SLOW K-NIM

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Abstract

Given n piles of tokens and a positive integer $k \leq n$, we study two impartial combinatorial games, $\text{NIM}^1_{n,\leq k}$ and $\text{NIM}^1_{n,=k}$. In the first (resp. second) game, each move consists of choosing at least 1 and at most k (resp. exactly k) non-empty piles and removing one token from each of them. We study the normal and misére versions of both games. For $\text{NIM}^1_{n,=k}$ we give explicit formulas of its Sprague-Grundy function for the cases $2 = k \leq n \leq 4$; for $\text{NIM}^1_{n,\leq k}$ we provide such formulas for $2 = k \leq n \leq 3$ and characterize the $\mathcal{P}$-positions for the cases $n \leq k+2$ and $n = k+3 \leq 6$.

1. Introduction

1.1. Impartial Combinatorial Games

An impartial combinatorial game is a two-person game in which each of two players has the same options of moves from every position. There is no hidden information and no moves of chance, and the two players take turns. Under the *normal* play convention, the player who makes the last move wins, or in other words, the player who has no legal move loses. Since there is no draw, exactly one player has a winning strategy. There is also an opposite winning convention, called *misére*, in which the player who makes the last move loses. We will discuss both versions in this paper.

An impartial combinatorial game is naturally modeled by a directed graph $G = (X, E)$ whose vertices $x \in X$ represent the positions and whose directed edges $(x, x') \in E$ represent the legal moves of the game. We will denote the move (x, x') from x to x' by $x \to x'$. We assume that graph G is acyclic, in particular, it contains no loops and oppositely directed parallel edges. We also assume that G is *potentially finite*, meaning that, although X may be infinite, for any $x \in X$ the set of positions that can be reached from x by a sequence of successive moves is finite. A position x from which there is no legal move is called *terminal*. We denote by X_T the set of all terminal positions. Since all such positions can be merged, we can assume that G has only one terminal position, $X_T = \{x_T\}$. We call G an *impartial combinatorial game* or simply a game, for short.

We assume that the reader is familiar with the basics of the Sprague-Grundy (SG) theory [9, 10, 16, 17]. A detailed introduction (and much more) can be found in [1, 8, 15]. In this paper we will need only very basic concepts: the SG function and $\mathcal{P}$-positions, for both the normal and misére versions. We begin with the normal one, postponing the misére version until Section 1.5.

1.2. Characterization of Positions

Given a position $x \in X$, it is called an $\mathcal{N}$-position (resp., a $\mathcal{P}$-position) if the player who makes a move from x (resp., the opponent) wins. Since exactly one of the two players has a winning strategy, X is partitioned by the $\mathcal{N}$- and $\mathcal{P}$-positions, that is, $X = X_{\mathcal{N}} \cup X_{\mathcal{P}}$. This partition is characterized by the following two properties:

($\mathcal{P}$N) There exists no move $x \to x'$ such that $x, x' \in X_{\mathcal{P}}$.

($\mathcal{P}$Y) For any $x \notin X_{\mathcal{P}}$ there exists a move $x \to x'$ such that $x' \in X_{\mathcal{P}}$.

Note that in graph theory, the vertex set $X_{\mathcal{P}} \subseteq X$ is called a *kernel*. Conditions $\mathcal{P}$N and $\mathcal{P}$Y show that $X_{\mathcal{P}}$ is independent and absorbing, respectively.

Sprague [16, 17] and Grundy [9] introduced a refinement of the concept of the $\mathcal{N}$- and $\mathcal{P}$-positions. Let $\mathbb{N}_0$ denote the set of non-negative integers. Given a subset $S \subset \mathbb{N}_0$, the *minimum excludant* $mex(S) \in \mathbb{N}_0$ is defined as the minimum of $\mathbb{N}_0 \setminus S$. In particular, $mex(\varnothing) = 0$. With this notation, the SG function $\mathcal{G} : X \to \mathbb{N}_0$ is defined recursively as follows:

- $\mathcal{G}(x) = 0$ if x is a terminal position;
- $\mathcal{G}(x) = mex\{\mathcal{G}(x') \mid (x, x') \in E\}$.

In other words, $\mathcal{G}(x)$ is the first missing integer among the SG values of all immediate successors of x in G. It is not difficult to see that the $\mathcal{P}$-positions of a game are exactly the zeros of its SG function, that is, $x \in X_{\mathcal{P}}$ if and only if $\mathcal{G}(x) = 0$. In general, the SG function is completely characterized by the following two properties:

(SGN) There exists no move $x \to x'$ such that $\mathcal{G}(x) = \mathcal{G}(x')$;

(SGY) For any position x with $\mathcal{G}(x) = i$ and any non-negative integer $j < i$, there exists a move $x \to x'$ with $\mathcal{G}(x') = j$.

In other words, no move keeps the SG value the same, but every smaller non-negative integer value can be reached by a move.

1.3. Nim and Its Versions

The ancient game of Nim is played as follows. There are n piles containing $x_1, \dots, x_n$ tokens. Two players alternate turns. In each move, a player chooses a non-empty pile $i \in [n] = \{1, \dots, n\}$ and removes any number of tokens from it, replacing x_i by x'_i such that $0 \le x'_i < x_i$. The game terminates when all piles are empty. The player who made the last move wins the normal version of the game and loses its misére version. Both versions were solved by Bouton in 1901 in his seminal paper [7]. In fact, he computed the SG function as well, although he did not define it for an arbitrary game, only for Nim.

In 1910, Moore [14] introduced a generalization Nim_k of Nim, which we call *Moore's* k-Nim and denote by $\text{Nim}_{n,\le k}$. Given an integer parameter k such that $0 < k \le n$, a move consists of choosing at least one and at most k piles and removing at least one token from each chosen pile, not necessarily the same number. Moore characterized the $\mathcal{P}$-positions, that is, the positions with SG value 0. Much later, in 1980, Jenkyns and Mayberry [13] characterized the positions with SG value 1 for any n and k and obtained an explicit formula for the SG function when $k = n - 1$. The results of [13] were further generalized in [2, 3] and then in [5, 6]. Yet, already for the case $n = 4$ and $k = 2$ no explicit formula for the SG function is known.

Boros et al. [4] introduced a variation called *Exact* k-Nim, denoted by $\text{Nim}_{n,=k}$. In this game a move consists of removing at least one token from exactly k piles (rather than from at most k). Explicit formulas for the SG function were obtained for $2k > n$ and $2k = n > 4$. Yet, already for the case $n = 5$ and $k = 2$ no explicit formula is known, not even for the $\mathcal{P}$-positions. Recently, results of [4] were generalized in [5, 6].

By definition, both games, Moore's and Excact k-Nim, turn into the standard Nim when $k = 1$, and into the trivial one-pile Nim when $k = n$. In the latter case the number of tokens in the pile is $\sum_{i=1}^{n} x_i$ for Moore's k-Nim and $\min_{i=1}^{n} x_i$ for Exact k-Nim.

