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AN ACHIEVEMENT GAME ON A CYCLE

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Abstract

Consider the following game played by Maker and Breaker on the vertices of the cycle C_n, with first move given to Breaker. The aim of Maker is to maximize the number of adjacent pairs of vertices that are both claimed by her, and the aim of Breaker is to minimize this number. The aim of this paper is to find this number exactly for all n when both players play optimally, answering a related question of Dowden, Kang, Mikalački and Stojaković.

1. Introduction

Consider the following game, called the 'Toucher-Isolator' game on a graph G, introduced by Dowden, Kang, Mikalački and Stojaković [4]. The two players, Toucher and Isolator, claim edges of G alternately with Toucher having the first move. Let $t(G)$ be the number of vertices that are incident to at least one of the edges claimed by Toucher. The aim of Toucher is to maximize $t(G)$ and the aim of Isolator is to minimize $t(G)$. Hence this is a 'quantitative' Maker-Breaker type of game.

For given G, let $u(G)$ be the number of isolated vertices, i.e. the number of vertices that are not incident to any of the edges claimed by Toucher at the end of the game when both players play optimally. In [4], the authors gave bounds for the size of $u(G)$ for general graphs G, and studied some particular examples as well which included cycles and paths. In particular, they proved that

$$\frac{3}{16}(n-3) \leq u(C_n) \leq \frac{n}{4}$$

and

$$\frac{3}{16}(n-1) \leq u(P_n) \leq \frac{n}{4},$$

where C_n is a cycle with n vertices and P_n is a path with n edges.

Note that these bounds imply that the asymptotic proportion of untouched vertices is between $\frac{3}{16}$ and $\frac{1}{4}$ in both cases. Dowden, Kang, Mikalački and Stojaković asked what the correct asymptotic proportion of untouched vertices is, and suggested that the correct answer could be $\frac{1}{5}$. In this paper we prove that this is the correct asymptotic proportion, and in fact we give the exact values of $u\left(C_n\right)$ and $u\left(P_n\right)$ for all n.

Theorem 1. *When $G = C_n$ with $n \geq 3$ and both players play optimally, there will be $\left\lfloor \frac{n+1}{5} \right\rfloor$ untouched vertices.*

Theorem 2. *When $G = P_n$ with $n \geq 2$ and both players play optimally, there will be $\left\lfloor \frac{n+4}{5} \right\rfloor$ untouched vertices.*

Although this paper is self-contained, for general background on Maker-Breaker type games; see Beck [2]. There are many other papers dealing with achievement games on graphs – see e.g. a classical paper of Chvátal and Erdős [3], and subsequent papers [1, 5].

For convenience we work on a 'dual version' of these games. Consider a game played on the vertices of a cycle C_n with two players Maker and Breaker claiming vertices in alternating turns, with first move given to Breaker. For this game, define the *score* to be the number of adjacent pairs of vertices claimed by Maker on the cycle. It is easy to see that this game is identical to the Toucher-Isolator game played on C_n, with Maker corresponding to Isolator and Breaker corresponding to Toucher. Indeed, this follows from the fact that claiming adjacent pairs of vertices on the dual game corresponds to claiming two edges whose endpoints meet in the original game, which is precisely the same as isolating the vertex where they meet.

When considering the dual version for the path, we have to be a bit more careful due to irregular behaviour at the endpoints. For that reason it turns out to be useful to define three different games which essentially only differ at the endpoints of the path. First we define a game $F(n)$ played on the elements of $\{1, \ldots, n\}$ with two players Maker and Breaker claiming elements in alternating turns with first move given to Maker. For this game, define the *score* to be the number of pairs $\{i, i+1\}$ such that both i and $i+1$ are claimed by Maker, and as usual Maker is aiming to maximize this score and Breaker is aiming to minimize this score. Let $\alpha(n)$ be the score attained when both Maker and Breaker play optimally.

Similarly as with C_n, the game $F(n-1)$ and the Toucher-Isolator game on P_n have a strong relation with Maker corresponding to Isolator and Breaker corresponding to Toucher. However, unlike with the game on the cycle, they are not exactly the same game as first and last vertex can be isolated by claiming only the first or last edge respectively, and also because Toucher has first move in the Toucher-Isolator game whereas Maker has first move in the game $F(n-1)$.

We also define the games $G(n)$ and $H(n)$ both played on $\{1, \ldots, n\}$, with players Maker and Breaker claiming elements in alternating turns with first move given to

Maker. In $G\left(n\right)$, we increase the score by one for each pair $\{i, i+1\}$ with both i and $i+1$ claimed by Maker, and the score is also increased by 1 if Maker claims the element 1. In a sense, this can be viewed as a game on the board $\{0, \ldots, n\}$ with 0 assigned to Maker initially. Similarly on $H\left(n\right)$, we increase the score by one for each pair $\{i, i+1\}$ with both i and $i+1$ claimed by Maker, and additionally the score is increased by 1 for claiming either of the elements 1 or n. Again, this can be viewed as a game on the board $\{0, \ldots, n+1\}$ with both 0 and $n+1$ assigned to Maker initially. Define $\beta\left(n\right)$ and $\gamma\left(n\right)$ to be the scores of these games when both players play optimally.

The idea behind defining these games is the following. If B is a game of the form $F\left(n\right)$, $G\left(n\right)$ or $H\left(n\right)$, and if Breaker plays her first move adjacent to Maker's first move, then the board B splits into two disjoint boards, which are of the form $F\left(m\right)$, $G\left(m\right)$ or $H\left(m\right)$ - however, note that these two boards are not in general of the same form, and not necessarily of the same form as the original board. Hence it turns out to be useful to analyze all of these games at the same time.

Consider the dual game played on C_n with vertex set $\{1, \ldots, n\}$, and recall that in this dual version the first move is given to Breaker. By the symmetry of the cycle, we may assume that Breaker claims n on her first move. Hence after this first move, the available winning lines that can increase the score are $\{1, 2\}, \ldots, \{n-2, n-1\}$. These are exactly the winning lines of the game $F\left(n-1\right)$, and since Maker has the next move it follows that the subsequent game is equivalent to the game $F\left(n-1\right)$. Hence $u\left(C_n\right) = \alpha\left(n-1\right)$, and thus it suffices to find the value of $\alpha\left(n\right)$ for all n.

To analyze the Toucher-Isolator game on P_n, define the game $H_b\left(n\right)$ in exactly the same way as $H\left(n\right)$, but with the first move given to Breaker, and let $\gamma_b\left(n\right)$ be the score of this game when both players play optimally. It is easy to see that the Toucher-Isolator game and $H_b\left(n\right)$ are equivalent in the same sense as the Toucher-Isolator game on C_n and $F\left(n-1\right)$ are. Hence it follows that $u\left(P_n\right) = \gamma_b\left(n\right)$, and thus it suffices to find the value of $\gamma_b\left(n\right)$.

