ATOMIC WEIGHT CALCULUS OF SPINDLY GAMES

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Abstract

In this paper, we introduce spindly rulesets—in each non-terminal position both players have one move and one of these moves is to end the game. SUBVERSION is played with an ordered pair (a, b) of non-negative integers. From (a, b), one player can only move to $(a, b - a)$ and the other to $(a - b, b)$. If one of the ordered pair is less than or equal to 0 then no further moves are possible in this pair. We show that every spindly position is equal to a position in SUBVERSION. We also use the atomic weight calculus to determine a given position's outcome class. Generally, atomic weight calculations can be very onerous, relying on the recursive computations of atomic weights of the parent game's followers. For spindly games, we reduce the calculations to a table look-up, moreover, given a SUBVERSION position (and hence any spindly game) (a, b), $a \geqslant b$, we present a procedure which utilizes the Euclidean algorithm, the continued fraction representation of $\frac{a}{b}$, to compute the atomic weight of (a, b).

1. Introduction

The quest is to analyze games whose components are simple. Here, ‘simple’ means in each non-terminated component, each player has at most one move. This paper is a small step toward that goal because we replace ‘simple’ by ‘very simple’—one of the two moves is to terminate the component. As an example, consider the game SUBVERSION: the position is an ordered pair of integers (a, b). The player Left can only move to $(a, b - a)$ and Right to $(a - b, b)$. If one of the ordered pair is less than or equal to 0 then no further moves are possible in this pair. SUBVERSION on $(5, 13)$ may not engender much interest but when playing with the pairs $(5, 13), (6, 19), (23, 11)$ it is not an easy decision to choose the correct component in which to play and win. In general, the exact values (canonical forms) are too complicated and bring no useful reductions that help human players. Instead, we use an approximation, atomic weight, of the value. Unfortunately, atomic weight calculations have exceptional cases that must be considered in each individual position. However, because of the special structure of these games, we develop an algorithm that reduces all the calculations, including the exceptions, to a table look-up. In addition, we show that every game with ‘very simple’ components is equivalent to replacing each component by an equivalent SUBVERSION position. This is (imperfectly) analogous to the situation for impartial games where every impartial position is equivalent to a single component of the game NIM.

In the rest of this section, the concepts are introduced with more care and connections with existing research are noted. See [2, 4, 6, 16] for further background. Section 2 gives the reduction of components to SUBVERSION positions and gives examples of other games in the class. Section 3 explains the use of atomic weights. Section 4 gives our main result, the reduction of the calculations to a table look-up. Section 5 presents the algorithm based on the continued fraction of a/b for the SUBVERSION position (a, b), then Section 6 presents several examples.

Combinatorial game theory (CGT) studies perfect information rulesets in which there are no chance devices (e.g. dice) and two players take turns moving alternately. The *options* of a game are all those positions which can be reached in one move. Using the standard notation for combinatorial game theory where Left (female) and Right (male) are the players, games can be expressed recursively as $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ where $G^{\mathcal{L}}$ are the Left options and $G^{\mathcal{R}}$ are the Right options of G. We distinguish between multiple meanings of the word *game* by using the words *ruleset* and *game*. The word *ruleset* has a concrete meaning related to some particular set of rules (what is called a “game” informally). The word *game*, by contrast, has the abstract mathematical meaning defined by Conway [4, 6]. When we speak of the *value* of a game, we are emphasizing that it is being considered in this latter sense, as an algebraic object which can be compared for equality with, or added to, other games.

We will often use the term *position* for a specific instance of a game.

In a *binary* game, on a turn each player has at most one move. In an *all-small* game either both players have a move or the game is over[1]. Thus, in a binary, all-small game each player has exactly one move or none at all[2]. We introduce *spindly* games: these are binary, all-small games in which at least one of the two options is to end the game.

Binary all-small games have already received attention [7, 15] when playing the *misère* convention (the last player to move loses). Under the *normal play* winning convention (last player to move wins) even finding a simple function that gives the outcome class can be difficult (see [13]). See [1, 3, 11] for the analysis of other all-small, but not necessarily binary, rulesets, and [12] for a general approach to a large subclass.

Often, in Combinatorial Game Theory, positions decompose into several sub-positions. Then, since each move in such a situation involves choosing an option of a single sub-position, the formalization of *disjunctive sum* is needed: $G+H = \{G^{\mathcal{L}}+H, G+H^{\mathcal{L}} \,|\, G^{\mathcal{R}}+H, G+H^{\mathcal{R}}\}$. An example of a combinatorial ruleset, where a player makes a choice of a sub-position (a pile) and moves in it, is the classic NIM, first studied by C. Bouton [5]. The values involved in NIM, needed in the second section of this paper, are called nimbers (or stars):

$$*k = \{0, *, \ldots, *(k-1) \,|\, 0, *, \ldots, *(k-1)\}.$$

The value of an all-small game is an infinitesimal—no all-small position is greater than a position in which Left has exactly one move and Right none (see [2, 4, 6, 16]). The only all-small number is 0. Even so, there are advantages that can be expressed as "the number of times that a player can wait and still win". For instance, in the game $\uparrow = \{0 \,|\, *\}$, if Left moves then she can win. But, if she *waits*, giving the turn to Right, she still can win; she can wait one move. Sometimes, the waiting effect can only be observed in the presence of a sufficiently *remote* star (☆). For instance, in the game $\{0, *, *2 \,|\, 0, *\}$, Left cannot wait, however, in $\{0, *, *2 \,|\, 0, *\} + *3$, she can wait. In the last example, it is enough to add $*n$, such that $n \geqslant 3$. This motivates the definition of *equivalence under remote star*: $G \sim_{☆} H$ if there exists $N \geq 0$ such that for all $n \geq N$, the outcome of $G + *n$ is equal to the outcome of $H + *n$. Considering this equivalence, all the games in the equivalence class of $\uparrow$ have the property that Left can wait once in the presence of a remote star.