In this paper we modify both games, Moore's k-NIM and Exact k-NIM, as follows. A move $x \to x'$ is called *slow* if at most one token can be taken from each pile, that is, $0 \leq x_i - x'_i \leq 1 \; \forall i \in [n]$. We define the games *Slow Moore's* k-NIM and *Slow Exact* k-NIM by restricting both players to their slow moves and keeping all other rules. We denote these two games by $\text{NIM}^1_{n,\leq k}$ and $\text{NIM}^1_{n,=k}$ and study them in Sections 2 and 3, respectively. For both games the case $k = 1$ is obvious.

Proposition 1. *If $k = 1$, Slow Moore's k-NIM and Slow Exact k-NIM coincide. Both are trivial: the SG function takes only values 0 or 1, which alternate after every move, depending thus only on the parity of the total number of tokens in all piles:*

$$\mathcal{G}(x) = \begin{cases} 0 \text{ if } \sum_{i=1}^{n} x_i \text{ is even;} \\ 1 \text{ if } \sum_{i=1}^{n} x_i \text{ is odd.} \end{cases}$$

The case $k = n$ is still trivial for Slow Exact k-NIM, but not for Slow Moore's k-NIM with $k > 2$; see Sections 2 and 3, respectively.

1.4. Non-decreasing Positions and Canonical Slow Moves

In all versions of NIM, positions are represented by non-negative integer vectors $x = (x_1, \ldots, x_n)$. Obviously, a permutation of n entries of x results in a position x' that is equivalent to x, in the sense that for any version of NIM the optimal results (defined in the previous section) of two games with the initial positions x and x' are the same.

A vector x, as well as the corresponding position, is called *non-decreasing* if $x_1 \leq \ldots \leq x_n$. Note that this property may not be respected by a move $x \to x'$: it may happen that x is non-decreasing, while x' is not. Moreover, this may happen even for a slow move $x \to x'$. Yet, in the latter case, the property "can be saved" (see Lemma 1 below).

A slow move $x \to x'$ will be called *canonical* if x is non-decreasing and for each inclusion-maximal sequence of (successive) equal entries of x (maybe, only one) this move reduces only successive left entries (with the smallest indices) if any. We can assign to each slow move $x \to x'$ a canonical slow move $y \to y'$ uniquely defined by the following procedure:

- permute, if necessary, entries of x to get a non-decreasing vector;
- keep, if necessary, permuting the equal (successive) entries until ones reduced by the move $x \to x'$ get the smallest indices.

The resulting move $y \to y'$ is canonical, by the above definition. Furthermore, it is equivalent to the original move $x \to x'$, in the sense that positions x' and y' (as well as x and y) are equivalent.

Another important notion that will be instrumental in characterizing the $\mathcal{P}$-positions and computing the SG functions for both games, Slow Moore's k-Nim and Slow Exact k-Nim, will be the parity vector of a position. For any $x = (x_1, \ldots, x_n)$ define its *parity vector* $p(x) = (p(x_1), \ldots, p(x_n))$, with entries e and o taken for $i = 1, \ldots, n$ according to the formula:

$$p(x_i) = \begin{cases} e \text{ if } x_i \text{ is even;} \\ o \text{ if } x_i \text{ is odd.} \end{cases}$$

Lemma 1. *For any slow move $x \to x'$ from a non-decreasing vector x, x' is also non-decreasing if and only if the move $x \to x'$ is canonical. Moreover, the parity vectors of x and x' are distinct, $p(x) \neq p(x')$.*

Proof. The first claim follows immediately from the definitions and properties mentioned above. To derive the second one, let us note that for any move $x \to x'$ the positions x and x' are distinct, and the corresponding parity vectors are distinct too whenever the move $x \to x'$ is slow. □

The second statement will be frequently used for verifying properties (SGN) and ($\mathcal{P}$N). From now on we will assume that any position vector x is non-decreasing and any move $x \to x'$ is canonical unless explicitly stated otherwise. In particular, we always assume that a move $x \to x'$ reduces any entry x_i of x by at most 1.

1.5. Normal and Misére Play

In the misére version of a game the player who has no legal move wins, in contrast to the normal version, while all other rules are the same for both. The misére SG function is denoted by $\mathcal{G}^-$ and is defined by the same recursion as the normal SG function $\mathcal{G}$, except that the initialization differs: $\mathcal{G}^-(x) = 1$ (while $\mathcal{G}(x) = 0$) for any terminal position $x \in X_T$.

One can easily reduce the misére version to the normal one by the following simple modification of the graph $G = (X, E)$: add a new position x^* to X and a new move $x \to x^*$ for each $x \in X_T \subset G$. Obviously, x^* is a unique terminal position of the resulting graph G^-. Thus, $\mathcal{G}^-$ is characterized by the same properties (SGN) and (SGY), with the following "slight" modification of the latter: (SGY) fails for $i = 1$ and $j = 0$ when $x \in X_T$ is a terminal position of G. More precisely, functions $\mathcal{G} : X \to \mathbb{N}_0$ and $\mathcal{G}^- : X \to \mathbb{N}_0$ are the SG functions of the normal and misére versions of a game $G = (X, E)$ if and only if for any non-terminal position $x \in X \setminus X_T$ and for all moves $x \to x'$ we have

(SGN±) $\mathcal{G}(x) \neq \mathcal{G}(x')$ and $\mathcal{G}^-(x) \neq \mathcal{G}^-(x')$ and

(SGY±) $\mathcal{G}(x')$ (resp., $\mathcal{G}^-(x')$) takes all integer values between 0 and $\mathcal{G}(x)-1$ (resp., $\mathcal{G}^-(x) - 1$).

Remark 1. Formally, (SGN±) holds for the terminal positions too, while (SGY±) holds for $\mathcal{G}$, but not for $\mathcal{G}^-$ for these positions. Indeed, a terminal position $x \in X_T$ is a position with $\mathcal{G}(x) = 0$ and $\mathcal{G}^-(x) = 1$ from which there are no moves, in particular, there is no move to a position x' with $\mathcal{G}^-(x') = 0$. Thus, (SGY±) cannot be extended to the terminal positions unless we add x^*. Yet, it is not convenient to define $\mathcal{G}$ and $\mathcal{G}^-$ using two different graphs (G and G^-). So, by definition, we allow (SGY±) to fail for $\mathcal{G}^-$ when x is a terminal position.

A position x will be called an *i-position* (resp., an *(i, j)-position*) if $\mathcal{G}(x) = i$ (resp., if $\mathcal{G}(x) = i$ and $\mathcal{G}^-(x) = j$). We denote the sets of all i- and (i, j)-positions by X_i and $X_{i,j}$, respectively. In particular, each terminal position is a $(0, 1)$-position.

In 1976, Conway [8, Chapter 12 page 178] introduced so-called *tame* games, which contain only $(0, 1)$-, $(1, 0)$-, and (ℓ, ℓ)-positions with $\ell \in \mathbb{N}_0$. We call the $(0, 1)$- and $(1, 0)$-positions *swap* positions. A related notion was introduced in [12]. A game is called *domestic* when it has no $(0, \ell)$- and $(\ell, 0)$-positions with $\ell > 1$. It is called *misérable* if from every non-swap position, either there are moves to both $(0, 1)$- and $(1, 0)$-positions, or there are no moves to swap positions at all. In other words, a move from x to $X_{0,1}$ exists if and only if there is a move from x to $X_{1,0}$.

