (z+2)) - 2$; hence, the Grundy number of this disjunctive sum is $x \oplus ((y \oplus (z+2)) - 2)$. Therefore, $\{x, y, z\}$ is a $\mathcal{P}$-position if and only if $x \oplus ((y \oplus (z+2)) - 2) = 0$ if and only if $x = (y \oplus (z+2)) - 2$ if and only if $(x+2) \oplus y \oplus (z+2) = 0$. $\square$

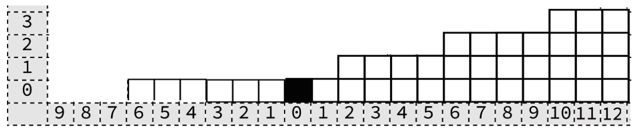

Figure 23: $\{6, 3, 12\}$ with $h_2(z) = \lfloor \frac{z+2}{4} \rfloor$.

An example of such a chocolate bar is shown in Figure 23.

Acknowledgements. The authors would like to express their thanks to the anonymous referee whose time and patience improved the quality of this work. Finally, many thanks to the Integers staff for the time which they put into this paper. This research was also supported by JSPS grant number 19H00225.

References

[1] M. H. Albert, R. J. Nowakowski and D. Wolfe, *Lessons In Play: An Introduction to Combinatorial Game Theory, Second Edition*, A K Peters/CRC Press, Natick, MA., United States, 2019.

[2] S. Nakamura and R. Miyadera, Impartial chocolate bar games, *Integers* **15** (2015), #G4.

[3] A. C. Robin, A poisoned chocolate problem, *Math. Gaz.* **73**, No. 466 (1989), 341-343.

[4] A. N. Siegel, *Combinatorial Game Theory*, volume 146 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2013.

[5] D. Zeilberger, Three-rowed CHOMP, *Adv. Appl. Math.* **26** (2001), 168-179.