We start by focusing on $F\left(n\right)$ and finding a lower bound for $\alpha\left(n\right)$. Since Maker is trying to maximize the score, it seems sensible for her to start by claiming some suitably chosen i, and then trying to claim as long a block of consecutive elements as possible. As long as $i \notin \{1, n\}$, she can certainly guarantee a block of length at least 2. Now suppose she has claimed a block of length t, and she cannot proceed in this way. This means that Breaker must have claimed the points next to the endpoints of this block (or one of the endpoints is 1 or n). Removing this block, together with the endpoints Breaker has claimed, leaves a path with $n-t-1$ elements containing at most $t-1$ elements claimed by Breaker, and no elements claimed by Maker.

This motivates the definition of the following game, which can be viewed as a delayed version of $F\left(n\right)$. Let $F\left(n, k\right)$ be the game played on $\{1, \ldots, n\}$, where at the start of the game Breaker is allowed to claim k points, and then the players claim elements alternately, with the score defined in the same way as for $F\left(n\right)$.

Thus $F(n)$ and $F(n,0)$ are identical games.

Let $\alpha(n,k)$ be the score attained when both players play optimally. It turns out that by following the strategy described above with a suitable choice of the initial move, we can prove a good enough lower bound for $\alpha(n,k)$, and almost the same argument also works for $\gamma_b(n)$.

One can observe from the proof of the lower bound of $\alpha(n,k)$ that allowing Maker to have multiple 'long blocks' would allow Maker to achieve a better score than the one stated in Theorem 2. This suggests that Breaker should claim an element next to the element claimed by Maker, and hence the initial board splits into two disjoint boards. Hence it is natural to consider games G that are disjoint unions of $F(l_1),\dots,F(l_r)$, $G(m_1),\dots,G(m_s)$ and $H(n_1),\dots,H(n_t)$.

The plan of the paper is as follows. In Section 2 we prove a lower bound for $\alpha(n,k)$ and deduce a lower bound for $\gamma_b(n)$. In Section 3 we prove an upper bound for the score of games G that are disjoint unions of games of the form $F(l)$, $G(m)$ and $H(n)$, and conclude the Theorems 1 and 2 from these upper and lower bounds.

2. The Lower Bound

Recall that $F(n,k)$ is defined to be the game played on $\{1,\dots,n\}$, where at the start of the game Breaker is allowed to claim k elements, and then the players claim elements in alternating order, and $\alpha(n,k)$ is the score attained when both players play optimally. We start by proving the following lower bound on $\alpha(n,k)$, which is later used to deduce a lower bound on $\gamma_b(n)$.

Lemma 1. *Let n be a positive integer and let $0 \le k \le n$. Then $\alpha(n,k) \ge \left\lfloor \frac{n-3k+2}{5} \right\rfloor$.*

Proof. Suppose that Breaker claims the elements $s_1,\dots,s_k$ on her first move. These elements split the path into $k+1$ (possibly empty) intervals of lengths $l_0,\dots,l_k$, with $l_i = s_{i+1} - s_i - 1$ (with the convention $s_0 = 0$ and $s_{k+1} = n+1$). By symmetry we may assume that l_0 is the longest interval.

If $l_0 \le 2$, then $n \le k + 2\cdot(k+1) = 3k+2$, and hence $\left\lfloor \frac{n-3k+2}{5} \right\rfloor = 0$. Thus the claim follows immediately in this case, and hence we may assume that $l_0 \ge 3$. We treat the cases $l_0 \ge 4$ and $l_0 = 3$ individually. In both cases the proof follows the same idea, however the choice of the initial move is slightly different for $l_0 = 3$ since an interval with only 3 elements is 'too short' for the general argument.

`Case 1.` $l_0 \ge 4$.

The aim for Maker is to build a long block of consecutive elements inside the interval. Initially, she claims the element 3. Assuming she has already claimed exactly the elements $\{t,\dots,t+r\}$, she claims one of $t+r+1$ or $t-1$, if possible. If not, she stops.

Consider the point when this process terminates, and suppose that at the point of termination she has claimed the set of elements $\{t, \ldots, t+r\}$. As one of these is the element 3, we must have $t+r \geq 3$ and $t \in \{1,2,3\}$. Also note that the element $t+r+1$ must be claimed by Breaker, and also either $t=1$ or the element $t-1$ is claimed by Breaker. Since $l_0 \geq 4$, it follows that the elements 2 and 4 are not claimed after Maker's first move. Because Breaker cannot claim both of these on her first move, it follows that Maker can always guarantee that $r \geq 1$.

Let $T_1 = \{t+r+2, \ldots n\}$ and let b be the number of elements claimed by Breaker in T_1. Note that Breaker has claimed $k+r+1$ elements in total, and one of these must be $t+r+1$. Furthermore, if $t > 1$ then one of them must be $t-1$ as well. Hence $b \leq k+r$, and if $t \geq 2$ we also have $b \leq k+r-1$. Also note that Maker has not claimed any elements in T_1.

Note that claiming the elements $\{t, \ldots, t+r\}$ increases the score by exactly r, and this is the only contribution for the score coming outside T_1. Thus the total score that Maker can attain is at least $r + \alpha(n-t-r-1, b)$. By induction, it follows that the score is at least

$$r + \left\lfloor \frac{n-t-r-1-3b+2}{5} \right\rfloor . \tag{1}$$

If $t=1$, it follows that $b \leq k+r$. Also, the condition $t+r \geq 3$ implies that $r \geq 2$. Hence (1) implies that Maker can guarantee that the score is at least

$$\left\lfloor \frac{n-3k+r+1-t}{5} \right\rfloor \geq \left\lfloor \frac{n-3k+2}{5} \right\rfloor ,$$

as required.

If $t \geq 2$, it follows that $b \leq k+r-1$. Recall that we always have $t \leq 3$ and $r \geq 1$. Hence (1) implies that Maker can guarantee that the score is at least

$$\left\lfloor \frac{n-3k-t+r+4}{5} \right\rfloor \geq \left\lfloor \frac{n-3k-3+1+4}{5} \right\rfloor = \left\lfloor \frac{n-3k+2}{5} \right\rfloor .$$

Hence we have $\alpha(n,k) \geq \left\lfloor \frac{n-3k+2}{5} \right\rfloor$, as required.

`Case 2.` $l_0 = 3$.

Again, Maker is aiming to claim as long a block of consecutive elements in $\{1,2,3\}$ as possible. Initially she claims the element 2. Since Breaker cannot pick both 1 and 3 on her first move, Maker can always guarantee that the length of this block is at least 2. If possible, she picks the last element on her third move.