For example, classic Nim is misérable. This observation is implicit already in [7]. Furthermore, **misérable games are tame** [11, 12]. This statement is also implicit in several earlier works, for example in [1]. Numerous examples of misérable, tame, and domestic games are given in [12, Section 6]. In [11] it was shown that the game *Euclid* is misérable.

A constructive criterion for verifying misérability was suggested in [12, Theorem 7]. In Section 2.3 we apply it to Slow Moore's $\text{Nim}^1_{n,\leq k}$ and prove that this game is misérable (and hence, tame) when $k = n - 1$ or $k = n$. In Section 3 we compute $\mathcal{G}$ and $\mathcal{G}^-$ for Slow Exact $\text{Nim}^1_{4,=2}$ and show that it has $(1, 2)$- and $(0, 3)$-positions. Although they are sparse, the game is not tame. Moreover, it is not even domestic. We also show that the subgame of $\text{Nim}^1_{4,=2}$ defined by the positions x with even sum of entries $x_1 + x_2 + x_3 + x_4$ is misérable.

2. Slow Moore's k-Nim

For $k = 1$, the SG function is given by Proposition 1, and we give a simple formula for the cases $k = n = 2$ and $k = 2, n = 3$ in Section 2.1. When $1 < k \leq n \leq k + 2$ and $n = k + 3$ with $k = 2$ or 3, we give a formula for the $\mathcal{P}$-positions in Section 2.2. In Section 2.3 we prove that the game is misérable (and hence, tame) when $1 \leq k = n - 1$ and $1 \leq k = n$.

2.1. Sprague-Grundy values for $1 < k \leq n \leq 3$

If $k = 2$ and $k \leq n \leq 3$, then the game is still simple: $\mathcal{G}(x)$ depends only on the parity vector $p(x)$.

Theorem 1. *The SG values $\mathcal{G}(x)$ in the cases $n = k = 2$ and $n = 3, k = 2$ are uniquely defined by the parity vector $p(x)$ as follows:*

(1) *For $n = k = 2$,*

$$\mathcal{G}(x) = \begin{cases} 0, & \text{if } p(x) = (e,e); \\ 1, & \text{if } p(x) = (e,o); \\ 2, & \text{if } p(x) = (o,o); \\ 3, & \text{if } p(x) = (o,e). \end{cases}$$

(2) *For $n = 3, k = 2$,*

$$\mathcal{G}(x) = \begin{cases} 0, & \text{if } p(x) \in \{(e,e,e),(o,o,o)\}; \\ 1, & \text{if } p(x) \in \{(e,e,o),(o,o,e)\}; \\ 2, & \text{if } p(x) \in \{(e,o,o),(o,e,e)\}; \\ 3, & \text{if } p(x) \in \{(e,o,e),(o,e,o)\}. \end{cases}$$

Proof. To verify that the mappings $\mathcal{G} : X \to \mathbb{N}_0$ given in (1) and (2) are the SG functions, we need to verify that they satisfy (SGN) and (SGY).

Let us start with (SGN). For (1) it follows immediately from Lemma 1. For (2) note that for each value of $\mathcal{G}$, the corresponding two parity vectors are complementary, that is, the Hamming distance between them is 3. Since $k = 2$, no move $x \to x'$ can convert one parity vector to the other in any such pair. Furthermore, no move can keep the parity vector by Lemma 1.

To verify (SGY), we have to check that for each x with $\mathcal{G}(x) = i$ and integer j such that $0 \leq j < i$, there exists a move $x \to x'$ such that $\mathcal{G}(x') = j$. Note that in each of the moves described below, it can be deduced from the parity vectors and the non-decreasing ordering that the moves are legal. We only show the case (2) and leave the simpler case (1) to the reader. In addition, we show only the case $\mathcal{G}(x) = 3$, since the other cases follow very similarly. The underlying idea is that a move on a particular stack in a complementary pair results in another complementary pair that then have the same SG value. Since the parity vectors in any two sets of complementary pairs have Hamming distances one or two, we can select on which stacks to play to achieve the desired SG value.

Assume that $\mathcal{G}(x) = 3$, then $p(x) = (e,o,e)$ or $p(x) = (o,e,o)$. Reducing x_2 gives x' and x'' with $p(x') = (e,e,e)$ and $p(x'') = (o,o,o)$, respectively, and $\mathcal{G}(x') = \mathcal{G}(x'') = 0$. Playing on both x_2 and x_3, we obtain x' and x'' with $p(x') = (e,e,o)$

and $p(x'') = (o, o, e)$, respectively, and $\mathcal{G}(x') = \mathcal{G}(x'') = 1$. Finally, playing on just x_3, we obtain x' and x'' with $p(x') = (e, o, o)$ and $p(x'') = (o, e, e)$, respectively, and $\mathcal{G}(x') = \mathcal{G}(x'') = 2$. This completes the proof. □

We note that Theorem 1 cannot be extended to the case $n = k = 3$. For example, $3 = \mathcal{G}((1, 1, 1)) \neq \mathcal{G}((1, 1, 3)) = 5$. Thus, $\mathcal{G}(x)$ is not uniquely defined by $p(x)$. However, we can still show that the $\mathcal{P}$-positions are determined by the parity vectors for certain values of n and k.

2.2. $\mathcal{P}$-positions for the Cases $n \leq k + 2$ and $n = k + 3 \leq 6$

In these cases the $\mathcal{P}$-positions are simply characterized by the parity vectors.

Theorem 2. *For $k > 1$, in the five cases listed below, the $\mathcal{P}$-positions of the game* $\text{Nim}^1_{n,\leq k}$ *are uniquely determined by the parity vectors as follows:*

(1) *for $n = k$: $x \in X_{\mathcal{P}}$ if and only if $p(x) \in Q_1 = \{(e, \dots, e)\}$;*

(2) *for $n = k + 1$: $x \in X_{\mathcal{P}}$ if and only if $p(x) \in Q_2 = \{(e, \dots, e), (o, \dots, o)\}$;*

(3) *for $n = k + 2$: $x \in X_{\mathcal{P}}$ if and only if $p(x) \in Q_3 = \{(e, \dots, e), (e, o \dots, o)\}$;*

(4) *for $n = 5$ and $k = 2$: $x \in X_{\mathcal{P}}$ if and only if*

$p(x) \in Q_4 = \{(e, e, e, e, e), (e, e, o, o, o), (o, o, e, e, o), (o, o, o, o, e)\}$;

(5) *for $n = 6$ and $k = 3$: $x \in X_{\mathcal{P}}$ if and only if*

$p(x) \in Q_5 = \{(e, e, e, e, e, e), (e, e, o, o, o, o), (o, o, e, e, o, o), (o, o, o, o, e, e)\}$.