In [16], it is shown that "remote enough" for G means that $*n$ is not an option of any position reachable from G. For spindly games, we may take ☆ to be $*2$ since $*n$ has n Left and Right options.

[1] In misère play, all-small games are not infinitesimals and have been called dicots.

[2] Whilst this may seem like a restricted class of games, it can be shown that any game is equivalent to a game with three or fewer options.

Consider the disjunctive sum of the SUBVERSION positions $(211, 155) + (5, 4) + (13, 17)$. After simplifying this position with the help of [17], we get the following "interesting" form:

$\{\{0^3|3.\downarrow||0|||\downarrow, \{\downarrow, \Downarrow*|||\Downarrow*, 3.\downarrow||3.\downarrow|6.\downarrow\}|3.\downarrow\}, \{0^3|3.\downarrow||0|||\{0^3|3.\downarrow||0|||\{0|||0^3|3.\downarrow||0$
$||||0, \{0^3|3.\downarrow||0\}\}|0||||0^2\}\}|\{\{0^3|3.\downarrow||0|||\{0|||0^3|3.\downarrow||0||||0, \{0^3|3.\downarrow||0\}\}|0||||0^2\}||||\downarrow,$
$\{\downarrow, \Downarrow*|||\Downarrow*, 3.\downarrow||3.\downarrow|6.\downarrow\}|3.\downarrow||\{*, \downarrow||\downarrow, \{\downarrow, \Downarrow*|||\Downarrow*, 3.\downarrow||3.\downarrow|6.\downarrow\}|3.\downarrow|||3.\downarrow, \{\downarrow, \{\downarrow, \Downarrow*|||$
$\Downarrow*, 3.\downarrow||3.\downarrow|6.\downarrow\}|3.\downarrow\}||||3.\downarrow\}|||0^2\}, \{\downarrow, \{\downarrow, \Downarrow*|||\Downarrow*, 3.\downarrow||3.\downarrow|6.\downarrow\}|3.\downarrow||||\downarrow, \{\downarrow, \Downarrow*|||\Downarrow*,$
$3.\downarrow||3.\downarrow|6.\downarrow\}|3.\downarrow||\{*, \downarrow||\downarrow, \{\downarrow, \Downarrow*|||\Downarrow*, 3.\downarrow||3.\downarrow|6.\downarrow\}|3.\downarrow|||3.\downarrow, \{\downarrow, \{\downarrow, \Downarrow*|||\Downarrow*, 3.\downarrow||3.\downarrow|$
$6.\downarrow\}|3.\downarrow\}||||3.\downarrow\}|||0^2\}\}$, where $\{0^k \mid G\}$ means $\{0 \parallel \{0^{k-1} \mid G\}\}$.

The above canonical form is a hard to parse. Clearly, a practical human interpretation is needed and can be found in the idea of "how long must a player wait before having to move to preserve the win?" This is what atomic weight measures. Atomic weight is a very useful and important tool. It has been used to prove a complexity result [8], and is featured in [8, 9, 10].

The game $\uparrow = \{0 \,|\, *\}$, the simplest positive all-small value, is the natural unit to make this "waiting measurement" in the sense that it introduces one waiting move. Therefore, good choices for representatives of the equivalence classes are multiples of up, $g.\uparrow$. If a game is equivalent under star to $3.\uparrow$, this means that Left can wait three times in the presence of a sufficiently remote star. However, there are "hot" games in the atomic sense like $\{5.\uparrow \,|\, 7.\downarrow\}$. Also, there are fractional behaviors such as $\{\Uparrow \,|\, \downarrow\}$ (Left needs two copies of this form to attain the possibility of waiting one time in the presence of a remote star). Because of this, the suitable set for the multipliers of $\uparrow$ is not $\mathbb{Z}$, but the set of all games themselves.

Using the Norton product to make sense of what it means to multiply games, the *atomic weight* of G is defined as the "coefficient" of $\uparrow$ in $g.\uparrow$, where $g.\uparrow$ is equivalent under remote star to G. There are four fundamental results that we will use in this work. See the technical details and proofs in [16], particularly the fact that, for each all-small G, $\mathrm{aw}(G) = g$ exists and is unique.

For this paper, we give the explicit definition.

Definition 1.1. The atomic weight of G, written $aw(G)$, is the game g if and only if $\downarrow\text{☆} < G - g.\uparrow < \uparrow\text{☆}$.

The atomic weight of G is another game g. Plotting the value of a game is not possible, but the mean value of a game is a number. For SUBVERSION, in Figure 1 we plot the mean values of the atomic weight of (a, b) for $0 \leq a < b < 500$. This shows some interesting patterns.

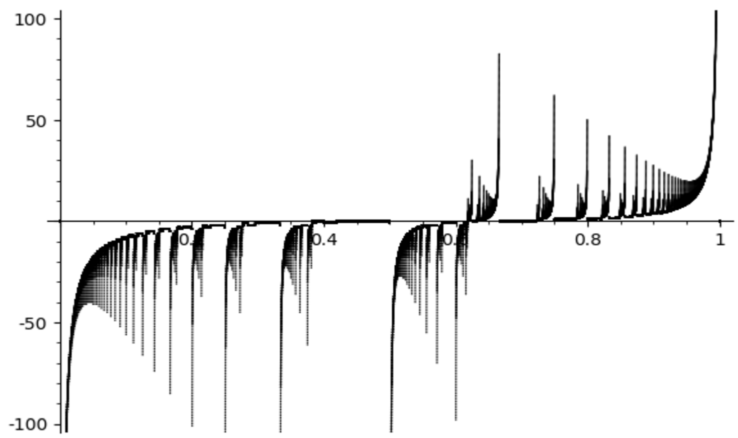


Figure 1: Mean Atomic Weight versus a/b, $0 \leq a \leq b \leq 500$, in Subversion

2. Universality of Subversion

For ease of notation, we will only use non-negative numbers in the description of a SUBVERSION position. For example, the options from $(3, 4)$ will be written as $(3, 1)$ and $(0, 4)$.