Thus at the end of this process, exactly one of the following are true:

1. Maker has claimed all three elements in $\{1,2,3\}$.

2. Maker has claimed two consecutive elements in $\{1,2,3\}$ and Breaker has claimed the third element in $\{1,2,3\}$.

In both cases, consider the other moves of the game that are played on $T_1 = \{5, \ldots, n\}$. Let a be the number of elements Maker claims in $\{1, 2, 3\}$. Note that in both cases Breaker claims all the other elements in $\{1, 2, 3, 4\}$ not claimed by Maker, and thus Breaker claims $4 - a$ elements in $\{1, 2, 3, 4\}$. Since Breaker claims in total $a + k$ elements, it follows that she claims $k + 2a - 4$ elements on T_1. Note that Maker has not yet claimed any elements in T_1, and hence Maker can increase the score on T_1 by $\alpha(n - 4, k + 2a - 4)$. Given that she has achieved a score of $a - 1$ outside T_1 with her block of a consecutive elements, it follows that the total score achieved is $a - 1 + \alpha(n - 4, k + 2a - 4)$.

By induction, it follows that the score achieved is at least

$$a - 1 + \left\lfloor \frac{n - 4 - 3(k + 2a - 4) + 2}{5} \right\rfloor = \left\lfloor \frac{n - 3k - a + 5}{5} \right\rfloor.$$

Since $a \in \{2, 3\}$, it follows that

$$\alpha(n, k) \geq \left\lfloor \frac{n - 3k + 2}{5} \right\rfloor,$$

as required.

Thus Lemma 1 holds by induction. □

Lemma 2. *$\gamma_b(n) \geq \left\lfloor \frac{n+4}{5} \right\rfloor$ for $n \geq 2$ and $\gamma_b(1) = 0$.*

Proof. When $n = 1$, the claim is trivial as the only move is given to Breaker. Now we consider the case $n \geq 2$.

At the start of the game, Maker is aiming to claim as long blocks of consecutive elements as possible near the endpoints. Once this is no longer possible, she starts using the same strategy as in Lemma 1. We start by describing this initial process formally.

Suppose that after Maker's k^{th} move the set of elements claimed by Maker is of the form $\{1, \ldots t\} \cup \{n - k + t + 1, \ldots, n\}$ for some $t \in \{0, \ldots, k\}$, with the convention that $\{1, \ldots, t\} = \emptyset$ when $t = 0$ and $\{n - k + t + 1, \ldots, n\} = \emptyset$ when $t = k$. Note that this certainly holds when $k = 0$, as Maker has not claimed any elements before her first move. If at least one of the elements $t + 1$ or $n - k + t$ is not yet claimed before Maker's $k + 1^{th}$ move, then Maker claims one of these elements which is still available, and thus the set of vertices claimed by Maker is of this form also after $k + 1$ moves. If both $t + 1$ and $n - k + t$ are claimed by Breaker, then the process stops.

This process terminates trivially, as Breaker must claim an element during the game. Suppose that when the process terminates, the set of vertices claimed by Maker is of the form $\{1, \ldots t\} \cup \{n - k + t + 1, \ldots, n\}$ for some k and $t \in \{0, \ldots, k\}$. Note that we must have $k \geq 1$, as Breaker cannot claim both elements 1 and n on her first move.

Let $T = \{t+2, \ldots, n-k+t-1\}$, and note that by the choice of k and t it follows that Maker has not yet claimed any elements in T. Because the process has terminated at this stage, it follows that Breaker must have claimed the elements $t+1$ and $n-k+t$. Since Breaker started the game, she has claimed $k+1$ elements in total, and thus $k-1$ of these elements must be in T.

Note that any increment of the score arising outside T occurs from the sets $\{1, \ldots, t\}$ and $\{n-k+t+1, \ldots, n\}$. On the other hand, since Maker has not claimed any elements in T and Breaker has claimed $k-1$ elements in T, the rest of the game on T corresponds to the game $F(n-k-2, k-1)$. Hence Maker can increase the score by at least $\alpha(n-k-2, k-1)$ in T.

It is easy to check that the contribution on the score arising from the intervals $\{1, \ldots, t\}$ and $\{n-k+t+1, \ldots, n\}$ is exactly $t+(k-t)=k$. Hence by Lemma 1, it follows that Maker can guarantee that the score is at least

$$k + \alpha(n-k-2, k-1) \geq k + \left\lfloor \frac{n-k-2-3(k-1)+2}{5} \right\rfloor = \left\lfloor \frac{n+k+3}{5} \right\rfloor .$$

Since $k \geq 1$, it follows that Maker can always guarantee that the score is at least $\left\lfloor \frac{n+4}{5} \right\rfloor$, which completes the proof. □

3. The Upper Bound

In this section, all congruences are considered modulo 5 unless otherwise stated, and hence we omit (mod 5) from the notation. Furthermore, we write $n \equiv 0$ or 1 instead of $n \equiv 0$ or $n \equiv 1$, and $n \not\equiv 0$ and 1 instead of $n \not\equiv 0$ and $n \not\equiv 1$.

Lemma 3. *Suppose T is a disjoint union of games $F(l_1), \ldots, F(l_r)$, $G(m_1), \ldots, G(m_s)$ and $H(n_1), \ldots, H(n_t)$, with Maker having the first move. Let $f(\underline{l}; \underline{m}; \underline{n})$ be the score of this game when both players play optimally. Let $N_1 = |\{i : l_i \equiv 3 \text{ or } 4\}|$, $N_2 = |\{i : m_i \equiv 0 \text{ or } 1\}|$, $N_3 = |\{i : n_i \neq 2 \text{ and } n_i \equiv 2 \text{ or } 3\}|$, $N_4 = |\{i : n_i = 2\}|$ and $N_5 = |\{i : n_i = 1\}|$. Let $\epsilon \in \{0,1\}$ be chosen such that $N_5 \equiv \epsilon$ (mod 2). Then we have*

$$f(\underline{l}; \underline{m}; \underline{n}) \leq \sum_{i=1}^{r} \left\lfloor \frac{l_i+2}{5} \right\rfloor + \sum_{i=1}^{s} \left\lfloor \frac{m_i+5}{5} \right\rfloor + \sum_{i=1}^{t} \left\lfloor \frac{n_i+8}{5} \right\rfloor - N_4 + \epsilon - \left\lfloor \frac{N_1+N_2+N_3+\epsilon}{2} \right\rfloor . \tag{2}$$

By looking at the proof of Lemma 1, it is reasonable for Breaker to claim one of the points next to the point Maker claimed on her first move, as in this case Breaker can restrict the length of intervals created by Maker. Such a first pair of moves splits the original board into two new boards, which motivates the idea of considering unions of disjoint boards. It might be tempting to say that Breaker can always follow Maker into the board where she plays her next move, and hence proceed by using an inductive proof. However, sometimes Breaker may gain an 'extra move' if one of these boards has no sensible moves left (i.e. the component is $F(1)$ or $F(2)$).