Proof. In each case we verify properties ($\mathcal{P}$N) and ($\mathcal{P}$Y) of Section 1.2. Let us begin with ($\mathcal{P}$N). According to Section 1.4, we may again restrict our consideration to the non-decreasing vectors and canonical slow moves. By Lemma 1, any such move keeps the order of the entries and changes the parity vector. For case (1) this is already enough. In each of the remaining four cases, (2) - (5), the Hamming distance d between any two parity n-vectors in Q_i is strictly greater than k. For the cases (2) – (5), we have, respectively:

$$d = n = k + 1, \quad d = n - 1 = k + 1, \quad d > 2 = k, \text{ and } d = 4 > 3 = k.$$

It remains to verify ($\mathcal{P}$Y) for the five cases. For each case $(i) = (1), \dots, (5)$ and any x such that $p(x) \notin Q_i$, we will find a (canonical) move $x \to x'$ such that $p(x') \in Q_i$.

(1) Here $p(x) \neq (e, \dots, e)$. Since the number of odd entries is at most n and $k = n$, there exists a move $x \to x'$ that reduces all odd entries of x. Thus, $p(x') = (e, \dots, e) \in Q_1$.

(2) Here $p(x) \notin Q_2$ means that $p(x)$ has both even and odd entries. Since $k = n-1$, there exists a move $x \to x'$ that reduces either all even entries or all odd ones, resulting in x' with $p(x') \in Q_2$.

(3) If all entries of x are odd, we reduce x_1 to move to x' with $p(x') = (e, o, \ldots, o)$. If exactly one entry x_i of x is even, then $i > 1$ since $(e, o, \ldots, o) \in Q3$. Reducing x_i and x_1 gives x' with $p(x') = (e, o, \ldots, o) \in Q_3$. Finally, if at least two entries of x are even, then we can reduce all odd entries, getting x' with $p(x') = (e, \ldots, e) \in Q_3$.

(4) If $p(x) = (o, o, o, o, o)$, reduce x_1 and x_2 to obtain x' with $p(x') = (e, e, o, o, o) \in Q_4$. If exactly one entry x_i of x is even, then $i \neq 5$ since $(o, o, o, o, e) \in Q_4$. If i is odd ($i = 1$ or $i = 3$), then $x_{i+1} > x_i$ and we can reduce x_{i+1} to obtain x' with $p(x') \in \{(e, e, o, o, o), (o, o, e, e, o)\} \subset Q_4$. If $i = 2$, reducing x_1 yields x' with $p(x') = (e, e, o, o, o) \in Q_4$, while for $i = 4$, reducing x_4 and x_5 gives x' with $p(x') = (o, o, o, o, e) \in Q_4$.

Now assume that exactly two entries x_i and x_j with $i < j$ are even. Then $(i, j) \notin \{(1, 2), (3, 4)\}$ since $(e, e, o, o, o), (o, o, e, e, o) \in Q_4$, and $i \neq 5$ since $i < j$. For $i = 1$ or $i = 3$, reducing x_{i+1} and x_j gives x' with $p(x') \in \{(e, e, o, o, o), (o, o, e, e, o)\} \subset Q_4$. If $i = 2$, we can play on x_1 and x_j to get x' with $p(x') = (e, e, o, o, o) \in Q_4$, while for $i = 4$ (and hence, $j = 5$), we reduce x_i to move to x' with $p(x') = (o, o, o, o, e) \in Q_4$. If three or four entries of x are even, we reduce all odd stack heights, resulting in x' with $p(x') = (e, e, e, e, e) \in Q_4$. Finally, not all five entries of x can be even, since $(e, e, e, e, e) \in Q_4$.

(5) This case is similar to case (4). If $p(x) = (o, o, o, o, o, o)$, reduce x_1 and x_2 to get x' with $p(x') = (e, e, o, o, o, o) \in Q_5$. Now assume that exactly one entry x_i in x is even. If i is odd, reduce x_{i+1}, and if i is even, reduce x_i, x_{i+1}, and x_{i+2} (where we define $i+1 = 1,\ i+2 = 2$ for $i = 6$). This leads to x' with $p(x') \in \{(e, e, o, o, o, o), (o, o, e, e, o, o), (o, o, o, o, e, e)\} \subset Q_5$.

If exactly two entries x_i and x_j with $i < j$ are even, then $(i, j) \notin \{(1, 2), (3, 4), (5, 6)\}$ since the corresponding parity vectors are in Q_5. Furthermore, $i \neq 5$ and $i \neq 6$ since $i < j$ and $(o, o, o, o, e, e) \in Q_5$. If i is odd, reduce x_{i+1} and x_j to obtain x' with $p(x') \in \{(e, e, o, o, o, o), (o, o, e, e, o, o), (o, o, o, o, e, e)\} \subset Q_5$. If $i = 2$, reduce x_1 and x_j, so $p(x') = (e, e, o, o, o, o) \in Q_5$. If $i = 4$, reduce x_i and the unique odd entry among the last two (the other one is the x_j that is even), moving to x' with $p(x') = (o, o, o, o, e, e)$.

If three, four, or five entries of x are even, we can move to x' with $p(x') = (e, e, e, e, e, e)$.

Thus, Q_i is the set of $\mathcal{P}$-positions in every case (i). □

We note that Theorem 2 cannot be extended to the case $n - k > 3$, that is, the $\mathcal{P}$-positions are not uniquely defined by the parity vectors. For example, for $n = 6$ and $k = 2$, our computations show that $(3, 3, 3, 4, 4, 4)$ is a $\mathcal{P}$-position, while $\mathcal{G}(1, 1, 1, 2, 2, 4) = 2$; also $(1, 3, 3, 3, 3, 3)$, $(1, 3, 3, 3, 5, 5)$, and $(1, 3, 5, 5, 5, 5)$ are $\mathcal{P}$-positions, while $\mathcal{G}(1, 1, 1, 1, 1, 3) = \mathcal{G}(1, 1, 1, 1, 1, 5) = 2$ and $\mathcal{G}(1, 1, 3, 3, 5, 5) = 3$.

2.3. Misérability

Let us now compare the normal and misére versions of $\text{NIM}^1_{n,\leq k}$. The case $k = 1$ is trivial: the game has only $(0, 1)$- and $(1, 0)$ positions, which alternate with every move (see Proposition 1). In particular, it is misérable. In this section we also prove misérability for the cases $1 < k \leq n \leq k + 1$.

Theorem 3. $\text{NIM}^1_{n,\leq k}$ *is misérable when* $1 < k = n-1$ *or* $1 < k = n$. *Furthermore, for* $1 < k = n$, $X_{0,1} = \{(0, \ldots, 0, 2b) | b \in \mathbb{N}_0\}$ *and* $X_{1,0} = \{(0, \ldots, 0, 2b+1) \mid b \in \mathbb{N}_0\}$*; for* $1 < k = n - 1$, $X_{0,1} = \{(a, \ldots, a, a + 2b) \mid a, b \in \mathbb{N}_0\}$ *and* $X_{1,0} = \{(a, \ldots, a, a + 2b + 1) | a, b \in \mathbb{N}_0\}$.