Theorem 2.1. *Every spindly position is equivalent to a* SUBVERSION *position.*

Proof. The *trunk* of a spindly game is obtained as follows: the original position is on the trunk; if G is on the trunk and $G = \{0 \mid G^R\}$ then G^R is on the trunk; if $G = \{G^L \mid 0\}$ then G^L is on the trunk. Note that the trunk ends with the game $G = *$ and both G^L and G^R are on the trunk and are called the left and right *terminals* respectively; the node above them is called the *stem.*

Given a spindly game G, we recursively identify positions on the the trunk with SUBVERSION positions then show that the two games are equivalent.

The left and right terminal positions are labelled $[1, 0]$ and $[0, 1]$, respectively, and the stem is labelled $[1, 1]$. Recursively, if H has the left option 0 and the trunk option is labelled $[a, b]$ then H is labelled $[a + b, b]$ and the left option is labelled $[a + b, 0]$. If H has the right option 0 and the trunk option labelled $[a, b]$ then H is labelled $[a, a + b]$ and the right option is labelled $[0, a + b]$. By this construction, the label on any node on the trunk has two positive entries.

`Claim:` A position labelled $[a, b]$ is equivalent to the SUBVERSION position (a, b).

Proof of Claim. We show that the two game trees are isomorphic.

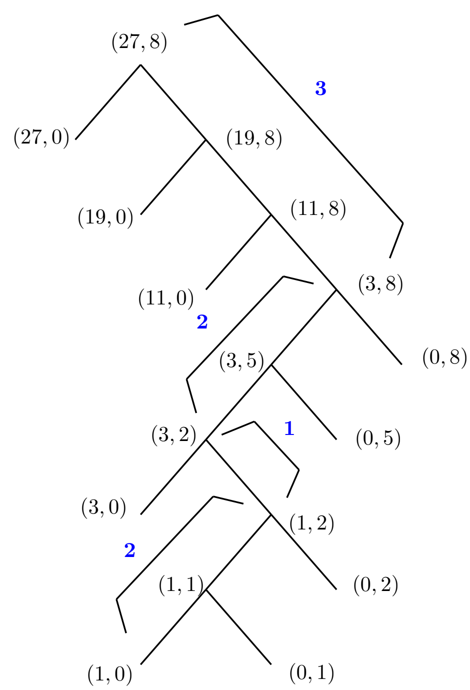
Clearly, the terminals and the stem have the same game trees as $(1, 0)$, $(0, 1)$ and

$(1, 1)$.

Given some $k > 1$, we may assume that all the trunk nodes of less height have isomorphic game trees to the purported SUBVERSION position.

Now consider a node on the trunk at height k with label $[a, b]$. Assume that the trunk option of $[a, b]$ is $[a, b - a]$. By construction, it follows that $a < b$ and the non-trunk, i.e., terminal, option is $[0, b]$ which is terminal and equal to the position $(0, b)$. By induction, the trunk option $[a, b - a]$ and $(a, b - a)$ have have isomorphic game trees. The argument is similar if the trunk option is $[a - b, b]$. The Claim now follows by induction. □

By considering it as a SUBVERSION position, a spindly tree can be described by a modified continued fraction. As an example, consider $(27, 8)$; Right has to subtract 8 three times from 27 before having an opportunity to win the game. This happens because $27 = 3 \times 8 + 3$. The tree of the game allows us to observe this fact:



After subtracting 8 three times from 27, we reach a turning point and we have to divide 8 by the remainder 3 to continue the process. Hence, if $a \geqslant b$, then the game tree of (a, b) is obtained by applying the Euclidean algorithm to the pair (a, b). Another way to think of this process is in terms of the continued fraction representation

$$\frac{a}{b} = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \cdots}}} = \big[n_1, n_2, n_3, \ldots, n_k\big].$$

Note that the n_1, n_2,..., n_k give us the lengths of the "tree arms" and the locations of the turning points. The continued fraction representation for $\frac{27}{8}$ is $\big[3, 2, 1, 2\big]$; this representation uniquely determines the shape of the game tree.

Alternatively, we use the list $\big[n_1, -n_2, n_3, -n_4, \ldots, \pm n_k\big]$ to represent a SUBVERSION position $(a, b), a \geqslant b$, where $n_1, n_2, n_3, \ldots, n_k$ are obtained with the Euclidean algorithm and the signs indicate the direction of the arms of the tree.

3. Atomic Weight Calculus

The following results can be found in [2, 4, 6, 16]. They are included to give a context for the subsequent results.

Theorem 3.1. *(Additivity of the atomic weight) We have* $\mathrm{aw}(G + H) = \mathrm{aw}(G) + \mathrm{aw}(H)$*.*

This result allows for the computation of the atomic weight of a disjunctive sum, knowing the atomic weights of the components.

Theorem 3.2. *(Two ahead rule)*

1. *If* $\mathrm{aw}(G) \geqslant 2$*, then* $G > 0$*.*
2. *If* $\mathrm{aw}(G) \geqslant 1$*, then* $G \mathrel{|\!\!\rhd} 0$*.*
3. *If* $\mathrm{aw}(G) \leqslant -2$*, then* $G < 0$*.*
4. *If* $\mathrm{aw}(G) \leqslant -1$*, then* $G \mathrel{\lhd\!\!|} 0$*.*

This result relates the atomic weight of G to its outcome.

The calculation of the atomic weight of a game in terms of its options follows the same scheme as games in general but there is an exceptional case that occurs where extra care is required.