Ignoring these extra moves completely would make the proof much shorter, but the bound obtained that way would not even be good enough asymptotically. Since Maker is free to alternate between these two boards, she has some control on the time of the game when Breaker is given this extra move. In particular, in this case we cannot assume that these extra moves are given at the start of the game, which was the case in Section 2. To keep track of these extra moves, we need to consider arbitrary disjoint unions of boards.

We start by briefly outlining the structure of the proof and explaining where the upper bound in (2) comes from. The proof is by induction on the sum of the lengths of the paths. The aim is to prove that for any possible initial move for Maker, there is a move for Breaker that can be used to show that (2) holds by induction. This move will in general depend on the position of the initial move modulo 5, however we have to be slightly more careful if the initial move is close to the endpoints of a board. For the same reason, one has to be careful with small components of the board as well.

Since there are 3 possible board types, 5 possible locations for the initial move (mod 5), and two possible cases for the size of the initial length of the component (depending on whether the initial length is involved in one of N_1, N_2 or N_3, or not), it follows that there are in some sense 30 cases to be considered. In addition, we have to cover small cases as well. Fortunately, some of these cases can be treated simultaneously, and in general the techniques used to prove various cases are identical or use very similar techniques.

In a sense, the key idea behind the proof is to come up with a suitable upper bound in (2) that is strong enough for an inductive argument to work. Once a suitable upper bound is chosen, identifying possible 'response moves' for Breaker is reasonably easy. Finally, the proof itself is mathematically not challenging, but it is reasonably tedious.

Why should we choose this particular upper bound in (2)? For $B = F(l)$, $G(m)$ or $H(n)$ (with $n \geq 3$) it turns out that Breaker can always guarantee that the score is at most $\left\lfloor \frac{l+2}{5} \right\rfloor$, $\left\lfloor \frac{m+5}{5} \right\rfloor$ or $\left\lfloor \frac{n+8}{5} \right\rfloor$, respectively. This explains the first three sums in the upper bound. Moreover, if $l \equiv 3$ or 4, $m \equiv 0$ or 1 or $n \equiv 2$ or 3 (and $n \geq 3$) it turns out that Breaker has a strategy which allows her to force Maker to either play the last non-trivial move (i.e. after which all components are either empty, $F(1)$ or $F(2)$), or Maker can only attain a score which is strictly less than this bound. Hence the quantity $N_1 + N_2 + N_3$ is measuring the number of these 'additional moves'. Given such an additional move, Breaker can make another component of the board slightly shorter, which either reduces the score by one or guarantees that she will gain an extra move from that board as well.

However, one has to be careful with small values of n. Indeed, it turns out that on $H(2)$ Maker can only increase the score by 1 (instead of 2), and Breaker cannot gain an extra turn. This is the reason behind the $(-N_4)$-term. Also, on $H(1)$ Maker can score 2 points (instead of 1), and Breaker gains an extra turn. Note that if the number of components of the form $H(1)$ is even, then Breaker can always claim a point on another component that is $H(1)$. If the number is odd, she can follow this pairing strategy until the number of such boards decreases to 1, in which case she has to use the extra move elsewhere. This is the reason behind the fact that only the parity of N_5 matters.

In a sense, dealing with boards of the form $H(n)$ is the hardest task due to irregular behavior of these boards when n is small. Hence we start the proof by considering these type of boards, and during the proof we also introduce some standard arguments that can be easily used when dealing with boards of the form $F(l)$ or $G(m)$. In those cases, we do not always give full justification.

Note that the bound (2) may not always be tight for arbitrary disjoint unions of boards, but by a similar argument as presented in Section 2 one could verify that it is tight for any of $F(l)$, $G(m)$ or $F(n)$, which is good enough for our purposes. The reason why the bound is not necessarily tight is the fact that sometimes Breaker could have a better place to play her extra move, rather than the 'worst case scenario' that is considered in the proof.

For convenience define

$$g(\underline{l};\underline{m};\underline{n}) = \sum_{i=1}^{r} \left\lfloor \frac{l_i+2}{5} \right\rfloor + \sum_{i=1}^{s} \left\lfloor \frac{m_i+5}{5} \right\rfloor + \sum_{i=1}^{t} \left\lfloor \frac{n_i+8}{5} \right\rfloor - N_4 + \epsilon - \left\lfloor \frac{N_1+N_2+N_3+\epsilon}{2} \right\rfloor,$$

$$y(\underline{l};\underline{m};\underline{n}) = \sum_{i=1}^{r} \left\lfloor \frac{l_i+2}{5} \right\rfloor + \sum_{i=1}^{s} \left\lfloor \frac{m_i+5}{5} \right\rfloor + \sum_{i=1}^{t} \left\lfloor \frac{n_i+8}{5} \right\rfloor$$

and

$$z(\underline{l};\underline{m};\underline{n}) = -N_4 + \epsilon - \left\lfloor \frac{N_1+N_2+N_3+\epsilon}{2} \right\rfloor.$$

For later purposes, it is convenient to observe that we may rewrite z as

$$z(\underline{l};\underline{m};\underline{n}) = -N_4 - \left\lfloor \frac{N_1+N_2+N_3-\epsilon}{2} \right\rfloor. \tag{3}$$

Proof. Define $N = \sum_{i=1}^{r} l_i + \sum_{i=1}^{s} m_i + \sum_{i=1}^{t} n_i$. The proof is by induction on N, and it is easy to check that the claim holds for all possible configurations when $N = 1$ or $N = 2$. Suppose the claim holds whenever $N \leq M - 1$ for some $M \geq 3$, and suppose that $\underline{l}$, $\underline{m}$, $\underline{n}$ are chosen so that $\sum_{i=1}^{r} l_i + \sum_{i=1}^{s} m_i + \sum_{i=1}^{t} n_i = M$.

We now split the proof into several cases depending on Maker's first move. In each case, let $S(T)$ be the maximum score that Maker can attain given this first move and given that Breaker plays optimally.

Case 1. Maker plays on $H(n_t)$.

For convenience set $n = n_t$. The game $H(n)$ is played on $\{1,\ldots,n\}$, and since both endpoints of the board are symmetric we may assume that Maker claims first an element j satisfying $j \leq \lceil \frac{n}{2} \rceil$. We prove that apart from small values of n, claiming one of $j-1$ or $j+1$ is a suitable choice for Breaker, where the choice is made depending on the value of $j \pmod 5$, as indicated in Table 1. If $j \geq 3$, after the first pair of moves it is easy to see that the $H(n)$-component of the board splits into a disjoint union of $H(a)$ and $G(b)$ for some a, b with $n = a+b+2$. However, as the boards $H(1)$ and $H(2)$ behave in a different way compared to other boards of the form $H(n)$, it turns out to be convenient to consider the cases $j=1$, $j=2$ and $(j,n) = (3,5)$ individually.