The proof is based on the characterization of misérability given in Lemma 2 below, for which we need the following definition. Given a game $G = (X, E)$ and two non-empty sets $X', X'' \subset X$, we say that X' is *movable* to X'' if for every $x' \in X'$ there is a move $x' \to x''$ with $x'' \in X''$.

Lemma 2 **(**[12, Theorem 7]**).** *A game* $G = (X, E)$ *is misérable if and only if there exist non-empty disjoint subsets* $X'_{0,1}, X'_{1,0} \subset X$ *satisfying the following conditions:*

(i) *both sets,* $X'_{0,1}$ *and* $X'_{1,0}$*, are independent, that is, in either of the two sets, there is no legal move between positions in that set.*

(ii) $X'_{0,1}$ *contains all terminal positions, that is,* $X_T \subseteq X'_{0,1}$*;*

(iii) $X'_{0,1} \setminus X_T$ *is movable to* $X'_{1,0}$*;*

(iv) $X'_{1,0}$ *is movable to* $X'_{0,1}$*;*

(v) *from every position* $x \notin X'_{0,1} \cup X'_{1,0}$*, either there exist moves to both* $X'_{0,1}$ *and* $X'_{1,0}$*, or there are no moves to* $X'_{0,1} \cup X'_{1,0}$.

noindent Moreover, $X'_{0,1} = X_{0,1}$ *and* $X'_{1,0} = X_{1,0}$ *if all five conditions hold.* □

Proof of Theorem 3. For both $k = n$ and $k = n - 1$, it is enough to show that the sets $X_{0,1}$ and $X_{1,0}$ defined in the statement of Theorem 3 satisfy all conditions of Lemma 2, which proves the misérability and also characterizes the swap positions. We will consider only the case $k = n - 1$, leaving the simpler one, $k = n$, to the reader.

Let $X'_{0,1} = \{(a, \ldots, a, a + 2b) \mid a, b \in \mathbb{N}_0\}$ and $X'_{1,0} = \{(a, \ldots, a, a + 2b + 1) \mid a, b \in \mathbb{N}_0\}$. Obviously, these two sets are non-empty and disjoint. We will show that they satisfy (i) - (v).

Since each move $x \to x'$ reduces at most $n-1$ entries, x and x' cannot be both in $X'_{0,1}$ or in $X'_{1,0}$, thus, (i) holds. The game has a unique terminal position $x = (0,\ldots,0) \in X'_{0,1}$, so (ii) holds. The moves $(a,\ldots,a,a+2b) \to (a,\ldots,a,a+2b-1)$ for non-terminal positons and $(a,\ldots,a,a+2b+1) \to (a,\ldots,a,a+2b)$ are legal, thus, (iii) and (iv) hold.

To prove (v), it is enough to show that there exists a move from $x \notin X'_{0,1} \cup X'_{1,0}$ to $X'_{0,1}$ if and only if there exists a move from x to $X'_{1,0}$. Let us prove the "only if" part. Assume that there is a move $\mathfrak{m}$ from x to $X'_{0,1}$. To find a move from x to $X'_{1,0}$, we consider two cases:

Case 1: $\mathfrak{m}$ is played on $x_1,\ldots,x_{n-1}$. Therefore, $x = (a+k_1,\ldots,a+k_{n-1},a+2b)$ for some a,b, where $0 \le k_\ell \le 1$ and $\sum_{\ell=1}^{n-1} k_\ell > 0$. Note that $b \ge 1$, since there is an entry $a+k_{\ell_0} > a$. If $k_\ell = 1$ for all ℓ then $x \in X'_{0,1}$, which is a contradiction. Since x is non-decreasing, $k_1 = 0$. In this case, the move $x \to (a,\ldots,a,a+2b-1)$ ends in $X'_{1,0}$.

Case 2: The move $\mathfrak{m}$ also reduces the last entry x_n. Without loss of generality we may assume that play is on the last $n-1$ entries $x_2,\ldots,x_n$, that is, $x = (a,a+k_1,\ldots,a+k_{n-2},a+2b+k_{n-1})$ for some a,b , where $0 \le k_\ell \le 1$ and $\sum_{\ell=1}^{n-1} k_\ell > 0$. (Note that such a move may not be canonical.) Then, there is $\ell_0 \le n-2$ such that $k_{\ell_0} = 1$, because otherwise $x \in X'_{0,1} \cup X'_{1,0}$. Note also that $b \ge 1$ when $k_{n-1} = 0$. Thus, the move $x \to (a,\ldots,a,a+2b-1)$ ends in $X'_{1,0}$ and is legal, irrespective of the value of k_{n-1}.

The "if" part can be proven by similar arguments and we leave it to the reader. Thus, by Lemma 2, the game is misérable; moreover, $X'_{0,1}$ and $X'_{1,0}$ are the two sets of its swap positions, namely, $X_{0,1} = X'_{0,1}$ and $X_{1,0} = X'_{1,0}$. □

Our computations show that the game $\text{Nim}^1_{4,\le 2}$ is neither tame nor domestic, as $(1,1,2,3)$ is a $(4,0)$-position. For the misére version, we do not even have a simple characterization for the $\mathcal{P}$-positions. In contrast, for the normal version such a characterization was given in Section 2.2 for any $\text{Nim}^1_{n,\le k}$ with $4 \le n \le k+2$.

3. Slow Exact *k*-Nim

By definition of $\text{Nim}^1_{n,=k}$, x is a terminal position if and only x has fewer than k positive entries. The game is trivial when $k = 1$ (see Proposition 1) and, unlike $\text{Nim}^1_{n,\le k}$, it is still trivial when $k = n$. The next statement is obvious.

Proposition 2. *The SG function $\mathcal{G}$ of game* $\textsc{Nim}^1_{n,=n}$ *takes only values* 0 *or* 1 *depending on the parity of the smallest pile:*

$$\mathcal{G}(x) = \begin{cases} 0 \text{ if } \min_{i=1}^n x_i \text{ is even;} \\ 1 \text{ if } \min_{i=1}^n x_i \text{ is odd.} \end{cases}$$

Furthermore, $\mathcal{G}^-(x) + \mathcal{G}(x) = 1$*; the* $(0,1)$*- and* $(1,0)$*-positions alternate with every move.*

Let us now consider $\textsc{Nim}^1_{4,=2}$. Somewhat surprisingly, for this game it is simpler to characterize $\mathcal{G}$ and $\mathcal{G}^-$ together rather than separately. Theorem 4 provides an explicit formula for the (i,j)-positions of $\textsc{Nim}^1_{4,=2}$, thus determining simultaneously the i-positions of the normal version and the j-positions of the misère one. As usual, $x = (x_1, x_2, x_3, x_4)$ with $0 \le x_1 \le x_2 \le x_3 \le x_4$ and we define $s = x_1 + x_2 + x_3 + x_4$. Note that $\textsc{Nim}^1_{n,=k}$ is split into k pairwise disjoint subgames $\mathrm{N}\ell$, $\ell = 0, 1, \ldots, k-1$, defined by the residuals $s \equiv \ell \mod k$, because each move reduces the sum of the stack heights by exactly k. Therefore, all options of a position in $\mathrm{N}\ell$ are also in $\mathrm{N}\ell$, so solving $\textsc{Nim}^1_{n,=k}$ is equivalent to solving all these subgames. We now state the result for $\textsc{Nim}^1_{4,=2}$.