Theorem 3.3. *(Atomic weight calculus) Let G be an all-small game and let*

$$w(G) = \{\mathrm{aw}(G^L) - 2 \mid \mathrm{aw}(G^R) + 2\}.$$

Then

- $aw(G) = w(G)$, *if* $w(G)$ *is not an integer;*
- $aw(G) = \begin{cases} \min(I) & \textit{if } G < \text{☆} \\ \max(I) & \textit{if } G > \text{☆} \\ 0 & \textit{otherwise} \end{cases}$, *if* $w(G)$ *is a integer,*
  where $I = \{x \in \mathbb{Z} \mid \operatorname{aw}(G^L) - 2 \lhd\!\!| \; x \lhd\!\!| \operatorname{aw}(G^R) + 2\}$.

This result allows for the recursive computation of the $\operatorname{aw}(G)$ from $\operatorname{aw}(G^{\mathcal{L}})$ and $\operatorname{aw}(G^{\mathcal{R}})$. The second case is, in general, much more complicated. Thus, when given a ruleset, it is important to determine if the positions with an atomic weight of the second type are finite in number; one can only hope for complete calculations if this is the case. This paper provides an example of such a ruleset.

4. Hereditary Behavior of the Atomic Weight of Spindly Forms

In general, given an all-small ruleset, the second case of Theorem 3.3 may appear in several different ways. However, the particular case of spindly has a very simple recursion. The class of spindly forms is constructed as follows:

- $0 = \{\,|\,\}$ is the only spindly game born on day 0;
- the spindly games born on day $n+1$ have the possible forms $\{0 \mid G^R\}$ or $\{H^L \mid 0\}$ where G^R and H^L are spindly games born on day n.

It is easy to see that $\operatorname{aw}(0) = \operatorname{aw}(*) = 0$. It will be useful to be able to distinguish between these (and other related forms). Throughout the rest of this paper, we will use the notion of *adorned zero*.

Definition 4.1. We say that G has *adorned atomic weight* 0^L (written as $\operatorname{aw}(G) = 0^L$) if

$$\operatorname{aw}(G) = 0,\ \operatorname{aw}(\{0 \mid G\}) = 1, \text{ and } \{G \mid 0\} = *.$$

Similarly, we say that G has *adorned atomic weight* 0^R (written as $\operatorname{aw}(G) = 0^R$) if

$$\operatorname{aw}(G) = 0,\ \operatorname{aw}(\{G \mid 0\}) = -1, \text{ and } \{0 \mid G\} = *.$$

We are now ready to establish a table of fundamental values.

Theorem 4.2. *If* G *is a spindly position, then the following table can be used to compute the atomic weight of* $\{0 \mid G\}$.

$\operatorname{aw}(G)$	$k \leqslant -4$	-3	-2	$-\frac{3}{2}$	$\{-2 \mid k\}$	-1	0^R or $G=0$	$G = *$	0^L	1	$\frac{3}{2}$	2	3	$\{k \mid 2\}$	$k \geqslant 4$
$\operatorname{aw}(\{0 \mid G\})$	$\{-2 \mid k+2\}$	$-\frac{3}{2}$	-1	0^L	0^L	If $G = \downarrow$, then 0. If $G = \{G^L \mid 0\}$, then -1. If $G = \{0 \mid G^R\}$, then 0^L.	0	1	1	2	3	3	4	$k+2$	$k+1$
$\{0 \mid G\}$						If $G = \downarrow$, then $*$.	$*$	$\uparrow$							

Similarly, the following table can be used to compute the atomic weight of $\{G \mid 0\}$.

$\mathrm{aw}(G)$	$k \leqslant -4$	-3	-2	$-\frac{3}{2}$	$\{-2 \mid k\}$	-1	0^R	0^L or $G=0$	$G = *$	1	$\frac{3}{2}$	2	3	$\{k \mid 2\}$	$k \geqslant 4$
$\mathrm{aw}(\{G \mid 0\})$	$k-1$	-4	-3	-3	$k-2$	-2	-1	0	-1	If $G = \uparrow$, then 0. If $G = \{G^L \mid 0\}$, then 0^R. If $G = \{0 \mid G^R\}$, then 1.	0^R	1	$\frac{3}{2}$	0^R	$\{k-2 \mid 2\}$
$\{G \mid 0\}$								$*$	$\downarrow$	If $G = \uparrow$, then $*$.					

Observation 4.3. The first table gives the atomic weight of $\{0 \mid G\}$, knowing the atomic weight of G. There are cases where we know the explicit form of $\{0 \mid G\}$ and these are given in row 3. The third row is necessary to distinguish between certain exceptional cases in the atomic weight calculations.

Throughout the rest of this paper, when we mention a *table entry*, we are referring to a game value g that can either be $\mathrm{aw}(G)$ or a special case. ♢

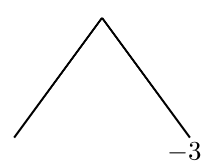
Proof. We prove that the values for the first table are correct by induction on birthday. (The values in the second table follow by game negation and the fact that atomic weight is a game homomorphism.) For the basis step of our induction, note that the table values are trivially correct for $G = 0$.

Entries 1,2,3.

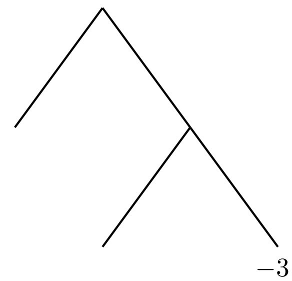
The listed atomic weights can be computed without issue from Theorem 3.3.

Entry 4.

By induction, the only way to obtain a game G with $\mathrm{aw}(G) = -\frac{3}{2}$ is from



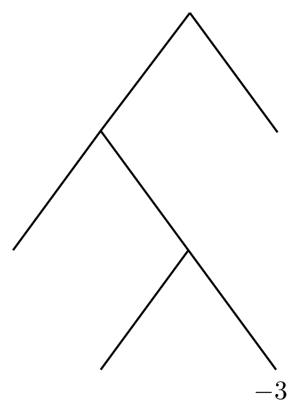
The picture means that there is one Left option $G^L = 0$ and one Right option G^R such that $\mathrm{aw}(G^R) = -3$. Therefore, the game $\{0 \mid G\}$ is



We claim that $\{0\,|\,G\} + {\downarrow}☆ < 0$. If Right plays first, he wins with the option $G + {\downarrow}☆$ because, by Theorem 3.2, an atomic weight of $-\frac{5}{2}$ constitutes a threat. Using similar reasoning, one can see that Left loses playing first.