If $4 \leq j \leq \lceil \frac{n}{2} \rceil$, then the board splits into $H(a)$ and $G(b)$ with $a \geq 3$. If $j=3$, then as indicated in Table 1 Breaker claims the element 2. Hence the boards splits into $G(1)$ and $H(n-3)$, which is one of $H(1)$ or $H(2)$ only if $n=5$, as $j \leq \lceil \frac{n}{2} \rceil$. Hence $j=1$, $j=2$ and $(j,n)=(3,5)$ are the only special cases which could change the number of boards of the form $H(1)$ or $H(2)$.

Denote the new set of parameters obtained after the first pair of moves as $\underline{l}'$, $\underline{m}'$ and $\underline{n}'$, and let s_i denote the increment of the score caused by Maker's first move. Throughout the proof it is convenient to define the quantities $d_1 = z(\underline{l};\underline{m};\underline{n}) - z(\underline{l}';\underline{m}';\underline{n}')$ and $d_2 = y(\underline{l};\underline{m};\underline{n}) - y(\underline{l}';\underline{m}';\underline{n}')$. Note that $g(\underline{l};\underline{m};\underline{n}) = d_1 + d_2 + g(\underline{l}';\underline{m}';\underline{n}')$.

By induction we know that $S(T) \leq g(\underline{l}';\underline{m}';\underline{n}') + s_i$. Since our aim is to prove that $S(T) \leq g(\underline{l};\underline{m};\underline{n})$, it suffices to prove that we always have $d_1 + d_2 \geq s_i$. In fact, we will prove that for all possible initial moves by Maker there exists a move for Breaker that satisfies $d_1 + d_2 \geq s_i$.

We start with the general case $j \geq 3$ and $n \geq 6$, and we deal with the special cases later.

	$F(n)$	Condition on a or b	$G(n)$	Condition on a or c	$H(n)$	Condition on a or b
$j \equiv 0$	$j+1$	$b \equiv 4$	$j-1$	$a \equiv 3$	$j-1$	$b \equiv 3$
$j \equiv 1$	$j-1$	$a \equiv 4$	$j-1$	$a \equiv 4$	$j-1$	$b \equiv 4$
$j \equiv 2$	$j+1$	$b \equiv 1$	$j+1$	$c \equiv 1$	$j+1$	$a \equiv 1$
$j \equiv 3$	$j-1$	$a \equiv 1$	$j-1$	$a \equiv 1$	$j-1$	$b \equiv 1$
$j \equiv 4$	$j-1$	$a \equiv 2$	$j+1$	$c \equiv 3$	$j+1$	$a \equiv 3$

Table 1: Choices for Breaker's first move depending on j

Case 1.1. $n \geq 6$, $j \geq 3$.

In this case we have $s_i = 0$, so it suffices to prove that $d_1 + d_2 \geq 0$. It is easy to see that N_1, N_4 and ϵ are unaffected in this case. Since N_2 certainly cannot increase and N_3 can decrease by at most 1, it follows that $d_1 \geq -1$.

Note that we have $d_2 = \left\lfloor \frac{n+8}{5} \right\rfloor - \left\lfloor \frac{a+8}{5} \right\rfloor - \left\lfloor \frac{b+5}{5} \right\rfloor$. By using the trivial upper and lower bounds $x - 1 \leq \lfloor x \rfloor \leq x$ and the fact that $n = a + b + 2$, it follows that $d_2 \geq \frac{n+3}{5} - \frac{a+b+13}{5} = \frac{-8}{5}$. As d_2 is an integer, it follows that $d_2 \geq -1$. We now split into two subcases based on the value of $n \pmod 5$ in order to improve our bounds on d_1 and d_2 to attain $d_1 + d_2 \geq 0$.

`Case 1.1.1.` $n \equiv 2$ or 3.

We start by improving the bound on d_2. Since $n \equiv 2$ or 3, it follows that $\left\lfloor \frac{n+8}{5} \right\rfloor \geq \frac{n+7}{5}$. Hence by using the trivial bounds for the other terms, we obtain that $d_2 \geq \frac{n+7}{5} - \frac{a+b+13}{5} = \frac{-4}{5}$. Since d_2 is an integer, it follows that $d_2 \geq 0$.

First suppose that $a \equiv 2$ or 3. Then N_3 cannot decrease, so in fact we have $d_1 \geq 0$. Hence we have $d_1 + d_2 \geq 0$, as required.

Now suppose that $b \equiv 0$ or 1. Then N_3 decreases by at most 1 and N_2 increases by 1. Hence the sum $N_2 + N_3$ certainly cannot decrease. Thus we also have $d_1 \geq 0$, and thus it follows that $d_1 + d_2 \geq 0$, as required.

Finally suppose that $a \not\equiv 2$ and 3 and $b \not\equiv 0$ and 1. Then we have $\left\lfloor \frac{a+8}{5} \right\rfloor + \left\lfloor \frac{b+5}{5} \right\rfloor \leq \frac{a+6}{5} + \frac{b+3}{5} = \frac{a+b+9}{5}$. Note that the equality holds if and only if $a \equiv 4$ and $b \equiv 2$, but by Table 1 it follows that this can never happen. Hence this inequality must be strict, and hence it follows that $d_2 > \frac{n+7}{5} - \frac{a+b+9}{5} = 0$. Hence we must have $d_2 \geq 1$, and combining this with the trivial bound $d_1 \geq -1$ it follows that $d_1 + d_2 \geq 0$, as required. This completes the proof of Case 1.1.1.

`Case 1.1.2.` $n \not\equiv 2$ and 3.

Since $n \not\equiv 2$ and 3, it follows that N_3 cannot decrease. Hence we must have $d_1 \geq 0$.

First suppose that $a \equiv 2$ or 3 and $b \equiv 0$ or 1. Then both N_2 and N_3 increase by 1, and hence it follows that $d_1 \geq 1$. Combining this with the trivial bound $d_2 \geq -1$ implies that $d_1 + d_2 \geq 0$, as required.