Theorem 4. *In* $\textsc{Nim}^1_{4,=2}$*, the pairs* (i,j) *take the eight values*

$$(0,1), (1,0), (0,0), (1,1), (2,2), (3,3), (0,3), \text{ and } (1,2)$$

according to the following classification:

(R0) *If* $x_1 = x_2$*, then* x *is a* $(0,1)$*- or* $(1,0)$*-position if* $x_2 + x_3$ *is even or odd, respectively.*

If $x_1 < x_2$*, then we have the following subcases:*

(RT) *The four tame* (i,i)*-values are taken when* s *is even or* $x_1 + x_4 > x_2 + x_3$*, in accordance with the parity rules:*

$$(\mathcal{G}(x), \mathcal{G}^-(x)) = \begin{cases} (0,0) \text{ if } p(x_1,x_2,x_3) = (e,e,e) \text{ or } (o,o,o); \\ (1,1) \text{ if } p(x_1,x_2,x_3) = (e,e,o) \text{ or } (o,o,e); \\ (2,2) \text{ if } p(x_1,x_2,x_3) = (e,o,o) \text{ or } (o,e,e); \\ (3,3) \text{ if } p(x_1,x_2,x_3) = (e,o,e) \text{ or } (o,e,o). \end{cases}$$

If s *is odd and* $x_1 + x_4 \le x_2 + x_3$ *(in fact, the inequality must be strict), then we have two more subcases:*

(RNT) *The non-tame values* $(0,3)$ *(resp.* $(1,2)$*) are taken when* $s \equiv 1 \mod 4$ *(resp.,* $s \equiv 3 \mod 4$*) and* $x_1 + x_4 + 1 = x_2 + x_3$*,* $x_1 + x_2$ *is odd, and* $x_3 < x_4$*.*

(RS) *Additional swap values* $(0,1)$ (*resp.,* $(1,0)$) *are taken when* $s \equiv 1 \mod 4$ (*resp.,* $s \equiv 3 \mod 4$) *and either* $x_1 + x_4 + 1 < x_2 + x_3$ *or* $x_1 + x_2$ *is even or* $x_3 = x_4$.

In addition, the subgame N0 is misèrable, and hence tame.

Before we prove the statement, we will collect some properties that will be referenced repeatedly in the proof.

Remark 2. Let $s = x_1 + x_2 + \cdots + x_n$.

(1) Specifically, $\text{Nim}^1_{n,=2}$ has two subgames, N0 and N1, where the sum s is even or odd, respectively. Since each move reduces s by 2, positions in N1 alternate between $s \equiv 1 \mod 4$ and $s \equiv 3 \mod 4$.

(2) In $\text{Nim}^1_{n,=2}$, the parity vector $p(x_1, x_2, \ldots, x_{n-1})$ changes in each move since there has to be play on at least one of $x_1, \ldots, x_{n-1}$.

(3) To move between *(RT)* and *(RS)*∪*(RNT)*, we must be in N1, so s is odd. In *(RT)*, we have $x_1+x_4 > x_2+x_3$, while in *(RS)*∪*(RNT)*, $x_1+x_4+1 \leq x_2+x_3$, so $x_1 + x_4 < x_2 + x_3$. The only way we can move between the two classes is if in *(RT)*, we have $x_1 + x_4 = x_2 + x_3 + 1$, and play is on x_1 and x_4 for a move from *(RT)* to *(RS)*∪*(RNT)*, and on x_2 and x_3 for a move in the other direction. Then we have in *(RT)* that $s = 2(x_2+x_3)+1 = 2(x_1+x_4)-1$ and

$$s \equiv 1 \mod 4 \Leftrightarrow x_2 + x_3 \text{ even } \Leftrightarrow x_1 + x_4 \text{ odd}$$

$$s \equiv 3 \mod 4 \Leftrightarrow x_2 + x_3 \text{ odd } \Leftrightarrow x_1 + x_4 \text{ even;}$$

in *(RS)*∪*(RNT)*, the roles of $x_1 + x_4$ and $x_2 + x_3$ are reversed.

Proof. It is easy to verify that the four classes partition the set of all positions $\{x = (x_1, x_2, x_3, x_4) \mid 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4\}$. The terminal positions are of the form $(0, 0, 0, \ell)$ for $\ell \in \mathbb{N}_0$. These positions are in (R0) and are $(0,1)$-positions, as they should be. They belong to the subgame N0 or N1 depending on whether ℓ is even or odd.

We now look at the non-terminal positions and consider each of the subgames N0 and N1 separately. Note that positions in N0 can only be in classes (R0) and (RT), while positions in N1 occur in all four classes. We start by showing that N0 is misèrable. Let $X'_{0,1}$ denote the $(0,1)$-positions of (R0) and $X'_{1,0}$ denote the $(1,0)$-positions.

To move within (R0), we need to play on either (x_1 and x_2) or on (x_3 and x_4), that is, on exactly one of x_2 and x_3, so the parity of x_2+x_3 changes. This is possible if $x_3 > 0$, a non-terminal position. The terminal positions all have $x_2+x_3 = 0$ even, so they are contained in $X'_{0,1}$. This shows the first four conditions of Lemma 2 are satisfied. Next we look at the positions in (RT), the non-swap positions in N0. To show that N0 is misèrable, we need to show that from each N0-position in (RT), there is either a move to both of the swap positions, or that there is no move to either one of them. To reach one of the swap positions from a non-swap position, we need to have $x_2 = x_1 + 1$. Since we have that x_1 and x_2 have the same parity for $(0,0)$- and $(1,1)$-positions, there is no move to (R0) from those tame positions. For $(2,2)$- and $(3,3)$-positions with $x_2 = x_1 + 1$, we can reach $(0,1)$- and $(1,0)$-positions in (R0) by playing on x_2 and on one of x_3 or x_4 in order to make $x_2 + x_3$ even or odd. If $x_2 > x_1 + 1$, then there is no move to (R0). By Lemma 2, N0 is misèrable, and therefore, tame.

We still have to show that the positions in (RT) have the claimed values for $\mathcal{G}$ and $\mathcal{G}^-$. Since each move in $\mathrm{NIM}^1_{4,=2}$ uniquely corresponds to a move in $\mathrm{NIM}^1_{3,\leq 2}$, the result follows immediately from Theorem 1(ii) for $\mathrm{NIM}^1_{3,\leq 2}$. Since tame positions have $\mathcal{G} = \mathcal{G}^-$, we have proved the claim for N0.