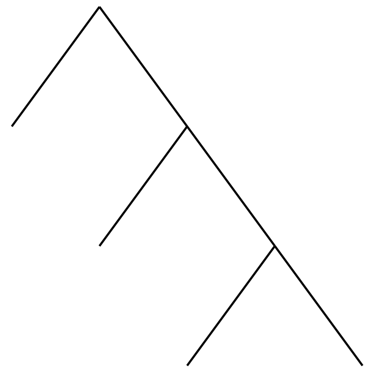
It is also true that $\{0\,|\,G\} + {\uparrow}☆ > 0$. If Right goes first and plays to $\{0\,|\,G\} + ☆$, Left wins with the option $\{0\,|\,G\}$. If Right plays first to $G + {\uparrow}☆$, Left wins with the option ${\uparrow}☆$ (recall $☆ = *m$, for $m \geq 2$ for spindly). Hence, by Definition 1.1, $\mathrm{aw}(\{0\,|\,G\}) = 0$.

Next, we consider the game $\{\{0\,|\,G\}\,|\,0\}$.



We will show that $\{\{0\,|\,G\}\,|\,0\} + * = 0$. To this end, note that if Left plays to $\{0\,|\,G\} + *$, Right wins by choosing $G + *$ ($\mathrm{aw}(G^R) = -3$ constitutes a threat; see Theorem 3.2). Similarly, if Right plays to $\{\{0\,|\,G\}\,|\,0\}$, then Left wins by playing to $\{0\,|\,G\}$.

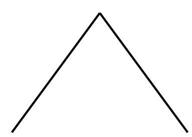
Finally, the game $\{0\,|\,\{0\,|\,G\}\}$ is



Playing each of the forthcoming games, it can be seen that $\{0\,|\,\{0\,|\,G\}\} + \Downarrow\!\star < 0$ and $\{0\,|\,\{0\,|\,G\}\} + \star > 0$. Hence, by Theorem 1.1, $\mathrm{aw}(\{0\,|\,\{0\,|\,G\}\}) = 1$. We conclude that $\mathrm{aw}(\{0\,|\,G\}) = 0^L$ and that the fourth entry of the table is accurate.

Entry 5.

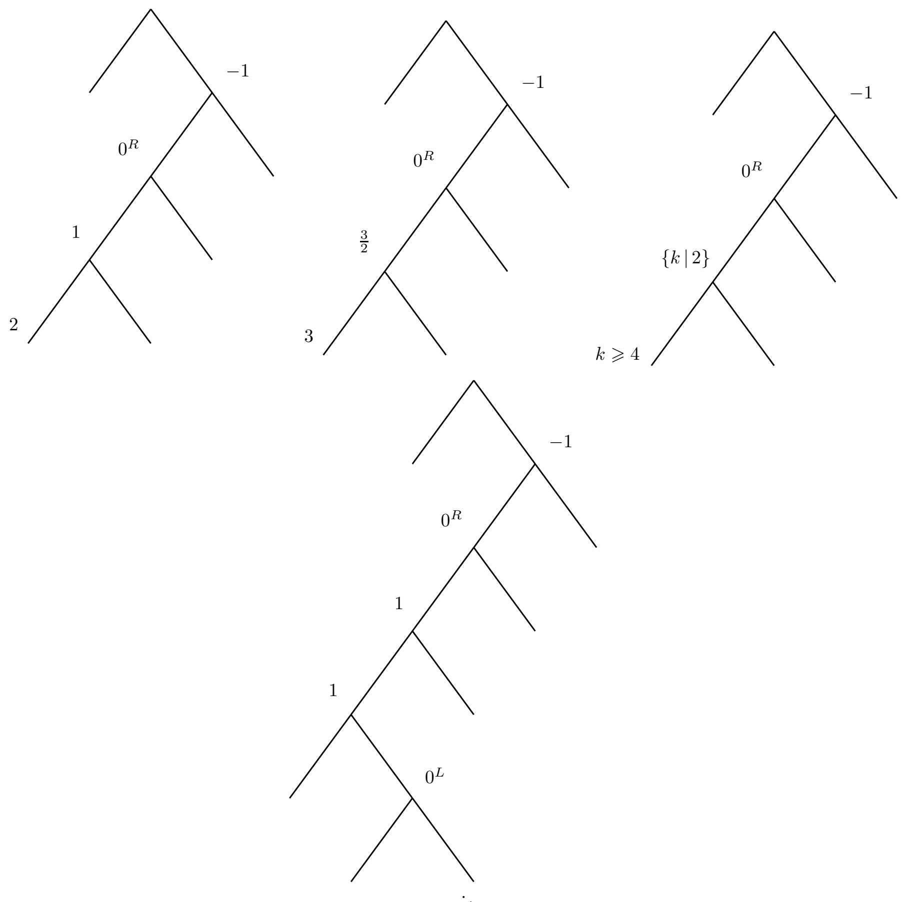
By induction, the only possibility for G such that $\mathrm{aw}(G) = \{-2\,|\,k\}$ is the following:



One can verify that $\mathrm{aw}(\{0\,|\,G\}) = 0^L$ in a manner similar to the way entry 4 was established.

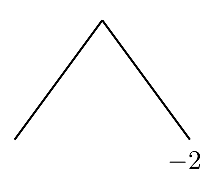
Entry 6.