Now suppose that $a \not\equiv 2$ and 3 or $b \not\equiv 0$ and 1. As in Case 1.1.1, in both cases we can improve the upper bound on $\left\lfloor \frac{a+8}{5} \right\rfloor + \left\lfloor \frac{b+5}{5} \right\rfloor$ to $\left\lfloor \frac{a+8}{5} \right\rfloor + \left\lfloor \frac{b+5}{5} \right\rfloor \leq \frac{a+b+11}{5}$, and note that the equality holds if and only if ($a \equiv 4$ and $b \equiv 0$) or ($a \equiv 2$ and $b \equiv 2$). However, note that by Table 1 both of these cases are impossible. Hence the inequality must be strict, and thus we have $d_2 > \frac{n+4}{5} - \frac{a+b+11}{5} = -1$. Hence it follows that $d_2 \geq 0$, and thus we have $d_1 + d_2 \geq 0$, which completes the proof of Case 1.1.2.

`Case 1.2.` $j = 1$.

Here we split into three cases based on the size of n. First, we consider the case $n \geq 3$ which should be viewed as the main part of the argument. Then we consider the cases $n = 2$ and $n = 1$ individually, as these behave in a slightly different way as the boards are small. The case $n = 1$ turns out to be very tedious and lengthy, and it does not really contain any interesting ideas either. In some sense, the only task in this case is to find out a good enough way for Breaker to use her additional move.

`Case 1.2.1.` $n \geq 3$.

Suppose Breaker claims the element 2. Since Maker claimed 1 on board $H(n)$ with $n \geq 3$, it follows that $s_i = 1$. Hence it suffices to prove that with this move Breaker can achieve $d_1 + d_2 \geq 1$. First of all, note that the board $H(n)$ was replaced by $G(n-2)$, which is non-empty as $n \geq 3$. As $n \geq 3$, and $n \equiv 2$ or 3 if and only if $n - 2 \equiv 0$ or 1, it follows that N_3 decreases by 1 if and only if N_2 increases by 1. In particular, it follows that $d_1 = 0$. On the other hand, it is easy to check that $d_2 = \left\lfloor \frac{n+8}{5} \right\rfloor - \left\lfloor \frac{(n-2)+5}{5} \right\rfloor = 1$. Hence we always have $d_1 + d_2 = 1$, which completes the proof of Case 1.2.1.

`Case 1.2.2.` $n = 2$.

Suppose Breaker claims the element 2. Since the board $H(2)$ has only two elements, it follows that all elements of the board are occupied after this pair of moves. Note that we certainly have $s_i = 1$, and $d_2 = \left\lfloor \frac{2+8}{5} \right\rfloor = 2$. On the other hand, it is clear that N_1, N_2, N_3 and ϵ remain unaffected while N_4 decreases by 1. Hence we have $d_1 = -1$, and thus $d_1 + d_2 = 1$ as required. This completes the proof of Case 1.2.2.

`Case 1.2.3.` $n = 1$.

Since $n = 1$, it follows that $s_i = 2$. First suppose $N_5 > 1$, and that Breaker chooses another board of the form $H(1)$ and claims the only element on that board. Hence N_5 decreases by 2, so ϵ remains unaffected and thus we have $d_1 = 0$. On the other hand, we have $d_2 = 2\left\lfloor \frac{1+8}{5} \right\rfloor = 2$, and hence it follows that $d_1 + d_2 = s_i$, as required.

Otherwise we must have $N_5 = 1$, and hence we certainly have $\epsilon = 1$. Since the total number of points on T is strictly more than 1, it follows that there exists another component B of T.

First suppose that $B = H(2)$ and that Breaker claims the element 1. Then N_2 increases by 1, N_4 decreases by 1 and ϵ is replaced by $1 - \epsilon$. Hence $N_1 + N_2 + N_3 - \epsilon$ increases by 2 and $-N_4$ increases by 1, so we have $d_1 = 0$. Note that $d_2 = \left\lfloor \frac{1+8}{5} \right\rfloor + \left\lfloor \frac{2+8}{5} \right\rfloor - \left\lfloor \frac{1+5}{5} \right\rfloor = 2$, and thus it follows that $d_1 + d_2 = s_i$, as required.

Now suppose that $B = H(m)$ with $m \geq 3$. Suppose that Breaker claims the element 1, and hence B is replaced by $G(m-1)$. Then N_4 remains unaffected, N_3 decreases by at most 1 and N_2 increases by at most one. Since ϵ changes from 1 to 0, it follows that $N_1 + N_2 + N_3 - \epsilon$ cannot decrease, and hence we have $d_1 \geq 0$. Note that we have $d_2 = \left\lfloor \frac{1+8}{5} \right\rfloor + \left\lfloor \frac{m+8}{5} \right\rfloor - \left\lfloor \frac{m+4}{5} \right\rfloor$, and thus we trivially have $d_2 \geq 1$.

If $m \equiv 1$, then $m - 1 \equiv 0$ and thus N_2 increases by 1 but N_3 does not decrease. Hence $N_1 + N_2 + N_3 - \epsilon$ increases by 2, and thus $d_1 \geq 1$. If $m \not\equiv 1$, then we certainly have $d_2 \geq 2$. Hence in either case we have $d_1 + d_2 \geq 2$, as required.

Next suppose that $B = G(m)$, and suppose that Breaker claims the element 1. Hence B is replaced by $F(m-1)$. As above, it is easy to deduce that N_4 remains unaffected and $N_1 + N_2 + N_3 - \epsilon$ cannot decrease, and hence we have $d_1 \geq 0$. We also have $d_2 = \left\lfloor \frac{1+8}{5} \right\rfloor + \left\lfloor \frac{m+5}{5} \right\rfloor - \left\lfloor \frac{m+1}{5} \right\rfloor$, and thus $d_2 \geq 1$.

If $m \equiv 4$, then $m - 1 \equiv 3$ and thus N_1 increases by 1 but N_2 does not decrease. Hence we can similarly deduce that $d_1 \geq 1$. Otherwise, it is easy to see that $d_2 \geq 2$. Hence in either case we have $d_1 + d_2 \geq 2$, as required. Note that the same argument also applies even when $m = 1$ (with the convention that $F(0)$ is an empty board).

Finally suppose that $B = F(m)$, and suppose that Breaker claims the element $m - 2$. Hence B is replaced by disjoint union of $F(m-3)$ and $F(2)$, but the component of the form $F(2)$ can be omitted as on this board Breaker can follow pairing strategy to avoid any increment in the score. Again, we know that N_1 cannot increase by more than 1, and hence $d_1 \geq 0$. We also have $d_2 = \left\lfloor \frac{1+8}{5} \right\rfloor + \left\lfloor \frac{m+2}{5} \right\rfloor - \left\lfloor \frac{m-1}{5} \right\rfloor$, and hence $d_2 \geq 1$.

If $m \equiv 3$, 4 or 5, then we certainly have $d_2 \geq 2$. If $m \equiv 1$ or 2, then $m - 3 \equiv 3$ or 4, and hence N_1 increases by 1. Hence $N_1 + N_2 + N_3 - \epsilon$ increases by 2, and thus we must have $d_1 \geq 1$. Hence in either case we have $d_1 + d_2 \geq 2$. This completes the proof of Case 1.2.3.