We now look at the subgame N1, for which the positions are in all four classes. Specifically, the swap positions consist of the set (R0) $\cup$ (RS). To prove the statement for N1, we have to verify the properties (SGN$\pm$) and (SGY$\pm$) of Section 1.5 for functions $\mathcal{G}$ and $\mathcal{G}^-$ defined by (R0), (RT), (RS), and (RNT) when s is odd. This is a routine but long case analysis. If successful, it implies that $\mathcal{G}$ and $\mathcal{G}^-$ are indeed the normal and misère SG functions of N1 and that the (i,j)-positions of N1 are defined in accordance with Theorem 4. For simplicity, we will refer to the claimed (i,j)-positions and SG values in the proof, although, technically speaking, this will be justified only when the proof is completed.

Let us begin with (SGN$\pm$), which claims that no move preserves the values of $\mathcal{G}$ and $\mathcal{G}^-$. By Remark 2, (1) and (2), a move from a position x to x' within one of (RT), (RS), and (RNT) cannot preserve the SG-values. As mentioned in the proof for N0, moves within (R0) are from one swap position to the other. Therefore, SG values could only be maintained by moves between the four classes.

Let's now consider the different possible SG values starting with values 2 and 3. Specifically, we need to check whether there is a possible move that would preserve the $\mathcal{G}^-$ values in moves between a $(1,2)$- and a $(2,2)$-position or between a $(0,3)$- and a $(3,3)$-position, respectively. For the $(0,3)$-position, $x_1 + x_2$ is odd, and for the $(3,3)$-position, the parity vector is either (e,o,e) or (o,e,o). To maintain the parities, moves have to be either on x_1 and x_2 or on x_3 and x_4, which is not possible by Remark 2(3). A similar argument shows that the $\mathcal{G}^-$-value 2 cannot be maintained.

Next we look at whether it is possible to preserve SG values 0 and 1. These values show up in all four classes and we need to consider the possible moves between them. Let's first consider moves between (RT) and (RNT)$\cup$(RS). If x in (RT) is a $(0,0)$-position, then x_2 and x_3 have the same parity. Since both $(0,3)$-positions and $(0,1)$-positions x' have $s' \equiv 1 \mod 4$, then by Remark 2, x must have $s \equiv 3 \mod 4$ and hence, $x_2 + x_3$ is odd, a contradiction. On the other hand, if we try to move from either a $(0,3)$- or a $(0,1)$-position x in (RNT)∪(RS) to a $(0,0)$-position x' in (RT), then we have that $x_2 + x_3$ is odd, and we have to play on both of these stacks, which violates the parity rule for x'. A similar argument shows that there are no moves between the $(1,1)$- and the $(1,2)$- and $(1,0)$-positions.

To move between (RT) and (R0), we need to have $x_2 = x_1 + 1$. This violates the parity rule for both $(0,0)$- and $(1,1)$-positions. On the other hand, to move from (R0) to (RT), we have to play on x_1 and not x_2. Therefore, the parities of x'_1 and x'_2 are different, so x' cannot be a $(0,0)$- or $(1,1)$-position.

For moves between (RNT) and (RS) note that positions with $\mathcal{G} = 0$ have $s \equiv 1 \mod 4$, so there cannot be a move between them by Remark 2 (2). Likewise, any positions with $\mathcal{G} = 1$ have $s \equiv 3 \mod 4$, so no moves exist between those positions.

It remains to show that there are no moves between positions of (R0) and (RS)∪(RNT) that maintain the SG values. Note that any such move requires that the position in (RS)∪(RNT) satisfies $x_2 = x_1 + 1$. So let's try to move from a $(0,1)$-position in (R0) to a $(0,1)$-position in (RS) (resp. $(0,3)$-position in (RNT)). Since $x_1 = x_2$ and s is odd, we have that $x_3 < x_4$. If $x_4 = x_3+1$, then $s = 2x_2+2x_3+1 \equiv 1 \mod 4$ since $x_2 + x_3$ is even. But then x' would have $s' \equiv 3 \mod 4$, so x' cannot be a $(0,1)$-position in (RS) nor a $(0,3)$-position in (RNT). Therefore $x_3 + 1 < x_4$. Now we know that $x'_1 = x_1 - 1$ and $x'_2 = x_2$, so $x'_1 + x'_2$ is odd. We also need to move on exactly one of x_3 and x_4, and $x_3 + 1 < x_4$ implies that $x'_3 + 1 \leq x'_4$ in either case, so $x'_1 + x'_4 + 1 = x_1 + x'_4 \geq x'_2 + x'_3 + 1 > x'_2 + x'_3$. Checking the conditions for (RS)∪(RNT) we see that there is no move to the desired positions. On the other hand, to move from a $(0,1)$-position in (RS) to (R0), we need to have $x_2 = x_1 + 1$, so neither the first nor the second condition of (RS) can be satisfied; thus $x_3 = x_4$ and $x = (x_2 - 1, x_2, x_3, x_3)$. Since $s = 2x_2 + 2x_3 - 1 \equiv 1 \mod 4$, $x_2 + x_3$ is odd. Then $x' = (x_2 - 1, x_2 - 1, x_3 - 1, x_3)$ with $x'_2 + x'_3$ is odd and the move is to a $(1,0)$-position in (R0). There is also no move from a $(0,3)$ position in (RNT) to a $(0,1)$-position in (R0), because $x_2 = x_1 + 1$ and $x_1 + x_4 + 1 = x_2 + x_3$ imply $x_3 = x_4$, a contradiction. Very similar arguments show that there are no moves between the $(1,0)$-positions and the $(1,2)$- and $(1,0)$-positions, respectively. This shows that (SGN±) holds.

We now verify (SGY±) and start with the case "$1 \to 0$". Then x is either a $(1,1)$-position, one of a pair of $(0,1)$ and $(1,0)$ swap positions, or a $(1,2)$-position. If x is a $(1,1)$-position in (RT), then play on x_3 and x_4 results in a $(0,0)$-position in (RT) since we maintain the conditions of (RT). For swap positions in (R0) and in (RNT), respectively, play on x_3 and x_4 (x_1 and x_2, respectively) results in the other type of swap position. For the $(1,2)$-position in (RNT), play on x_1 and x_2 leads to the $(0,3)$-position in (RNT). In each case, the SG values of $\mathcal{G}(x)$ and/or $\mathcal{G}^-(x)$ reduce from 1 to 0.

Next we verify (SGY$\pm$) for "$2 \to 0, 1$". First we consider $(2, 2)$-positions of (RT). We need to consider two cases, namely $x_2 > x_1 + 1$ and $x_2 = x_1 + 1$. In the first case, reducing x_2 and x_4 (resp., x_2 and x_3), we obtain a $(1, 1)$-position (resp., a (0,0)-position) in (RT) since we maintain the conditions of (RT), particularly, $x'_2 > x'_1$. In the second case, play on x_2 and x_4 (resp., x_2 and x_3) leads to a $(0, 1)$-position (resp. $(1, 0)$-position) in (R0). Note that $x_2 = x_1 + 1$ together with $x_1 + x_4 > x_2 + x_3$ implies that $x_3 + 1 < x_4$, which ensures that $x'_3 \le x'_4$. This is the only case where $\mathcal{G}(x) = 2$.