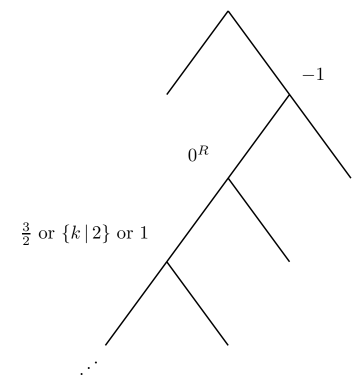
First, we note that $\{0\,|\,\downarrow\} = *$. Next, consider $G = \{G^L\,|\,0\}$ ($\mathrm{aw}(G) = -1$). By induction, there are 4 possibilities for $\{0\,|\,G\}$:



In all cases, $\{0 \,|\, G\} + \Uparrow\text{☆} > 0$ and $\{0 \,|\, G\} + \text{☆} < 0$. Considering the former possibility, suppose that Right plays first in $\{0 \,|\, G\} + \Uparrow\text{☆}$ to $G + \Uparrow\text{☆}$. Then, Left answers to $G^L + \Uparrow\text{☆}$. After, against Right's move to $G^L + \uparrow\text{☆}$, Left answers with $G^{LL} + \uparrow\text{☆}$. Finally, against Right's reply to $G^{LL} + \text{☆}$, Left answers with $G^{LL} + *$ and wins. Therefore, by Theorem 1.1, $\text{aw}(\{0 \,|\, G\}) = -1$.

By induction, there are two possibilities for $G = \{0 \,|\, G^R\}$ (with $\text{aw}(G) = -1$):





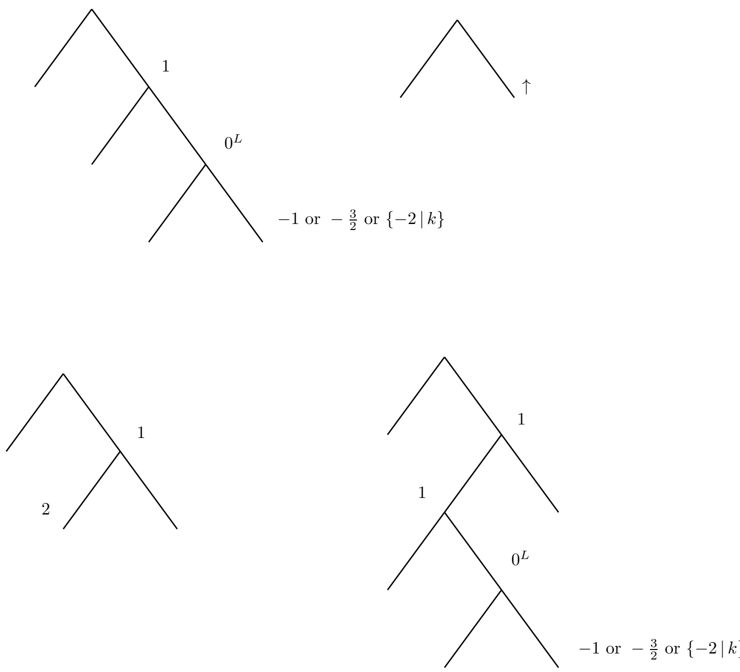
One can see in a manner similar to arguments used above that $\text{aw}(\{0 \,|\, G\}) = 0$, $\text{aw}(\{0 \,|\, \{0 \,|\, G\}\}) = 1$, and $\{\{0 \,|\, G\} \,|\, 0\} = *$. Hence $\text{aw}(\{0 \,|\, G\}) = 0^L$ for this case and thus the sixth entry of the table is correct.

Entries 7,8,9.

These entries are basic facts or follow by the definition of adorned atomic weight.

Entry 10.

By induction, the four possibilities for $\{0 \,|\, G\}$ such that $\text{aw}(G) = 1$ are the following:



As $\{0\,|\!\uparrow\} = \Uparrow\!*$, $\mathrm{aw}(\{0\,|\!\uparrow\}) = 2$. For all the others, we can play the games in order to check that $\{0\,|\,G\} + \downarrow\!☆ > 0$ and $\{0\,|\,G\} + 3.\!\downarrow\!☆ < 0$. Hence, by Theorem 1.1, $\mathrm{aw}(\{0\,|\,G\}) = 2$.

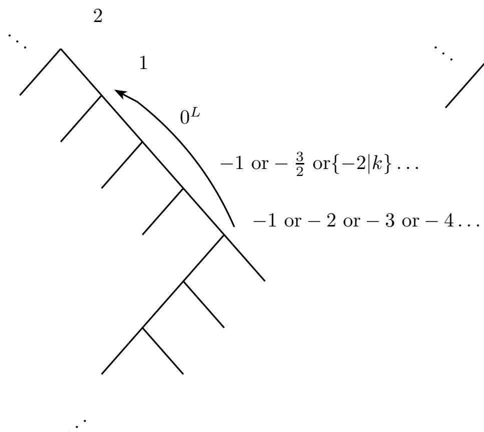
Entries 11,12,13,14,15.

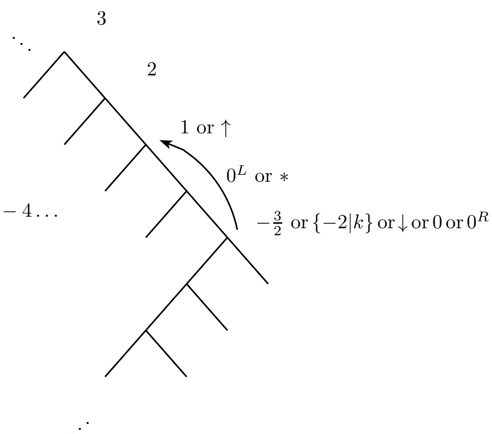
This is similar to the bottom left case of the previous entry.