`Case 1.3.` $j = 2$.

Since $j \leq \left\lceil \frac{n}{2} \right\rceil$, it follows that we must have $n \geq 3$. Hence we split into cases based on whether $n \geq 5$, $n = 4$ or $n = 3$.

`Case 1.3.1.` $n \geq 5$.

Suppose Breaker claims the element 1. Hence $s_i = 0$, and the board becomes a copy of $H(n-2)$. Since $n - 2 \geq 3$, it follows that N_4 and ϵ remain unchanged.

If $n \not\equiv 2$ and 3, then N_3 cannot decrease. Hence it follows that $d_1 \geq 0$. We also have $d_2 = \left\lfloor \frac{n+8}{5} \right\rfloor - \left\lfloor \frac{n+6}{5} \right\rfloor \geq 0$, and thus $d_1 + d_2 \geq 0$ as required.

If $n \equiv 2$ or 3, then N_3 decreases by at most 1 and hence we have $d_1 \geq -1$. We again have $d_2 = \left\lfloor \frac{n+8}{5} \right\rfloor - \left\lfloor \frac{n+6}{5} \right\rfloor$, and since $n \equiv 2$ or 3 it follows that $d_2 \geq 1$. Thus $d_1 + d_2 \geq 0$, which completes the proof of Case 1.3.1.

`Case 1.3.2.` $n = 4$.

Again suppose that Breaker claims the element 1. Hence $s_i = 0$, and since $4 \not\equiv 2$ and 3 it follows that N_1, N_2, N_3 and ϵ remain unaffected. On the other hand, by definition we know that N_4 increases by 1 as after this pair of moves the board becomes $H(2)$. Hence we have $d_1 = 1$. We also have $d_2 = \left\lfloor \frac{4+8}{5} \right\rfloor - \left\lfloor \frac{2+8}{5} \right\rfloor = 0$, and thus it follows that $d_1 + d_2 = 1 > 0$, which completes the proof of Case 1.3.2.

`Case 1.3.3.` $n = 3$.

Again suppose that Breaker claims the element 1, and thus we have $s_i = 0$. Note that after this pair of moves we are left with $H(1)$, and it is easy to verify that $d_2 = \left\lfloor \frac{3+8}{5} \right\rfloor - \left\lfloor \frac{1+8}{5} \right\rfloor = 1$.

It is clear that N_1, N_2 and N_4 remain unchanged. It is easy to observe that N_3 decreases by 1, and ϵ is replaced by $1-\epsilon$. Hence in the worst case $N_3-\epsilon$ decreases by 2, and thus by (3) it follows that $d_1 \geq -1$, and hence we have $d_1+d_2 \geq 0$. This completes the proof of Case 1.3.3.

`Case 1.4.` $n=5$ and $j=3$.

Suppose that Breaker claims the element 2. Hence the board $H(5)$ splits into $G(1)$ and $H(2)$, and we have $s_i=0$. Hence N_4 increases by 1, N_2 increases by 1 and N_1, N_3 and ϵ remain unaffected. Thus we must have $d_1 \geq 1$. On the other hand, note that $d_2 = \left\lfloor \frac{5+8}{5} \right\rfloor - \left\lfloor \frac{2+8}{5} \right\rfloor - \left\lfloor \frac{1+5}{5} \right\rfloor = -1$. Hence it follows that $d_1+d_2 \geq 0$, which completes the proof of Case 1.4.

This completes the proof of Case 1.

`Case 2.` Maker plays on $G(m_s)$.

For convenience set $n=m_s$. The game $G(n)$ is played on $\{1,\ldots,n\}$, and note that in this case the board is not symmetric. Hence we choose the labeling so that claiming the element 1 increases the score by 1, but claiming the element n does not.

Assume that Maker plays her first move in position j. As before we prove that claiming $j-1$ or $j+1$ is a suitable choice for Breaker, and this choice is again determined by $j \pmod 5$. We use the same notation as before, however in this case there are two options on how the board might split: the board either splits into components of the form $G(a)$ and $G(b)$ if Breaker claims $j-1$, or into components of the form $H(c)$ and $F(d)$ if Breaker claims $j+1$. In this case we only need to consider the cases $j=1$, $j=2$ and $j=n$ individually, and note that hence we may assume that $n \geq 4$. We start by checking the special cases, and we skip some of the details when they are identical to the arguments used in Case 1.

`Case 2.1.` $j=1$.

This is essentially identical to the proof of Case 1.2.1. Indeed, suppose Breaker claims the element 2. Hence after the first pair of moves the board becomes $F(n-2)$ and we have $s_i=1$. As in the proof of Case 1.2.1, we have $s_i=1$ and $d_2 = \left\lfloor \frac{n+5}{5} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor = 1$. Also as in Case 1.2.1, it follows that N_2 decreases by 1 if and only if N_1 increases by 1, and hence $d_1=0$. Thus $d_1+d_2=1$, which completes the proof of Case 2.1.

`Case 2.2.` $j=2$.

Suppose that Breaker claims the element 1. Note that in this case we do not claim the response move according to Table 1, as claiming the element 3 would generate $H(1)$ as one of the component. In this case the irregular behaviour of $H(1)$ would cause some difficulties.

It is easy to see that after the first pair of moves the board becomes $G\,(n-2)$, and we have $s_i = 0$. Note that N_2 can decrease by at most 1, and hence $d_1 \geq -1$. We also have $d_2 = \left\lfloor \frac{n+5}{5} \right\rfloor - \left\lfloor \frac{n+3}{5} \right\rfloor$, and thus we certainly have $d_2 \geq 0$.

If $n \equiv 0$ or 1, it is easy to verify that we have $d_2 = 1$, and hence $d_1 + d_2 \geq 0$ as required. Otherwise it follows that N_2 cannot decrease, and hence we must have $d_1 \geq 0$. Thus $d_1 + d_2 \geq 0$ holds in this case as well, which completes the proof of Case 2.2.

`Case 2.3.` $j = n$.

Suppose that Breaker claims the element $n-1$. After this pair of moves the board becomes $G\,(n-2)$, and by using exactly the same analysis as in Case 2.2 it follows that $d_1 + d_2 \geq 0$.

`Case 2.4.` $3 \leq j \leq n-1$.

Suppose that Breaker chooses the appropriate move indicated in Table 1 depending on the value of $j \pmod 5$. Note that depending on the value of j, the board may split into components of the form $G\,(a)$ and $G\,(b)$ or of the form $H\,(c)$ and $F\,(d)$. We now consider several cases, depending on the value value of $n \pmod 5$ and depending on how the board splits into two components. As in Case 1.1, we have the trivial lower bounds $d_1 \geq -1$ and $d_2 \geq -1$.