For $\mathcal{G}-$, we also need to consider $(1, 2)$-positions of (RNT), which satisfy $x_1 + x_4 + 1 = x_2 + x_3$, $s \equiv 3 \mod 4$, and $x_1 + x_2$ odd. By Remark 2(3), we also have that $x_2 + x_3$ is even, so the parity vector is of the form (o, e, e) or (e, o, o). Then play on x_2 and x_3 results in a $(1, 1)$-position in (RT) because $x'_1 + x'_4 = x_1 + x_4 = x_2 + x_3 - 1 = x'_2 + x'_3 + 1 > x'_2 + x'_3$ and we have the correct parity. On the other hand, play on x_2 and x_4 leads to a $(1, 0)$-position in (RS) because $x'_1 + x'_2$ is even and $s' \equiv 3 \mod 4$.

Finally we verify (SGY$\pm$) for "$3 \to 0, 1, 2$". First, consider the $(3, 3)$-positions of (RT), for which $x_1 + x_4 > x_2 + x_3$ and $p(x_1, x_2, x_3, x_4) \in \{(e, o, e, e), (o, e, o, o)\}$. Reducing x_3 and x_4 leads to a $(2, 2)$-position and maintains the required conditions for (RT). If $x_1 + 1 < x_2$, then either reducing x_2 and x_4 or x_2 and x_3 leads to a $(0, 0)$-position and a $(1, 1)$-position, respectively (since $x_1 < x_2$ implies $x_3 < x_4$). If $x_1 + 1 = x_2$, then play on either x_2 and x_4 or x_2 and x_3 results in the $(0, 1)$- and $(1, 0)$-positions of (R0).

For $\mathcal{G}-$, we also need to consider the $(0, 3)$-positions of (RNT), which satisfy $x_1 + x_4 + 1 = x_2 + x_3$, $s \equiv 1 \mod 4$, $x_1 + x_2$ is odd, and $x_3 < x_4$. By Remark 2(3), we also have $x_1 + x_4$ is even and $x_2 + x_3$ is odd. This means that $p(x_1, x_2, x_3, x_4) \in \{(o, e, o, o), (e, o, e, e)\}$. Reducing either x_2 and x_3 or x_2 and x_4 or x_3 and x_4, we obtain a $(1, 1)$-position, a $(1, 0)$-position in (RS), or a $(1, 2)$-position, respectively. In the first case we obtain $x'_1 + x'_4 = x_2 + x_3 - 1 > x'_2 + x'_3$, which places x' into (RT), and x' has the correct parity. In the other cases, $x'_1 + x'_4 + 1 = x'_3 + x'_2$, $s' \equiv 3 \mod 4$, and $x'_1 + x'_2$ is either even or odd. Finally, we note that all three types of moves are legal because for $(0, 3)$-positions in (RNT), we have $1 < x_2 < x_3 < x_4$. This completes the proof. □

We obtain the results for $\text{Nim}^1_{3,=2}$ as a special case by setting $x_1 = 0$ and then reindexing x_2 through x_4 as x_1 through x_3. Note that this reduces the tame values to just one parity case because $x_1 = 0$ is even.

Corollary 1. *In* $\text{Nim}^1_{3,=2}$*, the pairs* (i, j) *take the eight values*

$$(0, 1), (1, 0), (0, 0), (1, 1), (2, 2), (3, 3), (0, 3), \textit{and } (1, 2)$$

according to the following classification:

(R0) *If $x_1 = 0$, then x is a $(0,1)$- or $(1,0)$-position when x_2 is even or odd, respectively.*

If $x_1 > 0$, then we have the following subcases:

(RT) *The four "tame" (i,i)-values are taken when s is even or $x_1 + x_2 \leq x_3$, in accordance with the parity rule:*

$$(\mathcal{G}(x), \mathcal{G}^-(x)) = \begin{cases} (0,0) & \text{if } p(x_1,x_2) = (e,e); \\ (1,1) & \text{if } p(x_1,x_2) = (e,o); \\ (2,2) & \text{if } p(x_1,x_2) = (o,o); \\ (3,3) & \text{if } p(x_1,x_2) = (o,e). \end{cases}$$

If s is odd and $x_3 \leq x_1 + x_2$ (in fact, the inequality must be strict), then we have two more subcases:

(RNT) *The non-tame values $(0,3)$ (resp. $(1,2)$) are taken when $s \equiv 1 \mod 4$ (resp., $s \equiv 3 \mod 4$) and $x_3 + 1 = x_1 + x_2$, x_1 is odd, and $x_2 < x_3$.*

(RS) *Additional swap values $(0,1)$ (resp., $(1,0)$) are taken when $s \equiv 1 \mod 4$ (resp., $s \equiv 3 \mod 4$) and either $x_3 + 1 < x_1 + x_2$ or x_1 is even or $x_2 = x_3$.*

In addition, the subgame N0 is misèrable, and hence tame.

4. Summary and Future Work

We summarize the results we have obtained for Slow Moore's k-Nim in Table 1, where G refers to specification of the SG values, and P refers to specification of $\mathcal{P}$-values only. We do not list $k = 1$ as this game is just Nim. Overall we see that we have results for three families of games when k is close to n. For $n = 6$, the case $k = 2$ is missing, and for $n \geq 7$, we only know the results for the families of games. Thus, an open question is whether any patterns that are not just based on the parity vectors can be found for smaller values of k, for either the SG values or even just for the $\mathcal{P}$-positions.

Table 2 shows the results for Slow Exact k-Nim. Here we have additional SG values for the misère version, which will will indicate with G/G^- in the table. In the proofs for these results, we have been able to transfer results from Slow Moore's k-Nim to Slow Exact k-Nim when $k = 2$. Specifically, a move in $\text{Nim}^1_{n,=2}$ corresponds uniquely to a move in $\text{Nim}^1_{n-1,\leq 2}$. Unfortunately, this idea cannot be extended beyond $k = 2$, and therefore, we cannot obtain additional results from Slow Moore's k-Nim. We are left with a trivial result for the family of games with $n = k$ and two specific results, leaving many open questions. These cases will require

$n\backslash k$	2	3	4	5	6	$\cdots$
2	G					
3	G	P				
4	P	P	P			
5	P	P	P	P		
6	?	P	P	P	P	
7	?	?	?	$\ddots$	$\ddots$	$\ddots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		

Table 1: Characterizations for Slow Moore's k-Nim.

$n\backslash k$	2	3	4	5	6	$\cdots$
2	G/G^-					
3	G/G^-	G/G^-				
4	G/G^-	?	G/G^-			
5	?	?	?	G/G^-		
6	?	?	?	?	G/G^-	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		

Table 2: Characterizations for Slow Exact k-Nim.

additional structure beyond the parity vector to describe either the SG values or the $\mathcal{P}$-positions as was shown in the computational examples we provided.

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