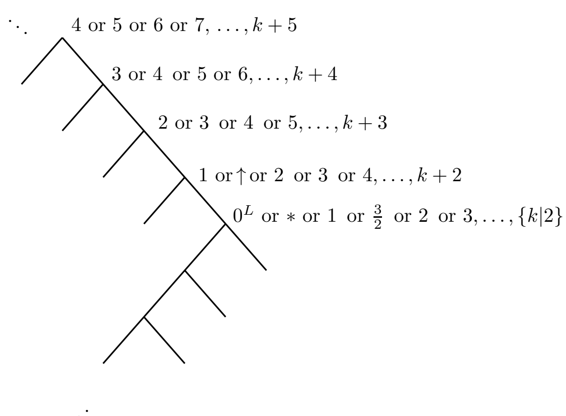
□

5. An Algorithm for the Atomic Weight Calculus of Spindly Forms

With the help of Theorem 4.2, it is possible to classify the turning points on the trunk:







Consider E, the possible entries of the table shown in Theorem 4.2. The classification of the turning points motivates the following partitions: $E = L_1 \sqcup L_2 \sqcup L_3$ and $E = R_1 \sqcup R_2 \sqcup R_3$, where $L_1 = \{\ldots, -3, -2, -1\}$, $L_2 = \{-\frac{3}{2}, \{-2 \,|\, k\}, \downarrow, 0, 0^R\}$, $L_3 = \{0^L, *, 1, \frac{3}{2}, 2, \ldots, \{k \,|\, 2\}\}$, and $R_1 = \{1, 2, 3, \ldots\}$, $R_2 = \{\frac{3}{2}, \{k \,|\, 2\}, \uparrow, 0, 0^L\}$, $R_3 = \{\{-2 \,|\, k\}, \ldots, -2, -\frac{3}{2}, -1, *, 0^R\}$.

To describe the turning situations, for $g \neq *$, consider the ceiling function $\lceil g \rceil = \min_{n\in\mathbb{Z}}\{n \geqslant g\}$ and the floor function $\lfloor g \rfloor = \max_{n\in\mathbb{Z}}\{n \leqslant g\}$. Note that for the special case when $g = *$, we define $\lceil * \rceil = \lfloor * \rfloor = 0$. Moreover, we also need the following two functions when $n \geqslant 2$:

$$\psi_L(n, g) = \begin{cases} n-2 & \text{if } g \in L_1 \\ n-1 & \text{if } g \in L_2 \\ n + \lceil g \rceil & \text{if } g \in L_3 \end{cases},$$

adorning 0^L if the result is 0 and

$$\psi_R(n,g) = \begin{cases} 2-n & \text{if } g \in R_1 \\ 1-n & \text{if } g \in R_2 \\ \lfloor g \rfloor - n & \text{if } g \in R_3 \end{cases},$$

adorning 0^R if the result is 0.

Finally, consider the class of objects

$$\Upsilon = \{ \ [n_1, -n_2, n_3, -n_4 \ldots] : g \mid n_i \in \mathbb{Z}^+ \ \text{ and } \ g \in E \ \}$$

and the function $\xi : \Upsilon \to \Upsilon$ defined by the following rules

(i) $\xi([\ \cdot\ ] : g) = [\ \cdot\ ] : g$

$$(ii) \quad \xi([n_1, -n_2, \ldots, n_k] : g) = \begin{cases} \big[n_1, \ldots, -n_{k-1}\big] : \mathrm{aw}(\{0|G\}) & \text{if } \ n_k = 1 \\ \big[n_1, \ldots, -n_{k-1}\big] : \uparrow & \text{if } \ n_k = 2 \text{ and} \\ & \quad (g = 0 \text{ or } g = 0^R \text{ or } g = \downarrow) \\ \big[n_1, \ldots, -n_{k-1}\big] : \psi_L(n_k, g) & \text{otherwise} \end{cases}$$

$$(iii) \quad \xi([n_1, -n_2, \ldots, -n_k] : g) = \begin{cases} \big[n_1, \ldots, n_{k-1}\big] : \mathrm{aw}(\{G|0\}) & \text{if } \ n_k = 1 \\ \big[n_1, \ldots, n_{k-1}\big] : \downarrow & \text{if } \ n_k = 2 \text{ and} \\ & \quad (g = 0 \text{ or } g = 0^L \text{ or } g = \uparrow) \\ \big[n_1, \ldots, n_{k-1}\big] : \psi_R(n_k, g) & \text{otherwise} \end{cases}$$

(Note that, using Theorem 4.2, $\mathrm{aw}(\{0|G\})$ and $\mathrm{aw}(\{G|0\})$ are computed using $g = \mathrm{aw}(G)$.)

Let (a, b), with $a \geqslant b$, be a spindly position with associated list $\big[n_1, -n_2, \ldots, \pm n_k\big]$ obtained using the Euclidean algorithm. By induction, it follows that $\mathrm{aw}(a, b)$ is the game component of $\xi^k\left(\big[n_1, -n_2, \ldots, \pm n_k\big] : 0\right)$.

6. Subversion Examples and Other Spindly Games

Example 6.1. Consider the spindly position $(4033, 936)$. Applying the Euclidean algorithm, we get $\big[4, -3, 4, -5, 3, -4\big]$. Now,

$$\big[4,-3,4,-5,3,-4\big] : 0 \xrightarrow{\xi} \big[4,-3,4,-5,3\big] : -3 \xrightarrow{\xi} \big[4,-3,4,-5\big] : 1$$

$$\xrightarrow{\xi} \big[4,-3,4\big] : -3 \xrightarrow{\xi} \big[4,-3\big] : 2 \xrightarrow{\xi} \big[4\big] : -1 \xrightarrow{\xi} \big[\ \cdot\ \big] : 2.$$

Thus, $\mathrm{aw}(4033, 936) = 2$. ♢

Example 6.2. Consider the spindly position $(189, 44)$. Applying the Euclidean algorithm, we get $\big[4, -3, 2, -1, 1, -2\big]$. Now,

$$\big[4,-3,2,-1,1,-2\big] : 0 \xrightarrow{\xi} \big[4,-3,2,-1,1\big] : \downarrow \xrightarrow{\xi} \big[4,-3,2,-1\big] : *$$

$$\xrightarrow{\xi} \big[4,-3,2\big] : \downarrow \xrightarrow{\xi} \big[4,-3\big] : \uparrow \xrightarrow{\xi} \big[4\big] : -2 \xrightarrow{\xi} \big[\,\cdot\,\big] : 2.$$

Thus, $\text{aw}(189, 44) = 2$.