`Case 2.4.1.` $n \equiv 0$ or 1 and $j \equiv 0$, 1 or 3.

As Breaker claims the element $j-1$, the board splits into components of the form $G\,(a)$ and $G\,(b)$. As in Case 1.1.1, by using the trivial upper and lower bounds for $\lfloor x \rfloor$ it follows that $d_2 \geq \frac{n+4}{5} - \frac{a+b+10}{5} = \frac{-4}{5}$. Since d_2 is an integer, it follows that $d_2 \geq 0$.

Note that from Table 1 we can conclude that $a \equiv 1$, 3 or 4. First suppose that $a \equiv 1$ or $b \equiv 0$ or 1. Then N_2 certainly does not decrease, so $d_1 \geq 0$. Hence $d_1 + d_2 \geq 0$, as required.

Otherwise we must have $a \equiv 3$ or 4 and $b \equiv 2$, 3 or 4. Hence we must have $\left\lfloor \frac{a+5}{5} \right\rfloor + \left\lfloor \frac{b+5}{5} \right\rfloor \leq \frac{a+2}{5} + \frac{b+3}{5} = \frac{n+3}{5}$. Since $n \equiv 0$ or 1, it follows that $d_2 \geq \frac{n+4}{5} - \frac{n+3}{5} > 0$, and thus $d_2 \geq 1$. Hence $d_1 + d_2 \geq 0$, which completes the proof of Case 2.4.1.

`Case 2.4.2.` $n \equiv 0$ or 1 and $j \equiv 2$ or 4.

As in the previous case we can deduce that $d_2 \geq 0$. Since Breaker claims the element $j+1$, the board splits into components of the form $H\,(c)$ and $F\,(d)$. Since $j > 2$, it follows that $j \geq 4$ and thus $c \geq 3$. Hence N_4 and ϵ are unaffected by the first pair of moves. Again, we will split into cases depending on whether or not one of $c \equiv 2$ or 3 or $d \equiv 3$ or 4 holds. The details follow exactly as in the previous case, and hence we omit the proof.

`Case 2.4.3.` $n \not\equiv 0$ and 1.

Now regardless of how the board splits into two components we can deduce that $d_1 \geq 0$, as none of the N_i's can decrease. Again, the rest of the proof is similar to the proof of Case 1.1.2 (with appropriate modifications similar to those done in Case 2.4.1). Hence we skip the details.

This completes the proof of Case 2.

Case 3. Maker plays on $F(l_r)$.

For convenience set $n = l_r$. The game $F(n)$ is played on $\{1, \ldots, n\}$, and this time the board is again symmetric. Hence we may assume that Maker plays her first move j in a position with $j \leq \left\lceil \frac{n}{2} \right\rceil$.This time the only special case that needs to be considered is $j = 1$, and again we prove that for $j \geq 2$ claiming $j-1$ or $j+1$ is a suitable choice for Breaker, and this choice is determined by $j \pmod 5$. Apart from the case $j = 1$, the board always splits into two boards of the form $F(a)$ and $G(b)$ for some a and b with $n = a + b + 2$. We use the same notation as in the earlier cases.

Case 3.1. $j = 1$.

Suppose Breaker claims the element 2. Then $s_i = 0$ and the board becomes $F(n-2)$. Hence $d_2 = \left\lfloor \frac{n+2}{5} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor$, which is certainly always non-negative. Since N_1 decreases by at most 1, it follows that $d_1 \geq -1$.

If $n \equiv 3$ or 4 then we have $d_2 \geq 1$ and hence $d_1 + d_2 \geq 0$, as required. Otherwise N_1 is certainly not decreasing, so $d_1 \geq 0$. Thus we again have $d_1 + d_2 \geq 0$, which completes the proof of Case 3.1.

Case 3.2. $j \neq 1$ and $n \equiv 3$ or 4.

The proof is identical to the proof of Case 1.1.1.

Case 3.3. $j \neq 1$ and $n \not\equiv 3$ and 4.

The proof is identical to the proof of Case 1.1.2.

This completes the proof of Claim 3, and hence Lemma 3 holds by induction.

□

Recall from the Introduction that $H_b(n)$ is the game played on the same board as $H(n)$, but with Breaker having the first move. Also recall that we have $u(P_n) = \gamma_b(n)$ and $u(C_n) = \alpha(n-1)$. We now deduce Theorems 1 and 2 from our earlier results.

Proof of Theorem 1. Note that Lemma 2 implies that $u(P_n) = \gamma_b(n) \geq \left\lfloor \frac{n+4}{5} \right\rfloor$. In order to prove the upper bound, consider the game $H_b(n)$ and suppose that Breaker claims the element n on her first move. After this initial move, the game is equivalent to the game on the same board as $G(n-1)$ with Maker having the first move. Hence it follows that $\gamma_b(n) \leq f(\emptyset; n-1; \emptyset)$, and thus Lemma 3 implies that $\gamma_b(n) \leq \left\lfloor \frac{(n-1)+5}{5} \right\rfloor$. Therefore we have $u(P_n) = \left\lfloor \frac{n+4}{5} \right\rfloor$, as required. □

Proof of Theorem 2. Recall that we have $u(C_n) = \alpha(n-1)$. Hence Lemma 1 implies that $u(C_n) \geq \left\lfloor \frac{(n-1)+2}{5} \right\rfloor$, and Lemma 3 implies that $u(C_n) \leq f(n-1; \emptyset; \emptyset) = \left\lfloor \frac{(n-1)+2}{5} \right\rfloor$. Thus it follows that $u(C_n) = \left\lfloor \frac{n+1}{5} \right\rfloor$, as required. □

In particular, for both $G = P_n$ and $G = C_n$ it follows that the asymptotic proportion of isolated vertices is $\frac{1}{5}$ when both players play optimally.

4. Conclusion

There are many questions that are open concerning the value of $u(G)$ for general G. In [4], the authors gave bounds for $u(G)$ that depended on the degree sequence of the graph G. As a consequence they concluded that if the minimum degree of G is at least 4, then $u(G) = 0$. They also noted that there exists a 3-regular graph with $u(G) > 0$, and they proved that the largest possible proportion of untouched vertices among all 3-regular graphs is between $\frac{1}{24}$ and $\frac{1}{8}$. It would be interesting to know what the exact value is. Their example for the proportion $\frac{1}{24}$ is not connected, so it would also be interesting to know what the maximal proportion is for connected 3-regular graphs.

They also proved that if T is a tree with n vertices, then it follows that $\left\lceil \frac{n+2}{8} \right\rceil \leq u(T) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. The upper bound is tight when T is a star, but they did not find a similar infinite family of examples for which the lower bound is tight. It would be interesting to know whether this lower bound is asymptotically correct.

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