Note that this example illustrates the importance of including the games $*$, $\downarrow$, and $\uparrow$ in the atomic weight calculations. If 0, -1, and 1 were used instead of $*$, $\downarrow$, and $\uparrow$, respectively, ambiguous situations would arise during the atomic weight computations. ♢

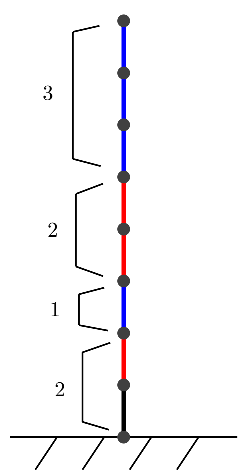
Example 6.3. Consider the spindly position $(17, 13)$. Applying the Euclidean algorithm, we get $\big[1,-3,4\big]$. Now,

$$\big[1,-3,4\big] : 0 \xrightarrow{\xi} \big[1,-3\big] : 3 \xrightarrow{\xi} \big[1\big] : -1 \xrightarrow{\xi} \big[\,\cdot\,\big] : -1.$$

Thus, $\text{aw}(17, 13) = -1$. ♢

Example 6.4. Revisiting the SUBVERSION example, $(211, 155)+(5, 4)+(13, 17)$, we can use the above calculations to conclude that $\text{aw}((211, 155) + (5, 4) + (13, 17)) = \text{aw}(211, 155) + \text{aw}(5, 4) + \text{aw}(13, 17) = -\frac{3}{2} - \frac{3}{2} + 1 = -2$. Hence, by Theorem 3.2, Right wins. ♢

Example 6.5. The game of LOP & CHOP is an all-small variation of HACKENBUSH [4] suggested by Berlekamp's YELLOW-BROWN HACKENBUSH [3] given by the following rules. Positions are strings of edges. The bottom edge is black and all the other edges are blue and red and blue is Left's color and red is Right's color. Edges may be removed as follows: if the top edge is the player's color then her only option is to remove it; if the top edge is not the player's color then her only option is the removal of the black edge. As in every HACKENBUSH variation, edges not connected to the ground disappear (here, it happens when a player removes the black edge). The SUBVERSION position $(27, 8)$ ($\big[3,-2,1,-2\big]$, in the list notation) has the following HACKENBUSH realization.



♢

Example 6.6. HEADS & TAILS, H_kT_j for short, are a class of partizan subtraction games ([14]) played with stacks of coins. The coins in a stack all face the same way, either all heads face up or all tails face up. In H_3T_2, Left can remove 1 coin from any stack with heads facing up; however, if she leaves a multiple of 3 coins in the stack, she must turn the stack over (so that tails are facing up). Left may remove the whole stack if it has tails facing up. Right removes 1 from any stack with tails facing up; however, if he leaves a multiple of 2 coins in the stack, he must turn the stack over (so that heads are facing up). Right may remove the whole stack if it has heads facing up.

Table 1 gives the atomic weights, including the values of the exceptional cases, of stacks up to size 12. It is now easy to note that both sequences of atomic weights are periodic with period length 6, starting at $n = 6$.

Heads up	0($*$)	-1($\downarrow$)	-2	-1($\downarrow$)	-2	-3	0($*$)	-1($\downarrow$)	-2	-1($\downarrow$)	-2	-3
Heap size	1	2	3	4	5	6	7	8	9	10	11	12
Tails up	0($*$)	1($\uparrow$)	0($*$)	1($\uparrow$)	0($*$)	1($\uparrow$)	$-3/2$	0^L	0($*$)	1($\uparrow$)	0($*$)	1($\uparrow$)

Table 1: Atomic Weights of H_3T_2

For example, in H_3T_2 on $10_H + 8_T + 7_T$, $\text{aw}(10_H + 8_T + 7_T) = -1 + 0 + (-3/2) = -5/2$ and so Right wins regardless of going first or second. ♢

7. Further Work

Note that Theorem 3.3 is a type of simplicity rule for atomic weight. However, in the second case of that result, the information $\text{aw}(G^{\mathcal{L}})$ and $\text{aw}(G^{\mathcal{R}})$ is not enough; the outcome of $G + ☆$ also matters. In spindly games, that situation is reduced to a finite number of cases.

Nimbers are examples of games where the form $\{\text{aw}(G^L) - 2 \,|\, \text{aw}(G^R) + 2\}$ is an integer; in fact, considering nimbers, that literal form is $\{-2 \,|\, 2\}$ and, among the integers that fit, 0 is the right choice. In the form $\{0, *, *2 \,|\, 0, *\}$, 1 is the correct choice. There are rulesets that have a finite nim dimension n, that is, they contain a position with game value $*2^n$ but not $*2^{n+1}$ (the nim dimension of spindly is 0 because only 0 and $*$ occur). We conjecture that CLOBBER has nim dimension 1 (no $*4$ is known). A very general question to explore is the following:

Given a ruleset, is there any relation between the possibility of reducing the second case of Theorem 3.3 to a finite number of situations and the nim dimension of *the ruleset?*

The version of the game analyzed in this article can be generalized quite naturally to larger ordered tuples, $\langle p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_m \rangle$, where a Left move is to choose some p_i and subtract it from some q_j, provided that $p_i \leqslant q_j$. Right's moves are similar. One could then ask the following question:

Is the behavior of GENERALIZED SUBVERSION *similar to the special case explored in this paper?*

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